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A HYBRID EXTRAGRADIENT METHOD FOR ASYMPTOTICALLY STRICT PSEUDO-CONTRACTIONS IN THE INTERMEDIATE SENSE

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Dedicated to Wataru Takahashi on the occasion of his retirement

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Abstract. In this paper we construct a new hybrid extragradient method for finding a common element of the fixed point set of an asymptotically strict pseudo-contraction in the intermediate sense and the solution set of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space. A strong convergence theorem of the proposed method is established and some of its special cases are also discussed.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty subset of H. A mapping $T : C \to H$ is L-*Lipschitz continuous* ($L > 0$) if $\|Tx - Ty\| \le L\|x - y\|$, for all $x, y \in C$. We denote by I the identity mapping of H. Recently, Sahu, Xu and Yao [16] introduced the class of asymptotically strict pseudo-contractions in the intermediate sense which are not necessarily Lipschitzian.

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Definition 1.1. *A mapping $S : C \to H$ is an asymptotically κ-strict pseudo-contraction in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that*

$$\limsup_{n\to\infty} \sup_{x,y\in C} [\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa\|(I-S^n)x - (I-S^n)y\|^2] \leq 0. \quad (1.1)$$

Throughout the paper we assume that

$$c_n := \max\{0, \sup_{x,y\in C} [\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa\|(I-S^n)x - (I-S^n)y\|^2]\}.$$

Then $c_n \geq 0$, for all $n \in \mathbf{N}$, $\lim_{n\to\infty} c_n = 0$ and (1.1) reduces to the relation

$$\|S^n x - S^n y\|^2 \leq (1+\gamma_n)\|x-y\|^2 + \kappa\|(I-S^n)x - (I-S^n)y\|^2 + c_n, \quad (1.2)$$

for all $n \in \mathbf{N}$ and $x, y \in C$. In particular, when $c_n \equiv 0$ (1.2), S is an *asymptotically κ-strict pseudo-contraction* with sequence $\{\gamma_n\}$ introduced by Kim and Xu [8].

The variational inequality problem for a mapping $A : C \to H$ due to Stampacchia [18] is to find an element $\bar{x} \in C$ such that $\langle A\bar{x}, y - \bar{x}\rangle \geq 0$, for all $y \in C$. The set of solutions of this variational inequality problem is denoted by $\Omega(A, C)$. The purpose of this paper is to establish an iterative method to approximate an element of $F(S) \cap \Omega(A, C)$, where $F(S) = \{x \in C : Sx = x\}$ denotes the set of fixed points of a self-mapping S of C.

A mapping A is *α-inverse-strongly monotone* [10] if there exists a positive constant α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \text{for all } x, y \in C.$$

Iiduka and Takahashi [6] constructed the following iterative scheme to generate a sequence converging strongly to an element of $F(S) \cap \Omega(A, C)$, where S is a nonexpansive mapping and A is an inverse-strongly monotone mapping: given an arbitrary $x_0 \in C$,

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n).$$

Zeng and Yao [22] proposed a new iterative method for a nonexpansive mapping S and a monotone and Lipschitz continuous mapping A and obtained a weak convergence theorem: given an arbitrary $x_0 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n). \end{cases}$$

In this paper, based on the extragradient method [9] and the modified Mann iteration [7, 8, 11, 12, 17], a new hybrid extragradient method for an asymptotically strict pseudo-contraction in the intermediate sense $S : C \to C$ and an inverse-strongly monotone mapping $A : C \to H$ in a Hilbert space is defined as follows: given a fixed $x_0 \in C$ and an arbitrary $x_1 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = \mu_n x_0 + (1-\mu_n) P_C(y_n - \lambda_n A y_n), \\ x_{n+1} = (1-\alpha_n-\beta_n) x_n + \alpha_n t_n + \beta_n S^n t_n. \end{cases}$$

Using this iteration, we obtain strong convergence of the sequence $\{x_n\}$ with limit $P_{F(S)\cap\Omega(A,C)}x_0$; see Section 3. Further, as an application, we study some special cases of this theorem in Section 4. Those results also extend some recent results; see, e.g., [2, 3, 5, 6, 22].

2. Preliminaries

We denote by $\rightharpoonup$ and $\to$ weak convergence and strong convergence, respectively. Let C be a nonempty subset of a real Hilbert space H. A mapping $A : C \to H$ is *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$, for all $x, y \in C$. An α-inverse-strongly monotone mapping is monotone and $(1/\alpha)$-Lipschitz continuous.

