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APPROXIMATION OF COMMON FIXED POINTS AND VARIATIONAL SOLUTIONS FOR ONE-PARAMETER FAMILY OF LIPSCHITZ PSEUDOCONTRACTIONS

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Dedicated to Wataru Takahashi on the occasion of his retirement

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Abstract. Let X be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of X and let $\mathcal{T}=\{T_t : t \in G\}$ be a one-parameter family of Lipschitz pseudocontractions on C such that each $T_t : C \to X$ satisfies the weakly inward condition. For any contraction $f : C \to C$, it is shown that the path $t \mapsto x_t,\ t \in [0, 1)$, in C, denoted by $x_t = \alpha_t T_t x_t + (1 - \alpha_t)f(x_t)$ is continuous and strongly converges to a common fixed point of $\mathcal{T}$, which is the unique solution of some variational inequality. On the other hand, if $\mathcal{T}=\{T_t : t \in G\}$ is a family of uniformly Lipschitz pseudocontractive self-mappings on C, it is also shown that the iteration process:

$$x_0 \in C,\ x_{n+1} = \beta_n(\alpha_n T_{r_n} x_n + (1 - \alpha_n)x_n) + (1 - \beta_n)f(x_n),\ n \ge 0,$$

strongly converges to the common fixed point of $\mathcal{T}$, which is the unique solution of the same variational inequality.

Key Words and Phrases: Viscosity approximation method, fixed point problem, variational inequality, Lipschitz pseudocontraction, strong convergence, smooth and uniformly convex Banach space.

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1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be its dual. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^*\rangle$. The (normalized) duality mapping J from X into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual X^* is defined by

$$J(x) = \{\varphi \in X^* : \langle x, \varphi\rangle = \|x\|^2 = \|\varphi\|^2\}, \quad \text{for all } x \in X.$$

It is known that the norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for each x, y in $U = \{x \in X : \|x\| = 1\}$ the unit sphere of X. It is said to be uniformly Gâteaux differentiable if for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and X is said to be uniformly smooth) if the limit in (1.1) is attained uniformly for $(x, y) \in U \times U$. Since the dual X^* of X is uniformly convex if and only if the norm of X is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The converse implication is false. A discussion of these and related concepts may be found in [5].

Recall also that if X is smooth then J is single-valued and continuous from the norm topology of X to the weak star topology of X^*, i.e., norm-to-weak* continuous. It is also well-known that if X has a uniformly Gâteaux differentiable norm, then J is uniformly continuous on bounded subsets of X form the strong topology of X to the weak star topology of X^*, i.e., uniformly norm-to-weak* continuous on each bounded subset of X. Moreover, if X is uniformly smooth then J is uniformly continuous on bounded subsets of X form the strong topology of X to the strong topology of X^*, i.e., uniformly norm-to-norm continuous on each bounded subset of X. See [5] for more details.

Let T be a mapping with domain $D(T)$ and range $R(T)$ in X. Denote by $\mathrm{Fix}(T)$ the set of all fixed points of T, that is, $\mathrm{Fix}(T) := \{x \in D(T) : Tx = x\}$. Following Morales [13], T is called strongly pseudocontractive if for some constant $k < 1$ and for all $x, y \in D(T)$,

$$(\lambda - k)\|x - y\| \le \|(\lambda I - T)x - (\lambda I - T)y\|$$

for all $\lambda > k$; while T is called a pseudocontraction if the last inequality holds for $k = 1$. The mapping T is called Lipschitz if there exists $L \ge 0$ such that $\|Tx - Ty\| \le L\|x - y\|$, for all $x, y \in D(T)$. The mapping T is called nonexpansive if $L = 1$ and is called a (strict) contraction if $L < 1$. We use Π_C to denote the collection of all contractions on C with a suitable contractive constant $\alpha \in [0, 1)$, that is,

$$\Pi_C := \{f : C \to C, \text{ a contraction with a suitable contractive constant}\}.$$

It is clear that every nonexpansive mapping is a pseudocontraction. The converse is not true in general. A counterexample can be found, e.g., in [22]. It follows from a result of Kato [11] that T is pseudocontractive if and only if there exists

$j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \quad \text{for all } x, y \in D(T).$$

Let D be a nonempty subset of C. A retraction from C to D is a mapping $Q : C \to D$ such that $Qx = x$ for all $x \in D$. A retraction Q from C to D is nonexpansive if Q is nonexpansive (i.e., $\|Qx - Qy\| \le \|x-y\|$ for all $x, y \in C$). A retraction Q from C to D is sunny if Q satisfies the property: $Q(Qx + t(x - Qx)) = Qx$ for each $x \in C$ and $t \ge 0$ whenever $Qx + t(x - Qx) \in C$. A retraction Q from C to D is sunny nonexpansive if Q is both sunny and nonexpansive.

