Fixed Point Theory, 11(2010), No. 2, 225-236
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

AN EXPLICIT ITERATIVE METHOD FOR FINDING A COMMON SOLUTION OF EQUILIBRIUM AND FIXED POINT PROBLEMS

RUDONG CHEN*, QINWEI FAN* AND HONG-KUN XU**,†

Dedicated to Wataru Takahashi on the occasion of his retirement

*Department of Mathematics, Tianjin Polytechnic University
Tianjin 300160, China
E-mails: tjcrd@yahoo.com.cn; fanqinwei2008@yahoo.com.cn

**Department of Applied Mathematics, National Sun Yat-sen University
Kaohsiung 80424, Taiwan
and
Department of Mathematics, College of Science
King Saud University, Riyadh 11451, Saudi Arabia
E-mail: xuhk@math.nsysu.edu.tw

Abstract. We provide an iterative method for finding a common solution of a finite family of equilibrium problems and of a fixed point problem, and prove its strong convergence. Our method extends an implicit method of Colao, et al. (Nonlinear Anal. 71 (2009), no. 7-8, 2708-2715) to an explicit method (in the case of a single nonexpansive mapping).

Key Words and Phrases: Equilibrium problem, fixed point, iterative method, nonexpansive mapping, projection, variational inequality.
2010 Mathematics Subject Classification: 47H09, 47H10, 47J20, 49J30.

1. Introduction

Equilibrium problems (EPs) have recently been received a great amount of investigation. An equilibrium problem can be formulated as finding a point x^* satisfying the property [2, 3]:

$$x^* \in C, \quad G(x^*, y) \geq 0, \quad y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and $G : C \times C \to \mathbb{R}$ is a so-called bifunction function.

The solution set of EP (1.1) is denoted as EP(G); namely,

$$\mathrm{EP}(G) := \{x \in C : G(x, y) \geq 0 \ \forall y \in C\}. \tag{1.2}$$

To solve EP (1.1), the following conditions on the bifunction G are assumed in literature:

†Corresponding author.

(A_1) $G(x,x) = 0$ for all $x \in C$;
(A_2) G is monotone, i.e., $G(x,y) + G(y,x) \leq 0$ for all $x, y \in C$;
(A_3) for each $x, y, z \in C$, $\lim_{t\to 0} G(tz + (1-t)x, y) \leq G(x,y)$;
(A_4) for each $x \in C$, $y \mapsto G(x,y)$ is convex and lower semicontinuous.

On the other hand, fixed point problems (FPPs) [11] have widely been investigated. A fixed point problem is formulating as finding a point $\hat{x}$ with the property:

$$T\hat{x} = \hat{x}, \tag{1.3}$$

where $T : C \to C$ is a (nonlinear) mapping. The solution set of (1.3) (or fixed point set of T) is denoted as $Fix(T)$. Of course, to solve FPP (1.3), the (possibly noncompact) operator T is assumed to be *nonexpansive*:

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C.$$

Recently, iterative methods have been developed to find a common solution of EPs and FPPs; namely, find an x^* with the property:

$$x^* \in \left(\bigcap_{j=1}^{N} EP(G_j)\right) \bigcap \left(\bigcap_{l=1}^{M} Fix(T_l)\right), \tag{1.4}$$

where $N, M \geq 1$ are integers, and where $\{G_j\}_{j=1}^N$ and $\{T_l\}_{l=1}^M$ are bifunctions and nonexpansive mappings on C, respectively. For the sake of simplicity, we will consider in this paper the case of a single nonexpansive (i.e., $M = 1$); namely, the problem of finding an x^* with the property:

$$x^* \in \left(\bigcap_{j=1}^{N} EP(G_j)\right) \cap Fix(W), \tag{1.5}$$

where W is a nonexpansive mapping on C.

