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ITERATES OF CESÀRO OPERATORS, VIA FIXED POINT PRINCIPLE

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Abstract. In a paper by F. Galaz Fontes and F.J. Solís (Iterating the Cesàro operators, Proc. Amer. Math. Soc., 136(2008), No. 6,2147-2153) the authors study the iterates of Cesàro operators on some subsets of $s(\mathbb{C})$ $(c(\mathbb{C}), c_0(\mathbb{C}), l^\infty(\mathbb{C}))$, on $(C[0,1],\mathbb{C})$ and on $C([0,\infty[,\mathbb{C})$. In this paper we study the iterates of Cesàro operators on $s(\mathbb{B})$, on $C([0,1],\mathbb{B})$ and on $C([0,\infty[,\mathbb{B})$, where $(\mathbb{B},\|\cdot\|)$ is a Banach space and $s(\mathbb{B})$ is the set of all sequences with elements in $\mathbb{B}$. We use the contraction principle on a metric space and on a gauge space and we prove the convergence of the sequence of iterates on the whole space (endowed with a weaker topology). Our proofs are suggested by the characterization theorem of weakly Picard operators on an L-space (I.A. Rus, Picard operators and applications, Sci. Math. Jpn., 58(2003), 191-219) and our method can be applied to a more general class of averaging operators.
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1. Introduction

Let $(\mathbb{B},\|\cdot\|)$ be a Banach space (real or complex). We denote by $s(\mathbb{B})$ the set of all sequences with elements in $\mathbb{B}$. In [7] the authors study the iterates of Cesàro operators on some subsets of $s(\mathbb{C})$ $(c(\mathbb{C}), c_0(\mathbb{C}), l^\infty(\mathbb{C}),\ldots)$, on $(C[0,1],\mathbb{C})$ and on $C([0,\infty[,\mathbb{C})$.

In this paper we study the iterates of Cesàro operators on $s(\mathbb{B})$, on $C([0,1],\mathbb{B})$ and on $C([0,\infty[,\mathbb{B})$ using the contraction principle on a metric space and on a gauge space. In [7] the authors used the uniform norm and they obtained the convergence of the sequence of iterates only on some subsets of the space. We use termwise convergence on $s(\mathbb{B})$ and pointwise convergence in $C([0,\infty[,\mathbb{B})$ and we obtain the convergence of the sequence of iterates on the whole space, so from this viewpoint our results are more general than the results of Theorem 1 and Theorem 4 in [7]. In section 5 we prove that our method can be applied to a wide class of averaging operators.

2. A fixed point theorem

Let X be a nonempty set and $A : X \to X$ an operator. We denote by F_A the fixed point set of A, i.e., $F_A := \{x \in X | A(x) = x\}$. By $A^0 := 1_X$, $A^1 := A, \ldots, A^n := A \circ A^{n-1}, n \in \mathbb{N}$, we denote the iterates of the operator A.

In this paper we use the following fixed point principle (see [5], [4] and [19]; see also [17], p.63).

Theorem 2.1. *Let $(X, (d_k)_{k\in\mathbb{N}})$ be a separated and complete gauge space. Let $A : X \to X$ be an operator such that for each $k \in \mathbb{N}$ there exists $\alpha_k \in]0,1[$ with*

$$d_k(A(x), A(y)) \le \alpha_k d_k(x, y), \quad \forall x, y \in X.$$

Then:

(i) $F_A = \{x^*\}$;
(ii) $A^n(x) \xrightarrow{d_k} x^*$ *as* $n \to \infty$, $\forall k \in \mathbb{N}$ *and* $\forall x \in X$.

Remark 2.2. If x^* is a fixed point for A, then the sequence of successive approximation is convergent to x^* without the assumption of completeness and $F_A = \{x^*\}$.

Remark 2.3. By definition (see [14] and [16]) an operator with the properties (i) and (ii) is a Picard operator.

3. Cesàro operator on $s(\mathbb{B})$

Let $(\mathbb{B}, \|\cdot\|)$ be a (real or complex) Banach space. If $x \in \mathbb{B}$, then $\tilde{x} := (x, \ldots, x, \ldots)$ is the constant sequence defined by element x of $\mathbb{B}$.

