ITERATES OF CESÁRO OPERATORS,
VIA FIXED POINT PRINCIPLE

SZILÁRD ANDRÁS AND IOAN A. RUS

Department of Applied Mathematics
Babeș-Bolyai University, Cluj Napoca, Romania
E-mails: andraszk@yahoo.com
iarus@math.ubbcluj.ro

Abstract. In a paper by F. Galaz Fontes and F.J. Solís (Iterating the Cesàro operators, Proc. Amer. Math. Soc., 136(2008), No. 6, 2147-2153) the authors study the iterates of Cesàro operators on some subsets of $s(\mathbb{C})$ ($c(\mathbb{C}), c_0(\mathbb{C}), l^\infty(\mathbb{C})$), on $C([0,1],\mathbb{C})$ and on $C([0,\infty],\mathbb{C})$. In this paper we study the iterates of Cesàro operators on $s(\mathbb{B})$, on $C([0,1],\mathbb{B})$ and on $C([0,\infty],\mathbb{B})$, where $(\mathbb{B}, \|\cdot\|)$ is a Banach space and $s(\mathbb{B})$ is the set of all sequences with elements in $\mathbb{B}$. We use the contraction principle on a metric space and on a gauge space and we prove the convergence of the sequence of iterates on the whole space (endowed with a weaker topology). Our proofs are suggested by the characterization theorem of weakly Picard operators on an $L$-space (I.A. Rus, Picard operators and applications, Sci. Math. Jpn., 58(2003), 191-219) and our method can be applied to a more general class of averaging operators.

Key Words and Phrases: Cesàro operators, iterate operators, fixed point, weakly Picard operators.

2010 Mathematics Subject Classification: 47B37, 40G05, 47H10, 54H25.

1. Introduction

Let $(\mathbb{B},\|\cdot\|)$ be a Banach space (real or complex). We denote by $s(\mathbb{B})$ the set of all sequences with elements in $\mathbb{B}$. In [7] the authors study the iterates of Cesàro operators on some subsets of $s(\mathbb{C})$ ($c(\mathbb{C}), c_0(\mathbb{C}), l^\infty(\mathbb{C})$, . . .), on $C([0,1],\mathbb{C})$ and on $C([0,\infty],\mathbb{C})$.

In this paper we study the iterates of Cesàro operators on $s(\mathbb{B})$, on $C([0,1],\mathbb{B})$ and on $C([0,\infty],\mathbb{B})$ using the contraction principle on a metric space and on a gauge space. In [7] the authors used the uniform norm and they obtained the convergence of the sequence of iterates only on some subsets of the space. We use termwise convergence on $s(\mathbb{B})$ and pointwise convergence in $C([0,\infty],\mathbb{B})$ and we obtain the convergence of the sequence of iterates on the whole space, so from this viewpoint our results are more general than the results of Theorem 1 and Theorem 4 in [7]. In section 5 we prove that our method can be applied to a wide class of averaging operators.
2. A fixed point theorem

Let $X$ be a nonempty set and $A : X \to X$ an operator. We denote by $F_A$ the fixed point set of $A$, i.e., $F_A := \{ x \in X | A(x) = x \}$. By $A^0 := 1_X$, $A^1 := A$, $\ldots$, $A^n := A \circ A^{n-1}$, $n \in \mathbb{N}$, we denote the iterates of the operator $A$.

In this paper we use the following fixed point principle (see [5], [4] and [19]; see also [17], p.63).

Theorem 2.1. Let $(X, (d_k)_{k \in \mathbb{N}})$ be a separated and complete gauge space. Let $A : X \to X$ be an operator such that for each $k \in \mathbb{N}$ there exists $\alpha_k \in [0,1]$ with

$$d_k(A(x), A(y)) \leq \alpha_k d_k(x,y), \quad \forall x, y \in X.$$

Then:

(i) $F_A = \{ x^* \}$;

(ii) $A^n(x) \xrightarrow{d_k} x^*$ as $n \to \infty$, $\forall k \in \mathbb{N}$ and $\forall x \in X$.

Remark 2.2. If $x^*$ is a fixed point for $A$, then the sequence of successive approximation is convergent to $x^*$ without the assumption of completeness and $F_A = \{ x^* \}$.

Remark 2.3. By definition (see [14] and [16]) an operator with the properties (i) and (ii) is a Picard operator.