One of the key tools to iteratively study problem (1.5) (or the more complicated (1.4)) is Combettes and Hirstoaga's ([9]) firmly nonexpansive mapping S_r associating with a bifunction G (see Lemma 2.2 in the next section). This enables that several implicit or explicit iterative methods have been invented for solving (1.4) and (1.5). For instance, Colao et al. [8] introduced an implicit method that generates a sequence $\{x_n\}$ via the implicit relation:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n) W S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^N_{r_{N,n}} x_n, \tag{1.6}$$

where $f : C \to C$ is a contraction, $\{\alpha_n\}$ is a sequence in the interval $(0,1)$, and for each $1 \leq j \leq N$, $\{r_{j,n}\}_{n=1}^\infty$ is a sequence of positive real numbers, and $S^j_{r_{j,n}}$ is the firmly nonexpansive mapping associating with the bifunction G_j.

It is the purpose of this paper to extend the implicit method (1.6) to an explicit one. Namely, we want to study the asymptotic behavior of the sequence $\{x_n\}$ which, starting with an initial guess $x_0 \in C$, is generated by the explicit iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) W S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^N_{r_{N,n}} x_n. \tag{1.7}$$

Under certain conditions on the sequences $\{\alpha_n\}$ and $\{r_{j,n}\}$, we will prove that the sequence $\{x_n\}$ converges in norm to the unique solution x^* of some variational inequality (to be specified in Theorem 3.5).

For some recent developments on this topic, the reader can consult with the articles [19, 13, 14, 18, 17, 20, 5, 6, 7, 24] and the references therein. Also, we notice that iterative methods for nonexpansive mappings have extensively been investigated, see [21, 22, 16], the recent articles [1, 4, 10, 12, 23], and the recent survey [15].

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying properties (A_1)-(A_4) listed in section one.

The following two lemmas are pertinent.

Lemma 2.1. *(Blum and Oettli* [3]*.) Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.2. *(Combettes and Hirstoaga* [9]*.) Given $r > 0$. Define a mapping $S_r : H \to C$ by*

$$S_r(x) = \left\{ z \in C : \ G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}, \quad x \in H. \tag{2.1}$$

Then the following hold:

(a) *S_r is single-valued;*
(b) *S_r is firmly nonexpansive, i.e., $\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle$ for any $x, y \in H$;*
(c) *$F(S_r) = EP(G)$;*
(d) *$EP(G)$ is closed and convex.*

Lemma 2.2 makes it possible to use nonexpansive mappings to iteratively approximate solutions of equilibrium problems.

We need the so-called *demiclosedness principle* for nonexpansive mappings. Recall that a mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Lemma 2.3. (cf. [11]) *Let C be a nonempty closed convex subset of a real Hilbert space and let $T : C \to H$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C. Then the conditions that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly imply that $x = Tx$.*

The following lemma, though elementary, is helpful in proving strong convergence of sequences even in infinite-dimensional spaces.

Lemma 2.4. *(cf.* [22, 16]*) Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) *either* $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. An Explicit Iterative Method

Suppose that $N \ge 1$ is a positive integer and $\{G_1, G_2, \cdots, G_N\}$ are bifunctions from $C \times C$ into $\mathbb{R}$, each of which satisfies properties $(A_1) - (A_4)$. Suppose, in addition, that $W : C \to C$ is a nonexpansive mapping with nonempty fixed point set $Fix(W)$. Our problem is to solve a system of equilibrium problems coupled with fixed point problems. More precisely, we want to find a point x^* with the property:

$$x^* \in \left(\bigcap_{j=1}^{m} \mathrm{EP}(G_j) \right) \cap Fix(W) =: F \tag{3.1}$$

upon assuming the existence of such a solution (i.e., assuming $F \ne \emptyset$).

We observe that to each G_j, we have a firmly nonexpansive mapping S_r^j which is defined via (2.1) in Lemma 2.2. To define our iterative method, we need to take an α-contraction $f : C \to C$, an initial guess $x_0 \in C$, an infinite sequence $\{\alpha_n\}_{n=0}^{\infty}$ in the interval $[0, 1]$, and, for each fixed integer $n \ge 1$, a finite sequence $\{r_{j,n}\}_{j=1}^{N}$ of positive real numbers. Assume x_n has been constructed, then the next iterate x_{n+1} is generated by the algorithm

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) W S_{r_{1,n}}^1 S_{r_{2,n}}^2 \cdots S_{r_{N,n}}^N x_n. \tag{3.2}$$

To discuss the convergence of Algorithm (3.2), we first establish several lemmas regarding the firmly nonexpansive mapping S_r given by (2.1) in Lemma 2.2 and the sequence $\{x_n\}$ generated by (3.2).