In what follows we consider the L-space $(s(\mathbb{B}), \xrightarrow{t})$, where $\xrightarrow{t}$ is the termwise convergence (for L-space see, for example, [14],[16],[17] and the references therein). Also, we consider on $s(\mathbb{B})$ the following family of pseudometrics $\mathcal{D} := \{d_k | k \in \mathbb{N}\}$, where $d_k(u, v) := \max\limits_{0\le n\le k} \|u_n - v_n\|$. Then the gauge space $(s(\mathbb{B}), \mathcal{D})$ is separated and complete. Moreover, for $(u^n)_{n\in\mathbb{N}}, u \in s(\mathbb{B})$ we have $u^n \xrightarrow{\mathcal{D}} u$ as $n \to \infty \Longrightarrow u^n \xrightarrow{t} u$ as $n \to \infty$. We consider on $s(\mathbb{B})$ the Cesàro operator $C : s(\mathbb{B}) \to s(\mathbb{B})$

$$(u_0, u_1, \ldots, u_n, \ldots) \mapsto \left(u_0, \frac{1}{2}(u_0 + u_1), \ldots, \frac{1}{n+1}(u_0 + u_1 + \ldots + u_n), \ldots\right).$$

Notice that $F_C = \{\tilde{x} | x \in \mathbb{B}\}$. For $x \in \mathbb{B}$ we consider $Y_x := \{u \in s(\mathbb{B}) | u_0 = x\}$. Then:

(a) Y_x is a closed subset of $(s(\mathbb{B}), \mathcal{D})$, for all $x \in \mathbb{B}$;
(b) $C(Y_x) \subset Y_x, \quad \forall x \in \mathbb{B}$;
(c) $s(\mathbb{B}) = \bigcup\limits_{x\in\mathbb{B}} Y_x$ is a partition of $s(\mathbb{B})$;
(d) $d_k(C(u), C(v)) \le \frac{k}{k+1} d_k(u, v), \quad \forall u, v \in Y_x$ and $\forall x \in \mathbb{B}$.

By Theorem 2.1 we get $C^n(u) \xrightarrow{\mathcal{D}} \tilde{x}$ as $n \to \infty$, $\forall u \in Y_x, x \in \mathbb{B}$. Thus, we have:

Theorem 3.1. $C^n(u) \xrightarrow{\mathcal{D}} \tilde{u_0}$ *as* $n \to \infty$, $\forall u \in s(\mathbb{B})$.

In terms of weakly Picard operators we can formulate Theorem 3.1 as follows:

Theorem 3.1. *The Cesàro operator $C : s(\mathbb{B}) \to s(\mathbb{B})$ is weakly Picard operator on $(s(\mathbb{B}), \xrightarrow{t})$ and $C^\infty(u) = \tilde{u_0}, \quad \forall u = (u_0, u_1, \ldots, u_m, \ldots) \in s(\mathbb{B})$.*

Remark 3.2. Our proof of Theorem 3.1 is suggested by the characterization theorem of weakly Picard operators (see [14]; see also [13], [15] and [8]).

Remark 3.3. The Cesàro operator $C : s(\mathbb{B}) \to s(\mathbb{B})$ is nonexpansive in $(s(\mathbb{B}), \mathcal{D})$, is contraction in each $(Y_x, \mathcal{D})$, for all $x \in \mathbb{B}$ and is graphic contraction in $(s(\mathbb{B}), \mathcal{D})$.

Remark 3.4. The above considerations are in connection with the theory of operators on an infinite dimensional cartesian product (see [18]).

Remark 3.5. In a similar way we can study the iterates of other summability operators (see [1], [2] and [10]).

4. Cesàro operator on $C([0,1], \mathbb{B})$

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space and

$$C([0,1], \mathbb{B}) := \{f : [0,1] \to \mathbb{B} | f - \text{continuous}\}.$$

For $u \in \mathbb{B}$ we denote by $\tilde{u}$ the constant function $t \mapsto u,\ t \in [0,1]$.