3. Cesàro operator on $s(\mathbb{B})$

Let $(\mathbb{B}, \| \cdot \|)$ be a (real or complex) Banach space. If $x \in \mathbb{B}$, then $\tilde{x} := (x, \ldots, x, \ldots)$ is the constant sequence defined by element $x$ of $\mathbb{B}$.

In what follows we consider the $L$-space $(s(\mathbb{B}), \overset{\ell}{\to})$, where $\overset{\ell}{\to}$ is the termwise convergence (for $L$-space see, for example, [14],[16],[17] and the references therein). Also, we consider on $s(\mathbb{B})$ the following family of pseudometrics $D := \{ d_k | k \in \mathbb{N} \}$, where $d_k(u,v) := \max_{0 \leq n \leq k} \| u_n - v_n \|$. Then the gauge space $(s(\mathbb{B}), D)$ is separated and complete.

Moreover, for $(u^n)_{n \in \mathbb{N}}, u \in s(\mathbb{B})$ we have $u^n \overset{D}{\to} u$ as $n \to \infty$ $\implies$ $u^n \overset{\ell}{\to} u$ as $n \to \infty$.

We consider on $s(\mathbb{B})$ the Cesàro operator $C : s(\mathbb{B}) \to s(\mathbb{B})$

$$(u_0, u_1, \ldots, u_n, \ldots) \mapsto \left( u_0, \frac{1}{2}(u_0 + u_1), \ldots, \frac{1}{n+1}(u_0 + u_1 + \ldots + u_n), \ldots \right).$$

Notice that $F_C = \{ \tilde{x} | x \in \mathbb{B} \}$. For $x \in \mathbb{B}$ we consider $Y_x := \{ u \in s(\mathbb{B}) | u_0 = x \}$. Then:

(a) $Y_x$ is a closed subset of $(s(\mathbb{B}), D)$, for all $x \in \mathbb{B}$;

(b) $C(Y_x) \subset Y_x$, $\forall x \in \mathbb{B}$;

(c) $s(\mathbb{B}) = \bigcup_{x \in \mathbb{B}} Y_x$ is a partition of $s(\mathbb{B})$;

(d) $d_k(C(u), C(v)) \leq \frac{k}{k+1} d_k(u,v)$, $\forall u, v \in Y_x$ and $\forall x \in \mathbb{B}$.

By Theorem 2.1 we get $\overset{\ell}{\to}$ $\tilde{x}$ as $n \to \infty$, $\forall u \in Y_x, x \in \mathbb{B}$. Thus, we have:

Theorem 3.1. $\overset{\ell}{\to}$ $\tilde{u}_0$ as $n \to \infty$, $\forall u \in s(\mathbb{B})$.

In terms of weakly Picard operators we can formulate Theorem 3.1 as follows:
Theorem 3.1: The Cesàro operator $C : s(\mathbb{B}) \to s(\mathbb{B})$ is weakly Picard operator on $(s(\mathbb{B}), L)$ and $C^\infty(u) = \bar{u}_0$, $\forall u = (u_0, u_1, \ldots, u_m, \ldots) \in s(\mathbb{B})$.

Remark 3.2. Our proof of Theorem 3.1 is suggested by the characterization theorem of weakly Picard operators (see [14]; see also [13], [15] and [8]).

Remark 3.3. The Cesàro operator $C : s(\mathbb{B}) \to s(\mathbb{B})$ is nonexpansive in $(s(\mathbb{B}), D)$, is contraction in each $(Y_x, D)$, for all $x \in \mathbb{B}$ and is graphic contraction in $(s(\mathbb{B}), D)$.

Remark 3.4. The above considerations are in connection with the theory of operators on an infinite dimensional cartesian product (see [18]).

Remark 3.5. In a similar way we can study the iterates of other summability operators (see [1], [2] and [10]).

4. Cesàro Operator on $C([0, 1], \mathbb{B})$

Let $(\mathbb{B}, \| \cdot \|)$ be a Banach space and

$C([0, 1], \mathbb{B}) := \{ f : [0, 1] \to \mathbb{B} | \text{f is continuous} \}$.

For $u \in \mathbb{B}$ we denote by $\bar{u}$ the constant function $t \mapsto u$, $t \in [0, 1]$.