A mapping $S : C \to C$ is called a *κ-strict pseudo-contraction*, introduced by Browder and Petryshyn [1], if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y)\|^2, \quad \text{for all } x, y \in C.$$

A 0-strict pseudo-contraction is nonexpansive and an asymptotically 0-strict pseudo-contraction is asymptotically nonexpansive [4]. A mapping $T : C \to C$ is *uniformly L-Lipschitzian* $(L > 0)$ if $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $n \in \mathbf{N}$ and for all $x, y \in C$. It is noticeable that every asymptotically κ-strict pseudo-contraction with sequence $\{\gamma_n\}$ is uniformly L-Lipschitzian with $L = \sup\left\{\frac{\kappa+\sqrt{1+(1-\kappa)\gamma_n}}{1+\kappa} : n \geq 1\right\}$, see [8].

A multi-valued mapping $T : H \to 2^H$ is *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if whenever $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in G(T)$, implies $f \in Tx$. Let $A : C \to H$ be a monotone and Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \text{ for all } u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone, and $0 \in Tv$ if and only if $v \in \Omega(A, C)$; see [14].

Suppose that C is a nonempty closed convex subset of a real Hilbert space H, Then for every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. We recall some properties of the metric projection in a Hilbert space.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H.*

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$, *for all $x, y \in H$.*
(ii) $\langle x - P_C x, P_C x - y\rangle \geq 0$, *for all $x \in H$, $y \in C$.*
(iii) *(see [19]) Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if*

$$\langle x - y, y - z\rangle \geq 0, \quad \text{for all } z \in C.$$

Notice that, if $A : C \to H$ is a monotone mapping, it follows from Lemma 2.1(ii) that

$$u \in \Omega(A, C) \iff u = P_C(I - \lambda A)u, \quad \text{for all } \lambda > 0.$$

We will need the following lemmas to prove our main results.

Lemma 2.2. [13] *Let X be an inner product space. For all $x, y, z \in X$ and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 2.3. [20, Lemma 2.5] *Let $\{s_n\}$ be a nonnegative sequence such that*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad \text{for all } n \ge 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *or equivalently,* $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$;

(ii) $\limsup_{n\to\infty} \beta_n \le 0$;

(iii) $\gamma_n \ge 0$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. [16, Lemma 2.6] *Let C be a nonempty subset of a Hilbert space and let $S : C \to C$ be an asymptotically κ-strict pseudo-contraction in the intermediate sense with sequence $\{\gamma_n\}$. Then, for all $x, y \in C$ and $n \ge 1$, we have that*

$$\|S^n x - S^n y\| \le \frac{1}{1-\kappa}\left[\kappa\|x-y\| + \sqrt{[1+(1-\kappa)\gamma_n]\|x-y\|^2 + (1-\kappa)c_n}\right].$$

Lemma 2.5. [16, Lemma 2.7] *Let C be a nonempty subset of a Hilbert space and let $S : C \to C$ be a uniformly continuous and asymptotically strict pseudo-contraction in the intermediate sense. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\lim_{n\to\infty}\|x_n - S^n x_n\| = 0$. Then $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.*

Lemma 2.6. (Demiclosedness principle [16, Proposition 3.1]) *Let C be a nonempty closed convex subset of a Hilbert space and let $S : C \to C$ be a continuous and asymptotically strict pseudo-contraction in the intermediate sense. Then $I - S$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - S^m x_n\| = 0$, then $(I-S)x = 0$.*

Lemma 2.7. [16, Proposition 3.2]) *Let C be a nonempty closed convex subset of a Hilbert space and let $S : C \to C$ be a continuous and asymptotically strict pseudo-contraction in the intermediate sense. Then $F(S)$ is closed and convex.*

3. Strong Convergence Theorem

In this section we shall present a strong convergence theorem for a new hybrid iterative method to find a common element of the fixed point set of an asymptotically strict pseudo-contraction in the intermediate sense and the solution set of the variational inequality for an inverse-strongly monotone mapping.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a ρ-inverse-strongly monotone mapping, and $S : C \to C$ a uniformly continuous and asymptotically κ-strict pseudo-contraction in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(S) \cap \Omega(A, C) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be the sequences generated by: given a fixed $x_0 \in C$ and an arbitrary $x_1 \in C$,*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = \mu_n x_0 + (1 - \mu_n) P_C(y_n - \lambda_n A y_n), \\ x_{n+1} = (1 - \alpha_n - \beta_n) x_n + \alpha_n t_n + \beta_n S^n t_n, \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subset [0, \infty)$ and $\{\mu_n\}$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ are such that $\alpha_n + \beta_n \leq 1$. Suppose that the following conditions hold:
(i) $\{\lambda_n\} \subset [a, b]$, for some $a, b \in (0, 2\rho)$;
(ii) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=1}^{\infty} \mu_n = \infty$;
(iii) $\{\alpha_n\} \subset [\kappa + \epsilon, 1]$, $\{\beta_n\} \subset [\delta, 1]$, for some $\epsilon, \delta \in (0, 1)$, $\sum_{n=1}^{\infty} \beta_n c_n < \infty$;
(iv) the series $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n|$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$ are convergent;
(v) $\lim_{n\to\infty} \sup_{u\in D} \|S^{n+1}u - S^n u\| = 0$, for every bounded subset D of C.
Then $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ converge strongly to the point $P_{F(S)\cap\Omega(A,C)} x_0$.