It is well known that in a smooth Banach space X, a retraction Q from C to D is a sunny nonexpansive retraction from C to D if and only if the following inequality holds:

$$\langle x - Qx, J(y - Qx)\rangle \le 0, \quad \text{for all } x \in C, \quad \text{for all } y \in D.$$

If C is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection P_C from H onto C is a sunny nonexpansive retraction. This however is not true for Banach spaces. It is known that if C is a closed convex subset of a uniformly smooth Banach space X and there is a nonexpansive retraction from X to C, then this retraction is sunny. See [17, 18, 21] for more details.

Let G be an unbounded subset of $[0, \infty)$ such that $t + h \in G$ for all $t, h \in G$ and $t - h \in G$ for all $t, h \in G$ with $t > h$ (for instance, $G = [0, \infty)$ or $G = \mathbb{N}$, the set of nonnegative integers). Recall that a one-parameter family $\mathcal{T} = \{T_t : t \in G\}$ of self-mappings of C is said to be a nonexpansive semigroup on C if the following conditions are satisfied:

(H1) $T_0 x = x$, for all $x \in C$;

(H2) $T_{t+s}x = T_t T_s x$, for all $t, s \in G$, $x \in C$;

(H3) for each $x \in C$, $T_t x$ is continuous in $t \in G$ when G has the relative topology of $[0, \infty)$;

(H4) for each $t \in G$, there holds $\|T_t x - T_t y\| \le \|x - y\|$, for all $x, y \in C$.

Denote by F the set of common fixed points of $\mathcal{T}$, i.e., $F = \{x \in C : T_s x = x, \ \text{for all } s \in G\}$.

Very recently, Yao and Noor [26] considered the viscosity approximation method for finding a common fixed point of a nonexpansive semigroup on a nonempty closed convex subset C of a reflexive Banach space X. They proved that the approximate solutions converge strongly to a common fixed point $Q(f)$ of the nonexpansive semigroup, which is just a solution of some variational inequality under some mild conditions.

In [26], Yao and Noor also studied the existence of $Q(f) \in F$ with $f \in \Pi_C$, which solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \le 0, \quad \text{for all } p \in F.$$

Let $f \in \Pi_C$ and $\{\alpha_s\}_{s \in G}$ be a net in the interval $(0, 1)$ such that $\lim_{s\to\infty} \alpha_s = 0$. By Banach's contraction principle, for each $s \in G$ we have a unique point $z_s \in C$ satisfying the equation

$$z_s = \alpha_s f(z_s) + (1 - \alpha_s) T_s z_s. \tag{1.2}$$

Theorem 1.1. ([26], Theorem 2) *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of X has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of X. Assume that $F \neq \emptyset$ and that $\mathcal{T}$ is uniformly asymptotically regular on bounded subsets of C, that is, for each bounded subset $\widetilde{C}$ of C and each $r \in G$, there holds*

$$\lim_{s \in G, s \to \infty} \sup_{x \in \widetilde{C}} \|T_r T_s x - T_s x\| = 0, \quad \text{uniformly in } r \in G. \tag{UARC}$$

Then the net $\{z_s\}$ defined by (1.2) converges strongly to a point in F. If we define $Q : \Pi_C \to F$ by

$$Q(f) = \lim_{s \to \infty} z_s, \quad f \in \Pi_C, \tag{1.3}$$

then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad f \in \Pi_C, \ p \in F.$$

In particular, if $f = u \in C$ is a constant, then the limit (1.3) defines the sunny nonexpansive retraction Q from C to F with

$$\langle Q(u) - u, J(Q(u) - p)\rangle \leq 0, \quad u \in C, \ p \in F.$$

Theorem 1.2. ([26], Theorem 3) *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of X has the fixed point property for nonexpansive mappings and X has a weakly sequentially continuous duality mapping. Let C be a nonempty closed convex subset of X. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0, 1)$ and $\{r_n\}$ be a sequence in G. Let $\{\alpha_n\}$ satisfy the control conditions (C1), (C2). Assume:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $r_n \to \infty$ $(n \to \infty)$;