We consider on $C([0,1], \mathbb{B})$ the Cesàro operator C, defined by (see [7], [2], [10])

$$C : C([0,1], \mathbb{B}) \to C([0,1], \mathbb{B})$$

$$C(f)(x) := \begin{cases} \frac{1}{x} \int\limits_0^x f(t)dt, & \text{for } x \in]0,1]; \\ f(0), & \text{for } x = 0. \end{cases}$$

The fixed point set of C is $F_C = \{\tilde{u} | u \in \mathbb{B}\}$.

In what follows we give a new proof of the following theorem.

Theorem 4.1. *$C^n(f) \xrightarrow{unif.} \tilde{f(0)}$ as $n \to \infty$, for all $f \in C([0,1], \mathbb{B})$.*

This theorem was proved in [7] by a different approach. Our technique can be applied for a wide class of general averaging operators. The main idea of the proof is that the Cesàro operator is a contraction on a well chosen subspace of $C([0,1], \mathbb{B})$ which is equipped with a suitable metric. We need the following two lemmas:

Lemma 4.2. *On the set $H_u := \{f \in C^1([0,1], \mathbb{B}) | f(0) = u\}$, the functional $d_1 : H_u \times H_u \to \mathbb{R}$ defined by*

$$d_1(f, g) = \min\{M \in \mathbb{R} | \|f(x) - g(x)\| \le Mx,\ \forall x \in [0,1]\}.$$

is a metric.

Proof. Let $f, g \in H_u$. Then $f - g \in H_0$, so there exists $l \in \mathbb{R}$ such that $l = \lim\limits_{x \to 0} \frac{\|f(x) - g(x)\|}{x}$. This implies that there exists $\delta > 0$ such that

$$\|f(x) - g(x)\| \le 2lx, \quad \forall x \in [0, \delta]. \tag{4.1}$$

On $[\delta, 1]$ the function $x \to \frac{\|f(x) - g(x)\|}{x}$ is continuous and this implies the existence of a constant K_0 with the property

$$\|f(x) - g(x)\| \le K_0 x, \quad \forall x \in [\delta, 1]. \tag{4.2}$$

From (4.1) and (4.2) we deduce that, for $M := \max\{2l, K_0\}$,

$$\|f(x) - g(x)\| \leq Mx, \quad \forall x \in [0,1], \tag{4.3}$$

Hence the set $\mathcal{M} = \{M \in \mathbb{R} | \|f(x) - g(x)\| \leq Mx, \ \forall x \in [0,1]\}$ is not empty. It is obvious that $\mathcal{M}$ is bounded from below and due to the continuity of f and g the infimum of $\mathcal{M}$ is reached for some $M \in \mathcal{M}$. This implies that d_1 is well defined.

From the definition we deduce $d_1(f,g) \geq 0$, for all $f, g \in H_u$. If $d_1(f,g) = 0$, we obtain $f(x) = g(x)$, for all $x \in (0,1]$, so $f = g$ (because $f, g \in H_u$). If $f, g, h \in H_u$ and $M_1 = d_1(f,g)$, $M_2 = d_1(g,h)$, then

$$\|f(x) - h(x)\| \leq \|f(x) - g(x)\| + \|g(x) - h(x)\| \leq (M_1 + M_2)x, \ \forall x \in [0,1].$$

This guaranties $d_1(f,h) \leq M_1 + M_2$, hence d_1 is a metric on H_u. □

Lemma 4.3. *If $(f_n)_{n\geq 0}$ is a convergent sequence in (H_u, d_1) and f^* is it's limit, then $f_n \xrightarrow{unif.} f^*$.*

Proof. $f_n \xrightarrow{d_1} f^*$ implies that

$$\|f_n(x) - f^*(x)\| \leq M_n x \leq M_n, \quad \forall x \in [0,1], \tag{4.4}$$

where $M_n = d_1(f_n, f^*)$. But $M_n \to 0$ as $n \to \infty$ and this implies the uniform convergence of the sequence $(f_n)_{n\geq 0}$ to f^*. □