Lemma 3.1. *Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying properties $(A_1) - (A_4)$ and let S_r be its associating firmly nonexpansive mapping given by (2.1) in Lemma 2.2. Then for $x \in H$ and $r, r' > 0$, we have*

$$\|S_{r'} x - S_r x\| \le \frac{|r - r'|}{r} \|x - S_r x\|. \tag{3.3}$$

Proof. Set $u = S_r x$ and $v = S_{r'} x$. Then, by (2.1), we have

$$G(u, v) + \frac{1}{r}\langle v - u, u - x\rangle \ge 0 \quad \text{and} \quad G(v, u) + \frac{1}{r'}\langle u - v, v - x\rangle \ge 0.$$

Adding up these inequalities and noting that $G(u, v) + G(v, u) \le 0$, we obtain

$$\langle v - u, \frac{1}{r}(u - x) + \frac{1}{r'}(x - v)\rangle \ge 0.$$

This yields

$$\|v - u\|^2 \le \left(1 - \frac{r'}{r}\right) \langle u - x, v - u\rangle$$

which in turns implies

$$\|v - u\| \le \left|1 - \frac{r'}{r}\right| \|u - x\|.$$

This is (3.3). □

In the rest of the paper, we always assume $F \neq \emptyset$, and by $\{x_n\}$ we always mean the sequence generated by Algorithm (3.2). Moreover, we set

$$S_n = S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^N_{r_{N,n}}. \tag{3.4}$$

Thus, Algorithm (3.2) can equivalently be rewritten as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) W S_n x_n. \tag{3.5}$$

Lemma 3.2. *We have that $\{x_n\}$ is bounded. Consequently, the sequences $\{f(x_n)\}$, $\{S_n x_n\}$, $\{W x_n\}$ and $\{W S_n x_n\}$ are all bounded.*

Proof. Take $p \in F$ to derive from (3.5) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(W S_n x_n - p)\| \\
&\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|W S_n x_n - p\| \\
&\leq \alpha_n \alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\
&= [1 - (1-\alpha)\alpha_n]\|x_n - p\| + \alpha_n \|f(p) - p\| \\
&\leq \max\left\{ \|x_n - p\|, \frac{1}{1-\alpha}\|f(p) - p\| \right\}.
\end{aligned}$$

It turns out by induction that

$$\|x_n - p\| \leq \max\left\{ \|x_0 - p\|, \frac{1}{1-\alpha}\|f(p) - p\| \right\}, \quad \forall n \geq 0.$$

This implies that $\{x_n\}$ is bounded. □

In what follows, we always assume the following conditions for the sequences $\{\alpha_n\}$ and $\{r_{j,n}\}$ which define Algorithm (3.2):

(C_1) $\lim_{n\to\infty} \alpha_n = 0$;
(C_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C_3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ and $\sum_{n=1}^{\infty} |r_{i,n+1} - r_{i,n}| < \infty$ for $1 \leq j \leq N$;
(C_4) $\lim_{n\to\infty} r_{j,n} = r_j > 0$ for $1 \leq j \leq N$.