(iv) $\mathcal{T}$ is a semigroup such that $F \neq \emptyset$ and satisfies the uniformly asymptotically regular condition

$$\lim_{r \in G, r \to \infty} \sup_{x \in \widetilde{C}} \|T_s T_r x - T_r x\| = 0, \quad \text{uniformly in } s \in G, \tag{UARC}$$

where $\widetilde{C}$ is any bounded subset of C. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{r_n} x_n, \end{cases} \tag{1.4}$$

converges strongly to $Q(f) \in F$, where $Q(f)$ is a solution of the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad f \in \Pi_C, \ p \in F.$$

It is worth pointing out that there actually are the same requirements in the proof of main results of Yao and Noor [26] (see the proof of Theorems 1-3 in [26]), that is, $\mathcal{T} = \{T_t : t \in G\}$ is a nonexpansive semigroup on C. We remark that the commutativity for the family $\mathcal{T}$ of nonexpansive mappings has played an important role in the proof of those main results of [26]. Their iterative algorithm is an important extension of the viscosity approximation method studied by many authors in the recent literature; see [1, 15, 24, 8, 2, 3, 4]. In the meantime, their results can be viewed as significant improvement and refinement of the corresponding results of Halpern [7], Reich [18], Moudafi [15], Xu [24] and some others.

On the other hand, Udomene [22] very recently investigated the path convergence, approximation of fixed points and variational solutions of Lipschitz pseudocontractions in Banach spaces.

Theorem 1.3 ([22], Theorem 6) *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of X, let $T : C \to X$ be a continuous pseudocontraction satisfying the weakly inward condition and let $f \in \Pi_C$. Suppose that every nonempty closed convex bounded subset of C has the fixed point property for nonexpansive self-mappings. If there exists $u_0 \in C$ such that the set*

$$B = \{x \in C : Tx = u_0 + \lambda(x - u_0) \text{ for some } \lambda > 1\} \tag{1.5}$$

is bounded, then the path $\{x_t\}, t \in [0, 1)$, described by

$$x_t = tTx_t + (1 - t)f(x_t)$$

converges strongly to a fixed point $x^* \in \mathrm{Fix}(T)$, which is the unique solution of the variational inequality

$$\langle (I - f)x^*, J(x^* - p)\rangle \leq 0, \quad p \in \mathrm{Fix}(T).$$

Theorem 1.4 ([22], Theorem 10) *Let C be a nonempty closed convex subset of a real reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let $T : C \to C$ be a Lipschitz pseudocontraction and let $f \in \Pi_C$. Suppose that every nonempty closed convex bounded subset of C has the fixed point property for nonexpansive self-mappings. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$x_{n+1} = \beta_n(\alpha_n Tx_n + (1 - \alpha_n)x_n) + (1 - \beta_n)f(x_n), \quad n \geq 0, \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$ satisfying the conditions:

(i) $\{\alpha_n\}$ is decreasing and $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \beta_n = 1$ and $\sum_{n=0}^{\infty}(1 - \beta_n) = \infty$;

(iii) (a) $\lim_{n\to\infty} \frac{1 - \beta_n}{\alpha_n} = 0$, (b) $\lim_{n\to\infty} \frac{\alpha_n^2}{1 - \beta_n} = 0$,
(c) $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{(1 - \beta_n)^2} = 0$, (d) $\lim_{n\to\infty} \frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}(1 - \beta_n)} = 0$.

If there exists some $u_0 \in C$ such that the set B described by (1.5) is bounded, then $\{x_n\}$ converges strongly to a fixed point $x^* \in \mathrm{Fix}(T)$, which is the unique solution of the variational inequality

$$\langle (I - f)x^*, J(x^* - p)\rangle \leq 0, \quad p \in \mathrm{Fix}(T).$$

We remark that Theorem 1.3 generalizes the recent results of Takahashi and Kim [20], Xu and Yin [25], Jung and Kim [9] to a more general class of mappings and to a more general class of Banach spaces. Theorem 1.4 also improves upon Schu's theorem [19] to some Banach spaces which include, for example, the L_p spaces with $1 < p < \infty$.