Proof of Theorem 4.1. We remark that $C^1([0,1],\mathbb{B}) = \bigcup_{u\in\mathbb{B}} H_u$ is a partition of $C^1([0,1],\mathbb{B})$ and each set H_u is an invariant set of the operator C, moreover each set H_u contains a unique fixed point of C. To complete the proof of Theorem 4.1 we need only to observe that the Cesàro operator is a contraction on (H_u, d_1). If $f, g \in H_u$ and $M = d_1(f,g)$, then $\|f(t) - g(t)\| \leq Mt, \quad \forall t \in [0,1]$.
Hence

$$\begin{aligned} \|C(f)(x) - C(g)(x)\| &= \left\| \frac{1}{x} \int_0^x (f(t) - g(t))dt \right\| \\ &\leq \frac{1}{x} \int_0^x \|f(t) - g(t)\|dt \leq \frac{M}{2}x, \ \forall x \in [0,1]. \end{aligned}$$

This inequality and the definition of the metric imply

$$d_1(Cf, Cg) \leq \frac{1}{2} d_1(f,g),$$

so the Cesàro operator is a contraction on (H_u, d_1). But $\tilde{u}$ is a fixed point of C in H_u, so it is the unique fixed point of C in H_u and the sequence of successive approximation converges to $\tilde{u}$. This implies that $C^n(f) \xrightarrow{d_1} \tilde{f(0)}$ as $n \to \infty$, and due to Lemma 4.3 $C^n(f) \xrightarrow{\text{unif.}} \tilde{f(0)}$ as $n \to \infty$. Using the density of $C^1([0,1],\mathbb{B})$ in $C([0,1],\mathbb{B})$ and the nonexpansive property of the Cesàro operator on $(C([0,1],\mathbb{B})$ we can conclude that $C^n(f) \xrightarrow{\text{unif.}} \tilde{f(0)}$ for all $f \in C([0,1],\mathbb{B})$. □

Remark 4.4. The Cesàro operator C is weakly Picard operator on the L-space $(C([0,1],\mathbb{B}), \xrightarrow{\text{unif.}})$ and $C^\infty(f) = \tilde{f(0)}, \forall f \in C([0,1],\mathbb{B})$.

Remark 4.5. The proof of Theorem 4.1 is suggested by the characterization theorem of weakly Picard operators (see [14]; see also [13] and [15]).

Remark 4.6. If $\mathbb{B} = \mathbb{C}$ we have a new proof for Theorem 3 from [7].

5. General averaging operators on $C([0,1],\mathbb{B})$

Denote by $w : [0,1] \to \mathbb{R}_+$ a continuous function with $w(x) > 0$, for $x > 0$ and define the averaging operator $C_w : C([0,1],\mathbb{B}) \to C([0,1],\mathbb{B})$ (see [3], [11], [12]) by

$$C_w(f)(x) := \begin{cases} \dfrac{\int_0^x w(t)f(t)dt}{\int_0^x w(t)dt}, & \text{for } x \in]0,1]; \\ f(0), & \text{for } x = 0. \end{cases} \tag{5.1}$$

Remark 5.1. For $w(x) = 1$, $\forall x \in [0,1]$, C_w is the Cesàro operator. Notice also that the fixed point set of C_w is $F_{C_w} = \{\tilde{u} | u \in \mathbb{B}\}$.

Using the same technique as in the proof of 4.1 we obtain the following results:

Theorem 5.2. *If for the continuous function $w : [0,1] \to \mathbb{R}_+$ $w(x) > 0$, for $x > 0$ and there exists $L < 1$ such that*

$$\int_0^x tw(t)dt \le Lx \int_0^x w(t)dt, \quad \forall x \in [0,1], \tag{5.2}$$

then $C_w^n(f) \xrightarrow{unif.} \tilde{f(0)}$ as $n \to \infty$, for all $f \in C([0,1],\mathbb{B})$.

Remark 5.3. If $w(x) = 1$, $\forall x \in [0,1]$, then 5.2 is satisfied with $L = \frac{1}{2}$.

Remark 5.4. If $w(x) = x^\alpha$, $\forall x \in (0,1]$ and $\alpha > -1$, then (5.2) is satisfied with $L = \frac{\alpha+1}{\alpha+2}$ and Theorem 5.2 can be used even if w is not continuous in 0.