Lemma 3.3. *We have that*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Proof. To prove (3.6), we need to estimate $\|S_{n+1}x_{n+1} - S_n x_n\|$. Towards this, we set, for each $1 \leq j \leq N$,

$$y^j_{n+1} := S^j_{r_{j,n+1}} \cdots S^N_{r_{N,n+1}} x_{n+1} \quad \text{and} \quad y^j_n := S^j_{r_{j,n}} \cdots S^N_{r_{N,n}} x_n.$$

Due to boundedness, we can find a constant $L > 0$ big enough so that

$$L \geq \|x_{n+1} - S^j_{r_{j,n+1}} x_{n+1}\| \quad \text{for all } 1 \leq j \leq N \text{ and } n \geq 0.$$

Then, by definition of S_i for $i \geq 1$ and by Lemma 3.1, we get

$$\begin{aligned}\|S_{n+1}x_{n+1} - S_n x_n\| &= \|S^1_{r_{1,n+1}} y^2_{n+1} - S^1_{r_{1,n}} y^2_n\| \\ &\leq \|S^1_{r_{1,n+1}} y^2_{n+1} - S^1_{r_{1,n}} y^2_{n+1}\| \\ &\quad + \|S^1_{r_{1,n}} y^2_{n+1} - S^1_{r_{1,n}} y^2_n\| \\ &\leq \frac{|r_{1,n+1} - r_{1,n}|}{r_{1,n}} \|y^2_{n+1} - S^1_{r_{1,n}}\| + \|y^2_{n+1} - y^2_n\| \\ &\leq \frac{L|r_{1,n+1} - r_{1,n}|}{r_{1,n}} + \|y^2_{n+1} - y^2_n\|. \end{aligned} \tag{3.7}$$

Similarly, we have

$$\begin{aligned}\|y^2_{n+1} - y^2_n\| &= \|S^2_{r_{2,n+1}} y^3_{n+1} - S^1_{r_{2,n}} y^3_n\| \\ &\leq \|S^2_{r_{2,n+1}} y^3_{n+1} - S^2_{r_{2,n}} y^3_{n+1}\| \\ &\quad + \|S^2_{r_{2,n}} y^3_{n+1} - S^1_{r_{2,n}} y^3_n\| \\ &\leq \frac{|r_{2,n+1} - r_{2,n}|}{r_{2,n}} \|y^3_{n+1} - S^2_{r_{2,n}}\| + \|y^3_{n+1} - y^3_n\| \\ &\leq \frac{L|r_{2,n+1} - r_{2,n}|}{r_{2,n}} + \|y^3_{n+1} - y^3_n\|. \end{aligned} \tag{3.8}$$

Continue this way and observe

$$\begin{aligned}\|y^N_{n+1} - y^N_n\| &= \|S^N_{r_{N,n+1}} x_{n+1} - S^N_{r_{N,n}} x_n\| \\ &\leq \|S^N_{r_{N,n+1}} x_{n+1} - S^N_{r_{N,n}} x_{n+1}\| + \|S^N_{r_{N,n}} x_{n+1} - S^N_{r_{N,n}} x_n\| \\ &\leq \frac{|r_{N,n+1} - r_{N,n}|}{r_{N,n}} \|x_{n+1} - S^N_{r_{N,n}} x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \frac{L|r_{N,n+1} - r_{N,n}|}{r_{N,n}} + \|x_{n+1} - x_n\|. \end{aligned} \tag{3.9}$$

Substituting (3.8)-(3.9) into (3.7) gives us that

$$\|S_{n+1}x_{n+1} - S_n x_n\| \leq L \sum_{j=1}^{N} \frac{|r_{j,n+1} - r_{j,n}|}{r_{j,n}} + \|x_{n+1} - x_n\|. \tag{3.10}$$

By virtue of Condition (C_4), we have $r > 0$ such that $r_{j,n} \geq r$ for all j and n. Also, we may assume that L is big enough so that