Let C be a nonempty closed convex subset of a smooth Banach space X and $f \in \Pi_C$. The purpose of this paper is to consider and analyze the modified version of Udomene's iterative scheme (1.6) for a family $\mathcal{T} = \{T_t : t \in G\}$ of Lipschitz pseudocontractive self-mappings on C, that is,

Algorithm 1.1. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0, 1)$ and let $\{r_n\}$ be a sequence in G with $r_n \to \infty$. For an arbitrarily initial $x_0 \in C$ define a sequence $\{x_n\}$ recursively by the following explicit iterative scheme:

$$x_{n+1} = \beta_n(\alpha_n T_{r_n} x_n + (1 - \alpha_n)x_n) + (1 - \beta_n)f(x_n), \quad n \geq 0. \tag{1.7}$$

If $G = \mathbb{N}$, and for all $n \in \mathbb{N}$, $T_n \equiv T$ a Lipschitz pseudocontractive self-mapping on C, then the iterative scheme (1.7) reduces to Udomene's iterative scheme (1.6). Further, whenever $f(x) = w \in C$ a constant, (1.6) reduces to Schu's iterative scheme [19].

In this paper, without the assumptions that every nonempty closed convex bounded subset of C has the fixed point property for nonexpansive self-mappings, that the family $\mathcal{T} = \{T_t : t \in G\}$ of Lipschitz pseudocontractions is a semigroup and that X admits a weakly sequentially continuous duality mapping, we first prove that x_t defined by $x_t = \alpha_t T_t x_t + (1-\alpha_t)f(x_t)$ strongly converges, as $t \to \infty$, to a common fixed point of $\mathcal{T}$ in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, Then we establish the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 1.1 under some control conditions in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Moreover, we deduce that these strong limits are the unique solution of the same variational inequality.

Our results are improvements, generalization and development of the previously known results in the literature including Schu [19], Takahashi and Kim [20], Xu and Yin [25], Jung and Kim [9], Moudafi [15], Xu [24], Yao and Noor [26], Jung [8], Ceng and Xu [2], Ceng, Xu and Yao [3] and Udomene [22].

2. Preliminaries

Let X be a real Banach space with the dual X^*. As usual, in a Banach space $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.

Let X be a Banach space and C be a nonempty subset of X. Then, for any $x \in C$, the set $I_C(x) = \{x + \lambda(z - x) : z \in C,\ \lambda \geq 1\}$ is called the inward set of x. A mapping $T : C \to X$ is said to satisfy the inward condition if $Tx \in I_C(x)$ for each $x \in C$, and is said to satisfy the weakly inward condition if $Tx \in cl[I_C(x)]$, the closure of $I_C(x)$, for each $x \in C$.

Before starting the main results of this paper, we include some lemmas which will be needed in the sequel. Lemma 2.1 is well known (see, e.g., [14]). The proof of Lemma 2.2 can be derived from Lemma 2.5 of [23].

Lemma 2.1. *Let X be a real Banach space. Then, for all $x, y \in X$ $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$, for all $j(x+y) \in J(x+y)$.*

Lemma 2.2. *Let $\{a_n\}_n$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\delta_n)a_n + \delta_n \sigma_n, \quad n \geq 0,$$

where $\{\delta_n\}_n \subset [0,1]$, $\{\sigma_n\}_n \subset [0,1]$ and $\sum_{n=0}^{\infty} \delta_n = \infty$, $\lim_{n\to\infty} \sigma_n = 0$.
Then, $\lim_{n\to\infty} a_n = 0$.

Recall that μ is said to be a mean on the set $\mathbb{N}$ of all positive integers if μ is a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. It is known that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(a) \leq \sup\{a_n : n \in \mathbb{N}\}, \text{ for every } a = (a_1, a_2, ...) \in l^\infty.$$

According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1}), \text{ for every } a = (a_1, a_2, ...) \in l^\infty.$$

Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. It is also known that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n, \text{ for every } a = (a_1, a_2, ...) \in l^\infty.$$

The following result is actually a variant of Lemma 1.2 in Reich [17].

Lemma 2.3. *Let C be a nonempty closed convex subset of a Banach space X with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in X. Let μ be a Banach limit and $p \in C$. Then*

$$\mu_n\|x_n - p\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2 \ \Leftrightarrow\ \mu_n\langle x - p, J(x_n - p)\rangle \leq 0, \text{ for all } x \in C.$$

3. Convergence of paths

We begin with some auxiliary results.