Theorem 5.5. *If $w : [0,1] \to (0,\infty)$ is a continuous function, then $C_w^n(f) \xrightarrow{unif.} \tilde{f(0)}$ as $n \to \infty$, for all $f \in C([0,1],\mathbb{B})$.*

Proof of Theorem 5.2. If $f \in H_u$, then from the continuity of w we have $C_w(f) \in H_u$. If $d_1(f, \tilde{f(0)}) = M$, then

$$-Mx \le \|f(x) - \tilde{f(0)}\| \le Mx, \quad \forall x \in [0,1]$$

and we obtain

$$\begin{aligned}\|C_w(f)(x) - C_w(f\tilde{(0)})(x)\| &= \frac{1}{\int_0^x w(t)dt} \left\| \int_0^x (f(t) - f\tilde{(0)}(t))w(t)dt \right\| \\ &\leq \frac{1}{\int_0^x w(t)dt} \int_0^x \|f(t) - f\tilde{(0)}(t)\| w(t)dt \\ &\leq \frac{M}{\int_0^x w(t)dt} \int_0^x tw(t)dt \leq MLx.\end{aligned}$$

Thus

$$d_1(C_w(f), f\tilde{(0)}) \leq Ld_1(f, f\tilde{(0)}).$$

The above inequality and the fact that $L < 1$ implies $C_w^n(f) \xrightarrow{\text{unif.}} f\tilde{(0)}$. □

To prove theorem 5.5 we need the following two lemmas

Lemma 5.6. *Suppose that the function $v : [0,1] \to [0,\infty)$ is continuously differentiable with $v(0) = 0$, $v(x) \neq 0$ for $x > 0$ and $v'(0) \neq 0$. Then, on the set $H_u := \{f \in C^1([0,1], \mathbb{B}) | f(0) = u\}$ the functional $d_v : H_u \times H_u \to \mathbb{R}$ defined by*

$$d_v(f,g) = \min\{M \in \mathbb{R} | \|f(x) - g(x)\| \leq Mv(x),\ \forall x \in [0,1]\}$$

is a metric.

Proof. $f - g \in H_0$, so there exists $l \in \mathbb{R}$ such that $l = \lim_{x\to 0} \frac{\|f(x)-g(x)\|}{x}$. $v \in C^1[0,1]$ and $v'(0) \neq 0$, so there exist $l = \lim_{x\to 0} \frac{x}{v(x)} = \frac{1}{v'(0)}$. This implies that $\lim_{x\to 0} \frac{\|f(x)-g(x)\|}{v(x)} = \frac{l}{v'(0)}$, so for $K_1 = \frac{2l}{v'(0)}$ there exists $\delta > 0$ such that,

$$\|f(x) - g(x)\| \leq K_1 v(x), \quad \forall x \in [0, \delta]. \tag{5.3}$$

On $[\delta, 1]$ the function $x \to \frac{\|f(x)-g(x)\|}{v(x)}$ is continuous and this implies the existence of a constant K_2 with the property

$$\|f(x) - g(x)\| \leq K_2 v(x), \quad \forall x \in [\delta, 1]. \tag{5.4}$$

From (5.3) and (5.4) we deduce that for $M = \max\{K_1, K_2\}$

$$\|f(x) - g(x)\| \leq Mv(x), \quad \forall x \in [0, 1]. \tag{5.5}$$

Hence the set $\mathcal{M} = \{M \in \mathbb{R} | \|f(x) - g(x)\| \leq Mv(x),\ \forall x \in [0,1]\}$ is not empty. It is obvious that $\mathcal{M}$ is bounded from below and due to the continuity of f and g the infimum of $\mathcal{M}$ is reached for some $M \in \mathcal{M}$. This implies that d_v is well defined.