$$L \geq \|WS_n x_n\| + \|f(x_n)\| \quad \text{for all } n.$$

We then infer that

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})WS_{n+1}x_{n+1} \\
&\quad -\alpha_n f(x_n)-(1-\alpha_n)WS_nx_n\| \\
&\le \|(1-\alpha_{n+1})(WS_{n+1}x_{n+1}-WS_nx_n)+(\alpha_n-\alpha_{n+1})WS_nx_n\| \\
&\quad +[\alpha_{n+1}(f(x_{n+1})-f(x_n))+(\alpha_{n+1}-\alpha_n)f(x_n)] \\
&\le (1-\alpha_{n+1})\|S_{n+1}x_{n+1}-S_nx_n\|+\alpha_{n+1}\alpha\|x_{n+1}-x_n\| \\
&\quad +|\alpha_n-\alpha_{n+1}|\|WS_nx_n-f(x_n)\| \\
&\le [1-(1-\alpha)\alpha_{n+1}]\|x_{n+1}-x_n\|+L\sum_{i=1}^{N}|\frac{r_{i,n+1}-r_{i,n}|}{r_{i,n}} \\
&\quad +|\alpha_n-\alpha_{n+1}|(\|WS_nx_n\|+\|f(x_n)\|) \\
&\le [1-(1-\alpha)\alpha_{n+1}]\|x_{n+1}-x_n\| \\
&\quad +L\left(\frac{1}{r}\sum_{i=1}^{N}|r_{i,n+1}-r_{i,n}|+|\alpha_n-\alpha_{n+1}|\right). \qquad (3.11)
\end{aligned}$$

Conditions (C_1)-(C_3) allow us to apply Lemma 2.4 to (3.11) to obtain (3.3). □

Lemma 3.4. *We have*

(i) *For each* $1 \le j \le N$,

$$\lim_{n\to\infty}\|x_n - S^j_{r_{j,n}}x_n\| = 0. \qquad (3.12)$$

(ii) *For each* $1 \le j \le N$ *and with* $r_j = \lim_{n\to\infty} r_{j,n}$,

$$\lim_{n\to\infty}\|x_n - S^j_{r_j}x_n\| = 0. \qquad (3.13)$$

(iii) $\lim_{n\to\infty}\|x_n - Wx_n\| = 0$.

Proof. (i) Since $S^j_{r_{j,n}}$ is firmly nonexpansive, we deduce that, for each $p \in F$,

$$\begin{aligned}
\|x_n - S^j_{r_{j,n}}x_n\|^2 &= \|(x_n-p)-(S^j_{r_{j,n}}x_n-p)\|^2 \\
&= \|x_n-p\|^2+\|S^j_{r_{j,n}}x_n-p\|^2-2\langle x_n-p, S^j_{r_{j,n}}x_n-p\rangle \\
&\le \|x_n-p\|^2+\|S^j_{r_{j,n}}x_n-p\|^2-2\|S^j_{r_{j,n}}x_n-p\|^2 \\
&= \|x_n-p\|^2-\|S^j_{r_{j,n}}x_n-p\|^2. \qquad (3.14)
\end{aligned}$$

We now use backward induction to prove (3.12); thus, we first prove that (3.12) holds for $j = N$. To see this, we compute

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)WS^1_{r_{1,n}}S^2_{r_{2,n}}\cdots S^N_{r_{N,n}}x_n-p\|^2 \\
&\le \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|WS^1_{r_{1,n}}S^2_{r_{2,n}}\cdots S^N_{r_{N,n}}x_n-p\|^2 \qquad (3.15) \\
&\le \|S^N_{r_{N,n}}x_n-p\|^2+\theta\alpha_n, \qquad (3.16)
\end{aligned}$$

where θ is a constant such that $\theta \ge \sup_{n\ge 0}\|f(x_n)-p\|^2$.

Combining (3.14) and (3.16) (with $j = N$), and using Lemma 3.3, we get

$$\|x_n - S^N_{r_{N,n}} x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \theta\alpha_n \to 0 \quad (\text{as } n \to \infty).$$

This proves (3.12) for $j = N$.

Assume now that (3.12) holds true for every $j = l+1, \cdots, N$. We next prove that (3.12) remains true for $j = l$. To see this, we use (3.15) to get

$$\|x_{n+1} - p\|^2 \le \|S^l_{r_{l,n}} \cdots S^N_{r_{N,n}} x_n - p\|^2 + \theta\alpha_n. \tag{3.17}$$