Proposition 3.1. *Let $\{\alpha_t\}_{t\in G}$ be a net in $[0,1)$. Let C be a nonempty closed convex subset of a Banach space X and let $\mathcal{T} = \{T_t : t \in G\}$ be a family of continuous pseudocontractions such that each $T_t : C \to X$ satisfies the weakly inward condition. Then for each contraction $f \in \Pi_C$, there exists a unique path $t \mapsto x_t \in C$, $t \in G$, satisfying*

$$x_t = \alpha_t T_t x_t + (1-\alpha_t) f(x_t). \tag{3.1}$$

Suppose additionally that the maps $t \mapsto \alpha_t \in [0,1)$ and $t \mapsto T_t x$ are continuous in $t \in G$ for each $x \in C$, respectively, when G has the relative topology of $[0,\infty)$, then the path $t \mapsto x_t \in C$, $t \in G$ is continuous.

Proof. Let $f \in \Pi_C$ with contractive constant $\alpha \in [0,1)$. Then, for each $t \in G$, the mapping $T_t^f : C \to X$ defined by $T_t^f(x) = \alpha_t T_t x + (1-\alpha_t) f(x)$ is a continuous strong pseudocontraction with constant $\alpha_t + (1-\alpha_t)\alpha \in [0,1)$. Since C is convex, $I_C(x)$ is convex for each $x \in C$. Indeed, let $x \in C$ and let $x+\lambda_1(z_1-x), x+\lambda_2(z_2-x) \in I_C(x)$. Then, for any $\beta \in [0,1]$,

$$\begin{aligned}
&(1-\beta)[x+\lambda_1(z_1-x)]+\beta[x+\lambda_2(z_2-x)]\\
&= x+[(1-\beta)\lambda_1 z_1+\beta\lambda_2 z_2-((1-\beta)\lambda_1+\beta\lambda_2)x]\\
&= x+((1-\beta)\lambda_1+\beta\lambda_2)[\tfrac{(1-\beta)\lambda_1}{(1-\beta)\lambda_1+\beta\lambda_2}z_1+\tfrac{\beta\lambda_2}{(1-\beta)\lambda_1+\beta\lambda_2}z_2-x] \in I_C(x)
\end{aligned}$$

since $(1-\beta)\lambda_1+\beta\lambda_2 \geq 1$ and $\frac{(1-\beta)\lambda_1}{(1-\beta)\lambda_1+\beta\lambda_2}z_1+\frac{\beta\lambda_2}{(1-\beta)\lambda_1+\beta\lambda_2}z_2 \in C$. Now, if $z=f(x)$ and $\lambda=1$ then $f(x)=x+1(f(x)-x) \in I_C(x)$. Thus, since $T_t x \in cl(I_C(x))$, we have that $T_t^f(x)=\alpha_t T_t x+(1-\alpha_t)f(x) \in cl(I_C(x))$. Therefore, T_t^f satisfies the weakly inward condition.

By Corollary 1 of [6], T_t^f has a unique fixed point $x_t \in C$, i.e.,

$$x_t=\alpha_t T_t x_t+(1-\alpha_t)f(x_t).$$

To prove the continuity of the path, we follow the same line of argument as in [14]. Let $t_0 \in G$. Then $\alpha_{t_0} \in [0,1)$ and for all $j(x_t-x_{t_0}) \in J(x_t-x_{t_0})$,

$$\begin{aligned}
\|x_t-x_{t_0}\|^2 &= \langle \alpha_t T_t x_t+(1-\alpha_t)f(x_t)-\alpha_{t_0}T_{t_0}x_{t_0}-(1-\alpha_{t_0})f(x_{t_0}), j(x_t-x_{t_0})\rangle\\
&= \alpha_t\langle T_t x_t-T_{t_0}x_{t_0}, j(x_t-x_{t_0})\rangle+(\alpha_t-\alpha_{t_0})\langle T_{t_0}x_{t_0}, j(x_t-x_{t_0})\rangle\\
&\quad +(1-\alpha_t)\langle f(x_t)-f(x_{t_0}), j(x_t-x_{t_0})\rangle-(\alpha_t-\alpha_{t_0})\langle f(x_{t_0}), j(x_t-x_{t_0})\rangle\\
&= \alpha_t\langle T_t x_t-T_t x_{t_0}, j(x_t-x_{t_0})\rangle+\alpha_t\langle T_t x_{t_0}-T_{t_0}x_{t_0}, j(x_t-x_{t_0})\rangle\\
&\quad +(\alpha_t-\alpha_{t_0})\langle T_{t_0}x_{t_0}-f(x_{t_0}), j(x_t-x_{t_0})\rangle\\
&\quad +(1-\alpha_t)\langle f(x_t)-f(x_{t_0}), j(x_t-x_{t_0})\rangle\\
&\leq \alpha_t\|x_t-x_{t_0}\|^2+\alpha_t\|T_t x_{t_0}-T_{t_0}x_{t_0}\|\|x_t-x_{t_0}\|\\
&\quad +|\alpha_t-\alpha_{t_0}|\|T_{t_0}x_{t_0}-f(x_{t_0})\|\|x_t-x_{t_0}\|+(1-\alpha_t)\alpha\|x_t-x_{t_0}\|^2\\
&= (\alpha_t+(1-\alpha_t)\alpha)\|x_t-x_{t_0}\|^2+|\alpha_t-\alpha_{t_0}|\|T_{t_0}x_{t_0}-f(x_{t_0})\|\|x_t-x_{t_0}\|\\
&\quad +\alpha_t\|T_t x_{t_0}-T_{t_0}x_{t_0}\|\|x_t-x_{t_0}\|,
\end{aligned}$$