We consider on $C([0, 1], \mathbb{B})$ the Cesàro operator $C$, defined by (see [7], [2], [10])

$C : C([0, 1], \mathbb{B}) \to C([0, 1], \mathbb{B})$

$C(f)(x) := \begin{cases} \frac{x}{n} \int_0^x f(t) dt, & \text{for } x \in [0, 1]; \\ f(0), & \text{for } x = 0. \end{cases}$

The fixed point set of $C$ is $F_C = \{ \bar{u} | u \in \mathbb{B} \}$.

In what follows we give a new proof of the following theorem.

Theorem 4.1. $C^n(f) \xrightarrow{\text{unif.}} f(0)$ as $n \to \infty$, for all $f \in C([0, 1], \mathbb{B})$.

This theorem was proved in [7] by a different approach. Our technique can be applied for a wide class of general averaging operators. The main idea of the proof is that the Cesàro operator is a contraction on a well chosen subspace of $C([0, 1], \mathbb{B})$ which is equipped with a suitable metric. We need the following two lemmas:

Lemma 4.2. On the set $H_u := \{ f \in C^1([0, 1], \mathbb{B}) | f(0) = u \}$, the functional $d_1 : H_u \times H_u \to \mathbb{R}$ defined by

$d_1(f, g) = \min\{ M \in \mathbb{R} | \| f(x) - g(x) \| \leq M x, \forall x \in [0, 1] \}$

is a metric.

Proof. Let $f, g \in H_u$. Then $f - g \in H_0$, so there exists $l \in \mathbb{R}$ such that $l = \lim_{x \to 0} \frac{f(x) - g(x)}{x}$. This implies that there exists $\delta > 0$ such that

$\| f(x) - g(x) \| \leq 2lx, \forall x \in [0, \delta]. \quad (4.1)$

On $[\delta, 1]$ the function $x \mapsto \frac{f(x) - g(x)}{x}$ is continuous and this implies the existence of a constant $K_0$ with the property

$\| f(x) - g(x) \| \leq K_0 x, \forall x \in [\delta, 1]. \quad (4.2)$
From (4.1) and (4.2) we deduce that, for $M := \max\{2l, K_0\}$,

$$\|f(x) - g(x)\| \leq Mx, \quad \forall x \in [0, 1],$$  \hfill (4.3)

Hence the set $\mathcal{M} = \{M \in \mathbb{R} \mid \|f(x) - g(x)\| \leq Mx, \forall x \in [0, 1]\}$ is not empty. It is obvious that $\mathcal{M}$ is bounded from below and due to the continuity of $f$ and $g$ the infimum of $\mathcal{M}$ is reached for some $M \in \mathcal{M}$. This implies that $d_1$ is well defined.

From the definition we deduce $d_1(f, g) \geq 0$, for all $f, g \in H_u$. If $d_1(f, g) = 0$, we obtain $f(x) = g(x)$, for all $x \in (0, 1]$, so $f = g$ (because $f, g \in H_u$). If $f, g, h \in H_u$ and $M_1 = d_1(f, g)$, $M_2 = d_1(g, h)$, then

$$\|f(x) - h(x)\| \leq \|f(x) - g(x)\| + \|g(x) - h(x)\| \leq (M_1 + M_2)x, \quad \forall x \in [0, 1].$$

This guarantees $d_1(f, h) \leq M_1 + M_2$, hence $d_1$ is a metric on $H_u$. \hfill \Box

**Lemma 4.3.** If $(f_n)_{n \geq 0}$ is a convergent sequence in $(H_u, d_1)$ and $f^*$ is its limit, then $f_n \xrightarrow{\text{unif.}} f^*$.

**Proof.** $f_n \xrightarrow{d_1} f^*$ implies that

$$\|f_n(x) - f^*(x)\| \leq M_n x \leq M_n, \quad \forall x \in [0, 1],$$  \hfill (4.4)

where $M_n = d_1(f_n, f^*)$. But $M_n \to 0$ as $n \to \infty$ and this implies the uniform convergence of the sequence $(f_n)_{n \geq 0}$ to $f^*$.