However, we also have

$$\begin{aligned}
\|S^l_{r_{l,n}} \cdots S^N_{r_{N,n}} x_n - p\| &\le \|S^l_{r_{l,n}} \cdots S^N_{r_{N,n}} x_n - S^l_{r_{l,n}} x_n\| + \|S^l_{r_{l,n}} x_n - p\| \\
&\le \|S^{l+1}_{r_{l+1,n}} \cdots S^N_{r_{N,n}} x_n - x_n\| + \|S^l_{r_{l,n}} x_n - p\| \\
&\le \|S^{l+1}_{r_{l+1,n}} \cdots S^N_{r_{N,n}} x_n - S^{l+1}_{r_{l+1,n}} x_n\| \\
&\quad + \|S^{l+1}_{r_{l+1,n}} x_n - x_n\| + \|S^l_{r_{l,n}} x_n - p\| \\
&\le \|S^{l+2}_{r_{l+2,n}} \cdots S^N_{r_{N,n}} x_n - x_n\| \\
&\quad + \|S^{l+1}_{r_{l+1,n}} x_n - x_n\| + \|S^l_{r_{l,n}} x_n - p\| \\
&\vdots \\
&\le \sum_{j=l+1}^{N} \|S^j_{r_{j,n}} x_n - x_n\| + \|S^l_{r_{l,n}} x_n - p\|.
\end{aligned} \tag{3.18}$$

Note that, by the induction assumption, we have

$$\sigma_n := \sum_{j=l+1}^{N} \|S^j_{r_{j,n}} x_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Consequently, we obtain from (3.18)

$$\|S^l_{r_{l,n}} \cdots S^N_{r_{N,n}} x_n - p\|^2 \le \|S^l_{r_{l,n}} x_n - p\|^2 + \sigma_n\delta, \tag{3.19}$$

where δ is a constant such that $\delta \ge \sup_{n\ge 0}(\sigma_n + 2\|S^l_{r_{l,n}} x_n - p\|)$. Now combining (3.14), (3.17) and (3.19), and using Lemma 3.3, we get

$$\begin{aligned}
\|x_n - S^l_{r_{l,n}} x_n\|^2 &\le \|x_n - p\|^2 - \|S^l_{r_{l,n}} x_n - p\|^2 \\
&= \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + \|x_{n+1} - p\|^2 - \|S^l_{r_{l,n}} \cdots S^N_{r_{N,n}} x_n - p\|^2 \\
&\quad + \|S^l_{r_{l,n}} \cdots S^N_{r_{N,n}} x_n - p\|^2 - \|S^l_{r_{l,n}} x_n - p\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \theta\alpha_n + \delta\sigma_n \to 0.
\end{aligned}$$

This completes the induction and hence we conclude that (3.12) is valid for all $1 \le j \le N$.

(ii) By Lemma 3.1, the fact that $r_{j,n} \to r_j > 0$ (as $n \to \infty$) and the boundedness of $\{x_n\}$, we get, for each $1 \le j \le N$,

$$\|S^j_{r_{j,n}} x_n - S^j_{r_j} x_n\| \le \frac{|r_{j,n} - r_j|}{r_j} \|x_n - S^j_{r_j} x_n\| \to 0 \quad (\text{as } n \to \infty).$$

This together with (3.12) implies (3.13).

(iii) It suffices to prove that $\|x_{n+1} - Wx_n\| \to 0$ since $\|x_{n+1} - x_n\| \to 0$ by Lemma 3.6. We have via (3.5) and with L_1 a constant such that $L_1 \ge \|f(x_n)\| + \|Wx_n\|$ for all n,

$$\begin{aligned} \|x_{n+1} - Wx_n\| &\le \alpha_n \|f(x_n) - Wx_n\| + (1-\alpha_n)\|WS_n x_n - Wx_n\| \\ &\le \alpha_n L_1 + \|S_n x_n - x_n\| \end{aligned} \tag{3.20}$$

However, by repeatedly using the triangle inequality and (3.12), it is easily found that

$$\|S_n x_n - x_n\| \le \sum_{j=1}^{N} \|S^j_{r_{j,n}} x_n - x_n\| \to 0 \quad (\text{as } n \to \infty).$$

Consequently, it follows from (3.20) that $\|x_{n+1} - Wx_n\| \to 0$ as required. □