Proof. For $x, y \in C$, since $\lambda_n < 2\rho$, we have

$$\begin{aligned} &\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \\ &= \|x - y\|^2 - 2\lambda_n \langle x - y, Ax - Ay \rangle + \lambda_n^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\rho)\|Ax - Ay\|^2 \leq \|x - y\|^2 \end{aligned} \tag{3.2}$$

which shows that $I - \lambda_n A$ is nonexpansive. Note that the set $F(S) \cap \Omega(A, C)$ is closed and convex by Lemma 2.7 and [21, Lemma 3.1]. The proof is divided into five steps.

Step 1. We will prove that $\{x_n\}$ is bounded. Let $p \in F(S) \cap \Omega(A, C)$ and $z_n = P_C(y_n - \lambda_n A y_n)$. Then $p = P_C(p - \lambda_n A p)$, $\langle Ap, y_n - p \rangle \geq 0$ and $\langle Ay_n - Ap, y_n - p \rangle \geq 0$, for all n. We have

$$\begin{aligned} &\|y_n - p\|^2 = \|P_C(x_n - \lambda_n A x_n) - P_C(p - \lambda_n A p)\| \leq \|x_n - p\|, \\ &\|z_n - p\|^2 = \|P_C(y_n - \lambda_n A y_n) - P_C(p - \lambda_n A p)\| \leq \|y_n - p\| \leq \|x_n - p\|, \\ &\|t_n - p\|^2 \leq \mu_n \|x_0 - p\|^2 + (1 - \mu_n)\|z_n - p\|^2 \leq \max\{\|x_0 - p\|^2, \|x_n - p\|^2\}. \end{aligned} \tag{3.3}$$

Recall that $\kappa < \alpha_n$. By Lemma 2.2, we obtain from (1.2), (3.1) and (3.3) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n-\beta_n)\|x_n - p\|^2 + \alpha_n\|t_n - p\|^2 + \beta_n\|S^n t_n - p\|^2 \\ &\quad - \alpha_n\beta_n\|t_n - S^n t_n\|^2 \\ &\leq (1-\alpha_n-\beta_n)\|x_n - p\|^2 + (\alpha_n+\beta_n+\gamma_n)\|t_n - p\|^2 \\ &\quad + \beta_n(\kappa-\alpha_n)\|t_n - S^n t_n\|^2 + \beta_n c_n \\ &\leq (1-\alpha_n-\beta_n)\|x_n - p\|^2 + (\alpha_n+\beta_n+\gamma_n)\|t_n - p\|^2 + \beta_n c_n \\ &\leq (1+\gamma_n)\left[\max\left\{\|x_0 - p\|^2, \|x_n - p\|^2\right\} + \beta_n c_n\right]. \end{aligned} \tag{3.4}$$

Next we shall prove by induction that for all $n \geq 1$,

$$\|x_{n+1} - p\|^2 \leq \left[\prod_{j=1}^{n}(1+\gamma_j)\right]\left[\max\left\{\|x_0 - p\|^2, \|x_1 - p\|^2\right\} + \sum_{i=1}^{n}\beta_i c_i\right]. \tag{3.5}$$

Indeed, if $n = 1$, (3.4) yields (3.5). Suppose that (3.5) holds for some integer $n \geq 1$. Then by (3.4) and the induction hypothesis,

$$\begin{aligned}\|x_{n+2} - p\|^2 &\leq (1+\gamma_{n+1})\left[\max\{\|x_0 - p\|^2, \|x_{n+1} - p\|^2\} + \beta_{n+1}c_{n+1}\right] \\ &\leq (1+\gamma_{n+1})\left\{\left[\prod_{j=1}^{n}(1+\gamma_j)\right]\left[\max\{\|x_0 - p\|^2, \|x_1 - p\|^2\}\right.\right. \\ &\quad \left.\left. + \sum_{i=1}^{n}\beta_i c_i\right] + \beta_{n+1}c_{n+1}\right\} \\ &\leq \left[\prod_{j=1}^{n+1}(1+\gamma_j)\right]\left[\max\{\|x_0 - p\|^2, \|x_1 - p\|^2\} + \sum_{i=1}^{n+1}\beta_i c_i\right].\end{aligned}$$

Hence (3.5) holds for for $n+1$.

Using the inequality $1 + t \leq e^t$, for $t \geq 0$, we derive from (3.5) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq e^{\sum_{j=1}^{n}\gamma_j}\left[\max\{\|x_0 - p\|^2, \|x_1 - p\|^2\} + \sum_{i=1}^{n}\beta_i c_i\right] \\ &\leq e^{\sum_{j=1}^{\infty}\gamma_j}\left[\max\{\|x_0 - p\|^2, \|x_1 - p\|^2\} + \sum_{i=1}^{\infty}\beta_i c_i\right], \quad n \in \mathbf{N}.\end{aligned}$$

Since $\sum\gamma_n < \infty$ and $\sum\beta_n c_n < \infty$, $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$.