**Proof of Theorem 4.1.** We remark that $C^1([0, 1], \mathbb{B}) = \bigcup_{u \in \mathcal{B}} H_u$ is a partition of $C^1([0, 1], \mathbb{B})$ and each set $H_u$ is an invariant set of the operator $C$, moreover each set $H_u$ contains a unique fixed point of $C$. To complete the proof of Theorem 4.1 we need only to observe that the Cesàro operator is a contraction on $(H_u, d_1)$. If $f, g \in H_u$ and $M = d_1(f, g)$, then $\|f(t) - g(t)\| \leq M t, \quad \forall t \in [0, 1]$.

Hence

$$\|C(f)(x) - C(g)(x)\| = \left\| \frac{1}{x} \int_0^x (f(t) - g(t))dt \right\| \leq \frac{1}{x} \int_0^x \|f(t) - g(t)\|dt \leq \frac{M}{2} x, \quad \forall x \in [0, 1].$$

This inequality and the definition of the metric imply

$$d_1(Cf, Cg) \leq \frac{1}{2} d_1(f, g),$$

so the Cesàro operator is a contraction on $(H_u, d_1)$. But $\tilde{u}$ is a fixed point of $C$ in $H_u$, so it is the unique fixed point of $C$ in $H_u$ and the sequence of successive approximation converges to $\tilde{u}$. This implies that $C^n(f) \xrightarrow{d_1} f(\tilde{0})$ as $n \to \infty$, and due to Lemma 4.3 $C^n(f) \xrightarrow{\text{unif.}} f(\tilde{0})$ as $n \to \infty$. Using the density of $C^1([0, 1], \mathbb{B})$ in $C([0, 1], \mathbb{B})$ and the nonexpansive property of the Cesàro operator on $(C([0, 1], \mathbb{B})$ we can conclude that $C^n(f) \xrightarrow{\text{unif.}} f(\tilde{0})$ for all $f \in C([0, 1], \mathbb{B})$. \hfill \Box

**Remark 4.4.** The Cesàro operator $C$ is weakly Picard operator on the $L$-space $(C([0, 1], \mathbb{B}), \xrightarrow{\text{unif.}})$ and $C^\infty_f = f(\tilde{0}), \forall f \in C([0, 1], \mathbb{B})$. 
Remark 4.5. The proof of Theorem 4.1 is suggested by the characterization theorem of weakly Picard operators (see [14]; see also [13] and [15]).

Remark 4.6. If $B = C$ we have a new proof for Theorem 3 from [7].

5. General averaging operators on $C([0, 1], B)$

Denote by $w : [0, 1] \to \mathbb{R}_+$ a continuous function with $w(x) > 0$, for $x > 0$ and define the averaging operator $C_w : C([0, 1], B) \to C([0, 1], B)$ (see [3], [11], [12]) by

$$C_w(f)(x) := \begin{cases} \int_0^x \frac{w(t)f(t)dt}{\int_0^x w(t)dt}, & \text{for } x \in [0, 1]; \\ f(0), & \text{for } x = 0. \end{cases}$$

(5.1)

Remark 5.1. For $w(x) = 1$, $\forall x \in [0, 1]$, $C_w$ is the Cesàro operator. Notice also that the fixed point set of $C_w$ is $F_{C_w} = \{ \tilde{u} | u \in B \}$.

Using the same technique as in the proof of 4.1 we obtain the following results:

Theorem 5.2. If for the continuous function $w : [0, 1] \to \mathbb{R}_+$ $w(x) > 0$, for $x > 0$ and there exists $L < 1$ such that

$$\int_0^x tw(t)dt \leq Lx \int_0^x w(t)dt, \quad \forall x \in [0, 1],$$

(5.2)

then $C^n_w(f) \xrightarrow{unif.} \tilde{f}(0)$ as $n \to \infty$, for all $f \in C([0, 1], B)$.

Remark 5.3. If $w(x) = 1$, $\forall x \in [0, 1]$, then 5.2 is satisfied with $L = \frac{1}{2}$.

Remark 5.4. If $w(x) = x^\alpha$, $\forall x \in (0, 1]$ and $\alpha > -1$, then (5.2) is satisfied with $L = \frac{\alpha+1}{\alpha+2}$ and Theorem 5.2 can be used even if $w$ is not continuous in 0.

Theorem 5.5. If $w : [0, 1] \to (0, \infty)$ is a continuous function, then $C^n_w(f) \xrightarrow{unif.} \tilde{f}(0)$ as $n \to \infty$, for all $f \in C([0, 1], B)$.