We are now in a position to state and prove the main result of this paper.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{G_j\}_{j=1}^N$ be a family of N bifunctions from $C \times C$ into $\mathbb{R}$, each of which satisfies properties (A_1)-(A_4), and let $W : C \to C$ be a nonexpansive mapping. Assume the common solution set F as defined in (3.1) is nonempty. Let $f : C \to C$ be an α-contraction. Moreover, assume that the sequences $\{\alpha_n\}$ and $\{r_{j,n}\}_{j=1}^N$ satisfy conditions (C_1)-(C_4). Then the sequence $\{x_n\}$ generated by Algorithm (3.2) converges in norm to the unique solution x^* of the variational inequality (VI):*

$$x^* \in F, \quad \langle (I - f)x^*, x - x^* \rangle \ge 0, \quad x \in F. \tag{3.21}$$

Alternatively, x^* is the unique fixed point of the contraction $P_F f$ (i.e., $x^* = (P_F f)x^*$).

Proof. Let x^* be the unique fixed point of the contraction $P_F f$; hence the unique solution of VI (3.21). To prove that $x_n \to x^*$ in norm, we estimate the distance from x_{n+1} to x^* in the following way (in the first inequality, we use the trivial inequality $\|u + v\|^2 \le \|u\|^2 + 2\langle v, u+v\rangle$ for all $u, v \in H$):

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(WS_n x_n - x^*) + \alpha_n (f(x_n) - x^*)\|^2 \\ &\le \|(1-\alpha_n)(WS_n x_n - x^*)\|^2 + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\ &\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle \\ &\quad + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \alpha \|x_n - x^*\| \|x_{n+1} - x^*\| \\ &\quad + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + \alpha_n \alpha (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ &\quad + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle. \end{aligned}$$

It turns out that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \frac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &= \left(1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}\right)\|x_n-x^*\|^2 + \frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n-x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\ &= (1-\beta_n)\|x_n-x^*\|^2+\beta_n\gamma_n, \end{aligned} \tag{3.22}$$

where

$$\beta_n = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} = O(\alpha_n) \quad (\text{as } n\to\infty)$$

and

$$\gamma_n = \frac{1}{2(1-\alpha)}\left(\alpha_n\|x_n-x^*\|^2 + 2\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\right).$$

By conditions (C_1)-(C_3), we easily find that

$$\lim_{n\to\infty}\beta_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}\beta_n = \infty. \tag{3.23}$$

Thus, in order to apply Lemma 2.4, it remains for us to prove that $\limsup_{n\to\infty}\gamma_n \le 0$; equivalently,

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, x_n-x^*\rangle \le 0. \tag{3.24}$$

To see (3.24), we take a subsequence $\{x_{n'}\}$ of $\{x_n\}$ in such a way that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, x_n-x^*\rangle = \lim_{n'\to\infty}\langle f(x^*)-x^*, x_{n'}-x^*\rangle.$$

Due to boundedness, we may further assume, with no loss of generality, that $x_{n'}\to\hat{x}$ weakly. The demiclosedness principle (Lemma 2.3) together with Lemma 3.4(ii)-(iii) ensures that, for every $1\le j\le N$, $\hat{x}\in Fix(W)\cap Fix(S^j_{r_j})$; hence, $\hat{x}\in F$. We therefore arrive at

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, x_n-x^*\rangle = \langle f(x^*)-x^*, \hat{x}-x^*\rangle \le 0$$

due to VI (3.21).

Finally, by virtue of (3.23) and (3.24), we can apply Lemma 2.4 to the relation (3.22) to conclude that $\|x_n-x^*\|\to 0$ as $n\to\infty$. □

The case of $N=1$ recovers a result of [20].

Corollary 3.6. *(Theorem 3.2 of* [20]*) Let C be a nonempty closed convex subset of H, $G: C\times C\to\mathbb{R}$ a bifunction satisfying (A_1)-(A_4), and $W: C\to C$ a nonexpansive mapping such that $EP(G)\cap Fix(W)\ne\emptyset$. Moreover, let $f: C\to C$ be a contraction. Let $\{x_n\}$ be generated by the iterative algorithm (with initial guess $x_0\in H$):*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)WS_{r_n}x_n,$$

where $\{\alpha_n\} \subset (0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy the conditions:*

(C_1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C_2) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(C_3) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ *converges in norm to the unique fixed point of the contraction* $P_{EP(G)\cap Fix(W)}f$.