Step 2. We will prove that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and A is Lipschitz continuous, $\{Ax_n\}$ and $\{Ay_n\}$ are bounded. Lemma 2.4 states that

$$\|S^n t_n - p\| \leq \frac{1}{1-\kappa}\left[\kappa\|t_n - p\| + \sqrt{[1+(1-\kappa)\gamma_n]\|t_n - p\|^2 + (1-\kappa)c_n}\right],$$

and thus $\{S^n t_n\}$ is also bounded. Therefore there exists a positive number M such that $\{\|z_n\|\}$, $\{\|t_n\|\}$, $\{\|Ax_n\|\}$, $\{\|Ay_n\|\}$, and $\{\|S^n t_n\|\}$ are all bounded by M. Since P_C and $I - \lambda_{n+1}A$ are nonexpansive, it follows that

$$\begin{aligned}\|y_{n+1} - y_n\| &\leq \|P_C(I - \lambda_{n+1}A)x_{n+1} - P_C(I - \lambda_{n+1}A)x_n\| \\ &\quad + \|P_C(I - \lambda_{n+1}A)x_n - P_C(I - \lambda_n A)x_n\| \\ &\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\leq \|x_{n+1} - x_n\| + M|\lambda_{n+1} - \lambda_n|\end{aligned}$$

and similarly,

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|P_C(I - \lambda_{n+1}A)y_{n+1} - P_C(I - \lambda_n A)y_n\| \\ &\leq \|y_{n+1} - y_n\| + |\lambda_{n+1} - \lambda_n|\|Ay_n\| \\ &\leq \|x_{n+1} - x_n\| + 2M|\lambda_{n+1} - \lambda_n|.\end{aligned}$$

Hence

$$\begin{aligned}\|t_{n+1} - t_n\| &= \|\mu_{n+1}x_0 + (1 - \mu_{n+1})z_{n+1} - \mu_n x_0 - (1 - \mu_n)z_n\| \\ &\leq |\mu_{n+1} - \mu_n|\|x_0\| + (1 - \mu_{n+1})\|z_{n+1} - z_n\| + |\mu_{n+1} - \mu_n|\|z_n\| \\ &\leq \|z_{n+1} - z_n\| + |\mu_{n+1} - \mu_n|(\|x_0\| + \|z_n\|) \\ &\leq \|x_{n+1} - x_n\| + K|\lambda_{n+1} - \lambda_n| + K|\mu_{n+1} - \mu_n|, \end{aligned} \tag{3.7}$$

where $K = 2M + \|x_0\|$. From (3.7) and

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n - \beta_n)x_n + \alpha_n t_n + \beta_n S^n t_n, \\ x_n &= (1 - \alpha_{n-1} - \beta_{n-1})x_{n-1} + \alpha_{n-1}t_{n-1} + \beta_{n-1}S^n t_{n-1},\end{aligned}$$

we compute

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\leq (1 - \alpha_n - \beta_n)\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)\|x_{n-1}\| \\ &\quad + \alpha_n\|t_n - t_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|t_{n-1}\| \\ &\quad + \beta_n\|S^n t_n - S^{n-1}t_{n-1}\| + |\beta_n - \beta_{n-1}|\|S^{n-1}t_{n-1}\| \\ &\leq (1 - \beta_n)\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)\|x_{n-1}\| \\ &\quad + K\alpha_n|\lambda_n - \lambda_{n-1}| + K\alpha_n|\mu_n - \mu_{n-1}| + |\alpha_n - \alpha_{n-1}|\|t_{n-1}\| \\ &\quad + \beta_n\|S^n t_n - S^{n-1}t_{n-1}\| + |\beta_n - \beta_{n-1}|\|S^{n-1}t_{n-1}\| \\ &\leq (1 - \beta_n)\|x_n - x_{n-1}\| + \beta_n\|S^n t_n - S^{n-1}t_{n-1}\| + K|\lambda_n - \lambda_{n-1}| \\ &\quad + K|\mu_n - \mu_{n-1}| + 2K|\alpha_n - \alpha_{n-1}| + 2K|\beta_n - \beta_{n-1}|.\end{aligned}$$

Lemma 2.3 asserts from Conditions (iii)-(v) that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, and therefore by (3.7),

$$\lim_{n\to\infty} \|t_{n+1} - t_n\| = 0. \tag{3.8}$$

Step 3. Observe that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$.