Proof of Theorem 5.2. If $f \in H_u$, then from the continuity of $w$ we have $C_w(f) \in H_u$. If $d_1(f, \tilde{f}(0)) = M$, then

$$-Mx \leq \|f(x) - \tilde{f}(0)\| \leq Mx, \quad \forall x \in [0, 1]$$
and we obtain
\[
\|C_w(f)(x) - C_w(f(\tilde{0}))(x)\| = \frac{1}{x} \int_{0}^{x} \left| \frac{\int_{0}^{x} (f(t) - f(\tilde{0})(t))w(t)dt}{w(t)} \right|
\]
\[
\leq \frac{1}{x} \int_{0}^{x} \|f(t) - f(\tilde{0})(t)\|w(t)dt
\]
\[
\leq \frac{M}{x} \int_{0}^{x} tw(t)dt \leq MLx.
\]
Thus
\[
d_1(C_w(f), f(\tilde{0})) \leq Ld_1(f, \tilde{f}(0)).
\]
The above inequality and the fact that \(L < 1\) implies \(C_w(f) \xrightarrow{\text{uniformly}} \tilde{f}(0)\). \(\square\)

To prove theorem 5.5 we need the following two lemmas

**Lemma 5.6.** Suppose that the function \(v : [0, 1] \to [0, \infty)\) is continuously differentiable with \(v(0) = 0, v(x) \neq 0\) for \(x > 0\) and \(v'(0) \neq 0\). Then, on the set \(H_u := \{f \in C^1([0, 1], \mathbb{B})|f(0) = u\}\) the functional \(d_v : H_u \times H_u \to \mathbb{R}\) defined by
\[
d_v(f, g) = \min\{M \in \mathbb{R}|\|f(x) - g(x)\| \leq Mv(x), \forall x \in [0, 1]\}
\]
is a metric.

**Proof.** \(f - g \in H_0,\) so there exists \(l \in \mathbb{R}\) such that \(l = \lim_{x \to 0} \frac{\|f(x) - g(x)\|}{x} v(x) \in C^1[0, 1]\) and \(v'(0) \neq 0\), so there exist \(l = \lim_{x \to 0} \frac{x}{v(x)} = \frac{1}{v'(0)}\). This implies that \(\lim_{x \to 0} \frac{\|f(x) - g(x)\|}{v(x)} = \frac{l}{v'(0)}\), so for \(K_1 = \frac{2l}{v'(0)}\) there exists \(\delta > 0\) such that,
\[
\|f(x) - g(x)\| \leq K_1 v(x), \quad \forall x \in [0, \delta]. \tag{5.3}
\]
On \([\delta, 1]\) the function \(x \to \frac{\|f(x) - g(x)\|}{v(x)}\) is continuous and this implies the existence of a constant \(K_2\) with the property
\[
\|f(x) - g(x)\| \leq K_2 v(x), \quad \forall x \in [\delta, 1]. \tag{5.4}
\]
From (5.3) and (5.4) we deduce that for \(M = \max\{K_1, K_2\}\)
\[
\|f(x) - g(x)\| \leq Mv(x), \quad \forall x \in [0, 1]. \tag{5.5}
\]
Hence the set \(\mathcal{M} = \{M \in \mathbb{R}|\|f(x) - g(x)\| \leq Mv(x), \forall x \in [0, 1]\}\) is not empty. It is obvious that \(\mathcal{M}\) is bounded from below and due to the continuity of \(f\) and \(g\) the infimum of \(\mathcal{M}\) is reached for some \(M \in \mathcal{M}\). This implies that \(d_v\) is well defined.

From the definition we deduce \(d_v(f, g) \geq 0,\) for all \(f, g \in H_u\). If \(d_v(f, g) = 0,\) we obtain \(f(x) = g(x),\) for all \(x \in (0, 1]\), so \(f = g\) (because \(f, g \in H_u\)). If \(f, g, h \in H_u\) and \(M_1 = d_v(f, g), M_2 = d_v(g, h),\) then
\[
\|f(x) - h(x)\| \leq \|f(x) - g(x)\| + \|g(x) - h(x)\| \leq (M_1 + M_2) v(x), \quad \forall x \in [0, 1].
\]
This guaranties $d_v(f, h) \leq M_1 + M_2$, hence $d_v$ is a metric on $H_u$. \hfill \Box

**Lemma 5.7.** If $(f_n)_{n \geq 0}$ is a convergent sequence in $(H_u, d_v)$ and $f^*$ is it’s limit, than $f_n \xrightarrow{\text{unif.}} f^*$.