From the definition we deduce $d_v(f,g) \geq 0$, for all $f, g \in H_u$. If $d_v(f,g) = 0$, we obtain $f(x) = g(x)$, for all $x \in (0,1]$, so $f = g$ (because $f, g \in H_u$). If $f, g, h \in H_u$ and $M_1 = d_v(f,g)$, $M_2 = d_v(g,h)$, then

$$\|f(x) - h(x)\| \leq \|f(x) - g(x)\| + \|g(x) - h(x)\| \leq (M_1 + M_2)v(x),\ \forall x \in [0,1].$$

This guaranties $d_v(f,h) \le M_1 + M_2$, hence d_v is a metric on H_u. □

Lemma 5.7. *If $(f_n)_{n\ge 0}$ is a convergent sequence in (H_u, d_v) and f^* is it's limit, than $f_n \xrightarrow{unif.} f^*$.*

Proof. $f_n \xrightarrow{d_1} f^*$ implies that

$$\|f_n(x) - f^*(x)\| \le M_n v(x) \le M_n M, \quad \forall x \in [0,1], \tag{5.6}$$

where $M_n = d_1(f_n, f^*)$ and $M = \max_{x\in[0,1]} v(x)$. But $M_n \to 0$ as $n \to \infty$ and this implies the uniform convergence of the sequence $(f_n)_{n\ge 0}$ to f^*. □

Proof of Theorem 5.5. We prove that the averaging operator C_w is a contraction on (H_u, d_v) where $v(x) = \int_0^x w(t)dt$, $x \in [0,1]$. Due to the assumptions on w the function v satisfies the conditions of Lemma 5.6. If $f, g \in H_u$ and $M = d_v(f,g)$, then

$$\|f(t) - g(t)\| \le M v(t), \quad \forall t \in [0,1].$$

Hence

$$\begin{aligned}\|C(f)(x) - C(g)(x)\| &= \frac{1}{v(x)} \left\| \int_0^x w(t)(f(t) - g(t))dt \right\| \\ &\le \frac{1}{v(x)} \int_0^x w(t)\|f(t) - g(t))\|dt \\ &\le \frac{M}{v(x)} \int_0^x v'(t)v(t)dt \le \frac{M}{2} v(x),\ \forall x \in [0,1].\end{aligned}$$

This inequality and the definition of the metric imply

$$d_v(Cf, Cg) \le \frac{1}{2} d_v(f,g).$$

Thus, the averaging operator is a contraction on (H_u, d_v). But $\tilde{u}$ is a fixed point of C_w in H_u, so it is the unique fixed point of C in H_u and the sequence of successive approximation converges to $\tilde{u}$. This implies that $C_w^n(f) \xrightarrow{d_w} \tilde{f(0)}$ as $n \to \infty$, and due to Lemma 5.7 $C_w^n(f) \xrightarrow{\text{unif.}} \tilde{f(0)}$ as $n \to \infty$. Using the density of $C^1([0,1], \mathbb{B})$ in $C([0,1], \mathbb{B})$ and the nonexpansive property of the averaging operator on $C([0,1], \mathbb{B})$ we can conclude that $C_w^n(f) \xrightarrow{\text{unif.}} \tilde{f(0)}$ for all $f \in C([0,1], \mathbb{B})$. □

6. Cesàro operator on $C([0, +\infty[, \mathbb{B})$

By a similar reasoning as in the proof of Theorem 4.1 we obtain the following theorem.

Theorem 6.1. $C^n(f) \xrightarrow{unif.} \tilde{f(0)}$ *as $n \to \infty$, for all $f \in C([0,k], \mathbb{B})$ and $k \in \mathbb{N}^*$.*

On the other hand let, us consider the Cesàro operator $C : C([0,\infty),\mathbb{B}) \to C([0,\infty),\mathbb{B})$. Since C has the Volterra property we can consider the restriction of C to $C([0,k],\mathbb{B})$ for each $k \in \mathbb{N}^*$. From theorem 6.1 we deduce that for each fixed $k \in \mathbb{N}^*$ $C^n(f) \xrightarrow{\text{unif.}} \tilde{f(0)}$ on $[0,k]$. This implies that $C^n(f) \xrightarrow{\text{unif.}} \tilde{f(0)}$ on every compact subset of $[0,\infty)$ so we have the following theorem.

Theorem 6.2. *On every compact subset of* $[0,\infty)$ $C^n(f) \xrightarrow{\textit{unif.}} \tilde{f(0)}$ *as* $n \to \infty$, *for all* $f \in C([0,\infty),\mathbb{B})$.

Remark 6.3. Theorem 6.2 is valid for the general averaging operators defined in (5.1) if $w : [0,\infty) \to [0,\infty)$ is a continuous function satisfying the assumptions of Theorem 5.2 or 5.5.

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