To see this, we need to prove that $\lim_{n\to\infty} \|x_n - S^n x_n\| = 0$. Indeed, it follows from (3.3) that

$$(\alpha_n + \beta_n)\|t_n - p\|^2 \le \mu_n\|x_0 - p\|^2 + (\alpha_n + \beta_n)\|x_n - p\|^2.$$

Since $\beta_n(\alpha_n - \kappa) \ge \epsilon\delta$, the inequality (3.4) yields

$$\begin{aligned}
&\epsilon\delta\|t_n - S^n t_n\|^2 \\
&\le (1 - \alpha_n - \beta_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\alpha_n + \beta_n + \gamma_n)\|t_n - p\|^2 + \beta_n c_n \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \mu_n\|x_0 - p\|^2 + \gamma_n\|t_n - p\|^2 + \beta_n c_n \\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - p\| - \|x_{n+1} - p\|) + \mu_n\|x_0 - p\|^2 \\
&\quad + \gamma_n\|t_n - p\|^2 + \beta_n c_n \\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \mu_n\|x_0 - p\|^2 + \gamma_n\|t_n - p\|^2 + \beta_n c_n.
\end{aligned}$$

Therefore (3.6) implies that

$$\lim_{n\to\infty} \|t_n - S^n t_n\| = 0. \tag{3.9}$$

From the definition of x_{n+1}, we have

$$\begin{aligned}
(\alpha_n + \beta_n)\|t_n - x_n\| &= \|(x_{n+1} - x_n) - \beta_n(S^n t_n - t_n)\| \\
&\le \|x_{n+1} - x_n\| + \beta_n\|S^n t_n - t_n\|.
\end{aligned}$$

Since $\alpha_n + \beta_n \ge \epsilon + \delta$, it follows from (3.6) and (3.9) that

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{3.10}$$

By Lemma 2.4,

$$\begin{aligned}
&\|S^n t_n - S^n x_n\| \\
&\le \frac{1}{1-\kappa}\left[\kappa\|t_n - x_n\| + \sqrt{[1 + (1-\kappa)\gamma_n]\|t_n - x_n\|^2 + (1-\kappa)c_n}\right] \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

which, together with (3.9) and (3.10), implies that

$$\|x_n - S^n x_n\| \le \|x_n - t_n\| + \|t_n - S^n t_n\| + \|S^n t_n - S^n x_n\| \to 0, \quad \text{as } n \to \infty.$$

Using (3.6) and Lemma 2.5, we obtain $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Step 4. We shall prove that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. According to (3.2) and (3.3), we have

$$\begin{aligned}
\|t_n - p\|^2 &\le \mu_n\|x_0 - p\|^2 + (1 - \mu_n)\|z_n - p\|^2 \\
&\le \mu_n\|x_0 - p\|^2 + (1 - \mu_n)\|P_C(y_n - \lambda_n Ay_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\le \mu_n\|x_0 - p\|^2 + (1 - \mu_n)[\|y_n - p\|^2 + \lambda_n(\lambda_n - 2\rho)\|Ay_n - Ap\|^2] \\
&\le \mu_n\|x_0 - p\|^2 + \|y_n - p\|^2 + a(b - 2\rho)(1 - \mu_n)\|Ay_n - Ap\|^2 \\
&\le \mu_n\|x_0 - p\|^2 + \|x_n - p\|^2 + a(b - 2\rho)(1 - \mu_n)\|Ay_n - Ap\|^2.
\end{aligned}$$

Combining this inequality with (3.4) yields

$$\|x_{n+1}-p\|^2 \le (1+\gamma_n)\|x_n-p\|^2+\mu_n(\alpha_n+\beta_n+\gamma_n)\|x_0-p\|^2 \\ + a(b-2\rho)(1-\mu_n)(\alpha_n+\beta_n+\gamma_n)\|Ay_n-Ap\|^2+\beta_n c_n$$

which asserts that

$$a(2\rho-b)(1-\mu_n)(\alpha_n+\beta_n+\gamma_n)\|Ay_n-Ap\|^2 \\ \le (1+\gamma_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2+\mu_n(\alpha_n+\beta_n+\gamma_n)\|x_0-p\|^2+\beta_n c_n \\ \le \gamma_n\|x_n-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)(\|x_n-x_{n+1}\|) \\ + \mu_n(\alpha_n+\beta_n+\gamma_n)\|x_0-p\|^2+\beta_n c_n.$$

Since $\mu_n \to 0$, $\alpha_n+\beta_n \ge \epsilon+\delta$ and $a, b \in (0, 2\rho)$, we obtain from (3.6) that

$$\lim_{n\to\infty}\|Ay_n-Ap\|=0. \tag{3.11}$$

Now, apply Lemma 2.1(i) to get

$$\|z_n-p\|^2 = \|P_C(y_n-\lambda_n Ay_n)-P_C(p-\lambda_n Ap)\|^2 \\ \le \langle (y_n-\lambda_n Ay_n)-(p-\lambda_n Ap), z_n-p\rangle \\ = \frac{1}{2}\big[\|(y_n-\lambda_n Ay_n)-(p-\lambda_n Ap)\|^2+\|z_n-p\|^2 \\ -\|(y_n-\lambda_n Ay_n)-(p-\lambda_n Ap)-(z_n-p)\|^2\big] \\ \le \frac{1}{2}\big[\|y_n-p\|^2+\|z_n-p\|^2-\|y_n-z_n\|^2+2\lambda_n\langle y_n-z_n, Ay_n-Ap\rangle \\ -\lambda_n^2\|Ay_n-Ap\|^2\big]$$