**Proof.** $f_n \xrightarrow{d_1} f^*$ implies that

\[
\|f_n(x) - f^*(x)\| \leq M_n v(x) \leq M_n M, \quad \forall x \in [0, 1],
\]

where $M_n = d_1(f_n, f^*)$ and $M = \max_{x \in [0, 1]} v(x)$. But $M_n \to 0$ as $n \to \infty$ and this implies the uniform convergence of the sequence $(f_n)_{n \geq 0}$ to $f^*$. \hfill \Box

**Proof of Theorem 5.5.** We prove that the averaging operator $C_w$ is a contraction on $(H_u, d_v)$ where $v(x) = \int_0^x w(t)dt, \ x \in [0, 1]$. Due to the assumptions on $w$ the function $v$ satisfies the conditions of Lemma 5.6. If $f, g \in H_u$ and $M = d_v(f, g)$, then

\[
\|f(t) - g(t)\| \leq M v(t), \quad \forall t \in [0, 1].
\]

Hence

\[
\|C(f)(x) - C(g)(x)\| = \frac{1}{v(x)} \left| \int_0^x w(t)(f(t) - g(t))dt \right|
\leq \frac{1}{v(x)} \int_0^x w(t)\|f(t) - g(t)\|dt
\leq \frac{M}{v(x)} \int_0^x v'(t)v(t)dt \leq \frac{M}{2} v(x), \quad \forall x \in [0, 1].
\]

This inequality and the definition of the metric imply

\[
d_v(Cf, Cg) \leq \frac{1}{2}d_v(f, g).
\]

Thus, the averaging operator is a contraction on $(H_u, d_v)$. But $\tilde{u}$ is a fixed point of $C_w$ in $H_u$, so it is the unique fixed point of $C$ in $H_u$ and the sequence of successive approximation converges to $\tilde{u}$. This implies that $C^n_w(f) \xrightarrow{d_w} \tilde{f}(0)$ as $n \to \infty$, and due to Lemma 5.7 $C^n_w(f) \xrightarrow{\text{unif.}} \tilde{f}(0)$ as $n \to \infty$. Using the density of $C^1([0, 1], \mathbb{B})$ in $C([0, 1], \mathbb{B})$ and the nonexpansive property of the averaging operator on $C([0, 1], \mathbb{B})$ we can conclude that $C^n_w(f) \xrightarrow{\text{unif.}} \tilde{f}(0)$ for all $f \in C([0, 1], \mathbb{B})$. \hfill \Box

6. Cesàro operator on $C([0, +\infty[, \mathbb{B})$

By a similar reasoning as in the proof of Theorem 4.1 we obtain the following theorem.

**Theorem 6.1.** $C^n(f) \xrightarrow{\text{unif.}} \tilde{f}(0)$ as $n \to \infty$, for all $f \in C([0, k], \mathbb{B})$ and $k \in \mathbb{N}^*$. 
On the other hand let, us consider the Cesàro operator $C : C([0, \infty), \mathbb{B}) \to C([0, k], \mathbb{B})$. Since $C$ has the Volterra property we can consider the restriction of $C$ to $C([0, k], \mathbb{B})$ for each $k \in \mathbb{N}^*$. From theorem 6.1 we deduce that for each fixed $k \in \mathbb{N}^*$ $C^n(f) \xrightarrow{\text{unif.}} f(0)$ on $[0, k]$. This implies that $C^n(f) \xrightarrow{\text{unif.}} f(0)$ on every compact subset of $[0, \infty)$ so we have the following theorem.

**Theorem 6.2.** On every compact subset of $[0, \infty)$ $C^n(f) \xrightarrow{\text{unif.}} f(0)$ as $n \to \infty$, for all $f \in C([0, \infty), \mathbb{B})$.

**Remark 6.3.** Theorem 6.2 is valid for the general averaging operators defined in (5.1) if $w : [0, \infty) \to [0, \infty)$ is a continuous function satisfying the assumptions of Theorem 5.2 or 5.5.

**References**


Received: October 22, 2009; Accepted: January 7, 2010.