Acknowledgement. The first author was supported in part by the National Science Foundation of China under Grant no. 10771050. The third author was supported in part by NSC 97-2628-M-110-003-MY3 (Taiwan).

References

[1] A. Aleyner, S. Reich, *Approximating common fixed points of nonexpansive mappings in Banach spaces*, Fixed Point Theory, **10**(2009), No. 1, 3-17.

[2] M. Bianchi, S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl., **90**(1996), 31-43.

[3] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63**(1994), 123-145.

[4] L. C. Ceng, A. Petruşel, J. C. Yao, *Weak convergence theorem by a modified extragradient method for nonexpansive mappings and monotone mappings*, Fixed Point Theory, **9**(2008), No. 1, 73-87.

[5] L C. Ceng, J. C. Yao, *Hybrid viscosity approximation schemes for equilibrium problem and fixed point problens of infinitely many nonexpansive mappings*, Appl. Math. Comp., **198**(2008), 729-741.

[6] F. Cianciaruso, G. Marino, L. Muglia, *Ishikawa iterations for equilibrium and fixed point problems for nonexpansive mappings in Hilbert spaces*, Fixed Point Theory, **9**(2008), No. 2, 449-464.

[7] V. Colao, G. Marino, H.K. Xu, *An iterative method for finding common solutions of equilibrium and fixed point problems*, J. Math. Anal. Appl., **344**(2008), 340352.

[8] V. Colao, G. Lopez-Acedo, G. Marino, *An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings*, Nonlinear Anal., **71** (2009), No. 7-8, 2708-2715.

[9] P.L. Combettes, S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117-136.

[10] Y.L. Cui, X. Liu, *Notes on Browder's and Halpern's methods for nonexpansive mappings*, Fixed Point Theory, **10**(2009), No. 1, 89-98.

[11] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, in: Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.

[12] S. He, H.K. Xu, *Variational Inequalities Governed by Boundedly Lipschitzian and Strongly Monotone Operators*, Fixed Point Theory, **10**(2009), No. 2, 245-258.

[13] A.N. Iusem, W. Sosa, *New existence results for equilibrium problems*, Nonlinear Anal., **52**(2003), No. 2, 621-635.

[14] A.N. Iusem, W. Sosa, *Iterative algorithms for equilibrium problems*, Optimization, **52**(2003), No. 3, 301-316.

[15] G. Lopez, V. Martin-Marquez, H.K. Xu, *Halpern's iteration for nonexpansive mappings*, in "Nonlinear Analysis and Optimization I: Nonlinear Analysis," Contemporary Mathematics, vol. 513, 2010, 211-230.

[16] G. Marino, H.K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **318** (2006), 43-52.

[17] J.W. Peng, J.C. Yao, *A modified CQ method for equilibrium problems, fixed points and variational inequality*, Fixed Point Theory, **9**(2008), No. 2, 515-531.

[18] S. Plubtieng, R. Punpaeng, *A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **336**(2007), 455-469.

[19] K. Shimoji, W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math., **5**(2001), 387-404.

[20] S. Takahashi, W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331**(2007), 506-515.

[21] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240-256.

[22] H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116** (2003), 659-678.

[23] H.K. Xu, *An alternative regularization method for nonexpansive mappings with applications*, in "Nonlinear Analysis and Optimization I: Nonlinear Analysis," Contemporary Mathematics, vol. 513, 2010, 239-263.

[24] Y. Yao, Y.C. Liou, C. Lee, M.M. Wong, *Convergence Theorem for Equilibrium Problems and Fixed Point Problems*, Fixed Point Theory, **10**(2009), No. 2, 347-363.

Received: December 31, 2009; Accepted: May 2, 2010.