which shows that

$$\|z_n-p\|^2 \le \|y_n-p\|^2-\|y_n-z_n\|^2+2\lambda_n\langle y_n-z_n, Ay_n-Ap\rangle \\ -\lambda_n^2\|Ay_n-Ap\|^2.$$

Using this inequality, by (3.3) we have

$$\|t_n-p\|^2 \le \mu_n\|x_0-p\|^2+(1-\mu_n)\|z_n-p\|^2 \\ \le \mu_n\|x_0-p\|^2+\|x_n-p\|^2-\|y_n-z_n\|^2+2\lambda_n\langle y_n-z_n, Ay_n-Ap\rangle \\ -\lambda_n^2\|Ay_n-Ap\|^2,$$

which, together with (3.10) and (3.11), yields

$$\|y_n-z_n\|^2 \le \mu_n\|x_0-p\|^2+\|x_n-p\|^2-\|t_n-p\|^2+2\lambda_n\langle y_n-z_n, Ay_n-Ap\rangle \\ -\lambda_n^2\|Ay_n-Ap\|^2 \\ \le \mu_n\|x_0-p\|^2+(\|x_n-p\|+\|t_n-p\|)\|x_n-t_n\| \\ -\lambda_n^2\|Ay_n-Ap\|^2 \to 0, \quad \text{as } n\to\infty.$$

Since $\mu_n \to 0$, we have $\|t_n-z_n\| = \mu_n\|x_0-z_n\| \to 0$. Consequently, $\|x_n-y_n\| \le \|x_n-t_n\|+\|t_n-z_n\|+\|z_n-y_n\| \to 0$, as claimed.

Step 5. Claim that $\lim_{n\to\infty}\|x_n - x^*\| = 0$, where $x^* = P_{F(S)\cap\Omega(A,C)}x_0$. To see this, we need to show that $\limsup_{n\to\infty}\langle x_0 - x^*, x_n - x^*\rangle \le 0$. Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle x_0 - x^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle x_0 - x^*, x_{n_j} - x^*\rangle$. Since $\{x_n\}$ is bounded, we may assume without loss generality that $\{x_{n_j}\}$ converges weakly to a point $\hat{x} \in C$, and thus $\{y_{n_j}\}$ also converges weakly to $\hat{x}$. Since A is Lipschitz continuous and $\|x_n - y_n\| \to 0$, we obtain $\|Ax_n - Ay_n\| \to 0$. Now, we show that $\hat{x} \in F(S)\cap\Omega(A,C)$ from which it follows that $\langle x_0 - x^*, \hat{x} - x^*\rangle \le 0$ by Lemma 2.1(iii). Since S is uniformly continuous and $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$, we have $\lim_{n\to\infty}\|x_n - S^m x_n\| = 0$, for all $m \in \mathbf{N}$; hence $\hat{x} \in F(S)$ by Lemma 2.6. To prove $\hat{x} \in \Omega(A,C)$, define a multi-valued function $T : H \to 2^H$ by

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v,w) \in G(T)$ so that $w \in Tv$ and $w - Av \in N_C v$. Hence

$$\langle v - u, w - Av\rangle \ge 0, \quad \text{for all } u \in C. \tag{3.12}$$

On the other hand, since $y_n = P_C(x_n - \lambda_n Ax_n)$, by Lemma 2.1(iii), we have that $\langle x_n - \lambda_n Ax_n - y_n, y_n - v\rangle \ge 0, \quad v \in C$, or equivalently,

$$\left\langle v - y_n, Ax_n + \frac{1}{\lambda_n}(y_n - x_n)\right\rangle \ge 0, \quad v \in C. \tag{3.13}$$

If we put $u = y_{n_j}$ in (3.12), then the monotonicity of A and (3.13) imply that

$$\begin{aligned}\langle v - y_{n_j}, w\rangle &\ge \langle v - y_{n_j}, Av\rangle \\ &\ge \langle v - y_{n_j}, Av\rangle - \left\langle v - y_{n_j}, Ax_{n_j} + \frac{1}{\lambda_{n_i}}(y_{n_j} - x_{n_j})\right\rangle \\ &= \langle v - y_{n_j}, Av - Ay_{n_j}\rangle + \langle v - y_{n_j}, Ay_{n_j} - Ax_{n_j}\rangle \\ &\quad - \left\langle v - y_{n_j}, \frac{1}{\lambda_{n_j}}(y_{n_j} - x_{n_j})\right\rangle \\ &\ge \langle v - y_{n_j}, Ay_{n_j} - Ax_{n_j}\rangle - \left\langle v - y_{n_j}, \frac{1}{\lambda_{n_j}}(y_{n_j} - x_{n_j})\right\rangle.\end{aligned}$$

Then take the limit as $j \to \infty$ to get $\langle v - \hat{x}, w\rangle \ge 0$. Since T is maximal monotone, $\hat{x} \in T^{-1}0$ and hence $\hat{x} \in \Omega(A,C)$. This shows that $\hat{x} \in F(S)\cap\Omega(A,C)$ and so $\langle x_0 - x^*, \hat{x} - x^*\rangle \le 0$. Therefore

$$\limsup_{n\to\infty}\langle x_0 - x^*, x_n - x^*\rangle = \langle x_0 - x^*, \hat{x} - x^*\rangle \le 0. \tag{3.14}$$

We have

$$\begin{aligned}\|t_n - x^*\|^2 &= \|\mu_n(x_0 - x^*) + (1-\mu_n)(z_n - x^*)\|^2 \\ &\le (1-\mu_n)^2\|z_n - x^*\|^2 + 2\langle \mu_n(x_0 - x^*), t_n - x^*\rangle \\ &\le (1-\mu_n)\|x_n - x^*\|^2 + 2\mu_n\langle x_0 - x^*, t_n - x^*\rangle,\end{aligned}$$

and so by (3.4) this implies that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le (1 - \alpha_n - \beta_n)\|x_n - x^*\|^2 + (1 - \mu_n)(\alpha_n + \beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\mu_n(\alpha_n + \beta_n)\langle x_0 - x^*, t_n - x^*\rangle + \gamma_n\|t_n - x^*\|^2 + \beta_n c_n \\
&= [1 - \mu_n(\alpha_n + \beta_n)]\|x_n - x^*\|^2 + \mu_n(\alpha_n + \beta_n)[2\langle x_0 - x^*, t_n - x^*\rangle] \\
&\quad + \gamma_n\|t_n - x^*\|^2 + \beta_n c_n.
\end{aligned}$$

By hypotheses, $\sum \mu_n(\alpha_n + \beta_n) = \infty$, $\sum \gamma_n < \infty$ and $\sum \beta_n c_n < \infty$. Moreover, it follows from (3.14) that

$$\begin{aligned}
\limsup_{n\to\infty}\langle x_0 - x^*, t_n - x^*\rangle &\le \lim_{n\to\infty}\langle x_0 - x^*, t_n - x_n\rangle + \limsup_{n\to\infty}\langle x_0 - x^*, x_n - x^*\rangle \\
&\le 0.
\end{aligned}$$

We conclude from Lemma 2.3 that $\lim_{n\to\infty}\|x_n - x^*\| = 0$ and hence $\{y_n\}$ and $\{t_n\}$ also converge strongly to x^*. This completes the proof. □

4. Applications

In this section we apply Theorem 3.1 to demonstrate some special cases.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ a τ-strict pseudo-contraction and $S : C \to C$ a uniformly continuous and asymptotically κ-strict pseudo-contraction in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(S) \cap F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$. Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be the sequences defined by (3.1), where $A = I - T$. Suppose that Conditions (i)-(v) as in Theorem 3.1 hold, where $\rho = (1 - \tau)/2$. Then $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ converge strongly to the point $P_{F(S)\cap F(T)}x_0$.*

Proof. Observe that A is a $[(1 - \tau)/2]$-inverse-strongly monotone mapping. Indeed, since for all $x, y \in C$,

$$\begin{aligned}
\|Tx - Ty\|^2 &= \|(I - A)x - (I - A)y)\|^2 \\
&= \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle Ax - Ay, x - y\rangle
\end{aligned}$$

and

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \tau\|(I - T)x - (I - T)y)\|^2,$$

it follows that

$$\langle Ax - Ay, x - y\rangle \ge \frac{1}{2}(1 - \tau)\|Ax - Ay\|^2.$$

For any $\lambda > 0$, by (2.1) we have

$$\begin{aligned}
Tu = u \quad &\Longleftrightarrow \quad u = u - \lambda Au = P_C(u - \lambda Au) \\
&\Longleftrightarrow \quad \langle Au, y - u\rangle \ge 0, \quad \text{for all } y \in C.
\end{aligned}$$

The desired conclusion follows from Theorem 3.1. □

We remark that Condition (v) in Theorem 3.1 is not required, in particular, if S is nonexpansive.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ a τ-strict pseudo-contraction and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be the sequences generated by: given a fixed $x_0 \in C$ and an arbitrary $x_1 \in C$,*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = \mu_n x_0 + (1-\mu_n) P_C(y_n - \lambda_n A y_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n t_n + \beta_n S t_n, \end{cases}$$

where $A = I - T$, $\{\lambda_n\} \subset [0,\infty)$, and $\{\mu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n \le 1$. Suppose that Conditions (i)-(iv) as in Theorem 3.1 hold, where $\rho = (1-\tau)/2$. Then $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ converge strongly to the point $P_{F(S)\cap F(T)} x_0$.

Proof. The proof is the same as that of Theorem 3.1 when $\kappa = 0$, $\gamma_n = 0$ and $c_n = 0$, and hence is omitted. □

It is well known (see [15]) that if $B : H \to 2^H$ is a maximal monotone mapping, then for each $u \in H$ and $\lambda > 0$ there is a unique $z \in H$ such that $u \in (I + \lambda B)(z)$. The (single-valued) function $J_\lambda^B := (I + \lambda B)^{-1}$ thus defined is called the *resolvent* of B of parameter λ. The mapping $J_\lambda^B : H \to H$ is nonexpansive and $J_\lambda^B(z) = z$ if and only if $0 \in B(z)$.

Theorem 4.3. *Let H be a real Hilbert space, $A : H \to H$ a ρ-inverse-strongly monotone mapping, $B : H \to 2^H$ a maximal monotone mapping and J_r^B the resolvent of B, for $r > 0$, such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be the sequences generated by: given a fixed $x_0 \in H$ and an arbitrary $x_1 \in H$,*

$$\begin{cases} y_n = x_n - \lambda_n A x_n, \\ t_n = \mu_n x_0 + (1-\mu_n)(y_n - \lambda_n A y_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n t_n + \beta_n J_r^B t_n, \end{cases}$$

where $\{\lambda_n\} \subset [0,\infty)$, and $\{\mu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n \le 1$. Suppose that Conditions (i)-(iv) as in Theorem 3.1 hold. Then $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ converge strongly to the point $P_{A^{-1}0\cap B^{-1}0} x_0$.

Proof. This is the case of Theorem 3.1 when $S = J_r^B$ and $P_H = I$ such that $\kappa = 0$, $\gamma_n = 0$ and $c_n = 0$. Then $\Omega(A, C) = A^{-1}0$ and $F(J_r^B) = B^{-1}0$ and so the desired result follows. □

References

[1] F. E. Browder, W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl., **20** (1967), 197-228.

[2] L. C. Ceng, S. Huang, *Modified extragradient methods for strict pseudo-contractions and monotone mappings*, Taiwanese Journal of Mathematics, **13** (2009), 1197-1211.

[3] L. C. Ceng, N. C. Wong, J. C. Yao, *Fixed point solutions of variational inequalities for a finite family of asymptotically nonexpansive mappings without common fixed point assumption*, Comput. Math. Appl., **56** (2008), 2312-2322.

[4] K. Goebel, W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35** (1972), No. 1, 171-174.

[5] S. Huang, *Fixed points of a sequence of asymptotically nonexpansive mappings*, Fixed Point Theory, **9** (2008), No. 2, 465-485.

[6] H. Iiduka, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal., **61** (2005), 341-350.

[7] T. W. Kim, H. K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal., **61** (2005), 51-60.

[8] T. H. Kim, H. K. Xu, *Convergence of the modified Mann's iteration method for asymptotically strict pseudocontractions*, Nonlinear Anal., **68** (2008), 2828-2836.

[9] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Matecon., **12** (1976), 747-756.

[10] F. Liu, M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal., **6** (1998), 313-344.

[11] N. Nadezhkina, W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, Journal of Optimization Theory and Applications, **128** (2006), 191-201.

[12] M. A. Noor, Y. Yao, R. Chen, Y. C. Liou, *An iterative method for fixed point problems and variational inequality problems*, Mathematical Communications, **12** (2007), 121-132.

[13] M. O. Osilike, D. I. Igbokwe, *Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations*, Comput. Math. Appl., **40** (2000), 559-567.

[14] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., **149** (1970), 75-88.

[15] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14** (1976), 877-898.

[16] D. R. Sahu, H. K. Xu, J. C. Yao, *Asymptotically strict pseudo-contractive mappings in the intermediate sense*, Nonlinear Anal., **70** (2009), 3502-3511.

[17] V. K. Singh, S. Kumar, *Iteration process with errors for local strongly H-accretive type mappings*, Fixed Point Theory, **9** (2008), No. 1, 351-362.

[18] G. Stampacchia, *Formes bilineaires coercitivies sur les ensembles convexes*, C. R. Acad. Sci. Paris, **258** (1964), 4413-4416.

[19] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[20] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66** (2002), 240-256.

[21] J. C. Yao, *Variational inequalities with generalized monotone operators*, Mathematics of Operations Research, **19** (1994), 691-705.

[22] L. C. Zeng, J. C. Yao, *Strong Convergence Theorem by an Extragradient Method for Fixed Point problems and Variational Inequality Problems*, Taiwanese Journal of Mathematics, **10** (2006), 1293-1303.

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