so that

$$\|x_t-x_{t_0}\| \leq \frac{|\alpha_t-\alpha_{t_0}|}{(1-\alpha_t)(1-\alpha)}\|T_{t_0}x_{t_0}-f(x_{t_0})\|+\frac{\alpha_t}{(1-\alpha_t)(1-\alpha)}\|T_t x_{t_0}-T_{t_0}x_{t_0}\|.$$

This completes the proof. □

Proposition 3.2. *Let $\{\alpha_t\}_{t\in G}$ be a net in $[0,1)$ and let C be a nonempty closed convex subset of a Banach space X. Let $\mathcal{T}=\{T_t : t \in G\}$ be a family of pseudocontractions such that for each contraction $f \in \Pi_C$, the equation*

$$x=\alpha_t T_t x+(1-\alpha_t)f(x)$$

has a solution $x_t \in C$ for every $t \in G$. Then the following hold:

(i) If for some $u \in C$, the path $y_t=\alpha_t T_t y_t+(1-\alpha_t)u$ is bounded, then for any contraction $f \in \Pi_C$, the path $\{x_t\}$ described by (3.1) is bounded.

(ii) If $\mathcal{T}$ has a common fixed point in C, then the path $\{x_t\}$ is bounded.

(iii) If $x^* \in F=\{x \in C : T_t x=x, \ \text{for all } t \in G\}$ then for all $j(x_t-x^*) \in J(x_t-x^*)$, we have

$$\langle x_t-f(x_t), j(x_t-x^*)\rangle \leq 0.$$

(iv) If $0 \le \alpha_s \le \alpha_t < 1$ *then*

$$\|x_t - T_t x_t\| \le \frac{1+\alpha}{1-\alpha}[\|x_s - T_s x_s\| + \|T_t x_s - T_s x_s\|].$$

Proof. (i) Let the path $\{y_t\}_{t\in G}$ given by $y_t = \alpha_t T_t y_t + (1-\alpha_t)u$, for some $u \in C$, be bounded. Then the set $\{f(y_t)\}_{t\in G}$ is bounded. Let $j(x_t - y_t) \in J(x_t - y_t)$. From the estimates

$$\begin{aligned}\|x_t - y_t\|^2 &= \alpha_t\langle T_t x_t - T_t y_t, j(x_t - y_t)\rangle + (1-\alpha_t)\langle f(x_t) - u, j(x_t - y_t)\rangle \\ &\le \alpha_t\|x_t - y_t\|^2 + (1-\alpha_t)\|f(x_t) - u\|\|x_t - y_t\|,\end{aligned}$$

we have that $\|x_t - y_t\| \le \|f(x_t) - u\| \le \alpha\|x_t - y_t\| + \|f(y_t) - u\|$. Thus, $\|x_t - y_t\| \le \frac{1}{1-\alpha}\|f(y_t) - u\|$ and, hence, $\{x_t\}$ is bounded.

(ii) Let $x^* \in F(\mathcal{T})$, and let $j(x_t - x^*) \in J(x_t - x^*)$. Then

$$\begin{aligned}\|x_t - x^*\|^2 &= \alpha_t\langle T_t x_t - x^*, j(x_t - x^*)\rangle + (1-\alpha_t)\langle f(x_t) - x^*, j(x_t - x^*)\rangle \\ &\le \alpha_t\|x_t - x^*\|^2 + (1-\alpha_t)\|f(x_t) - x^*\|\|x_t - x^*\|,\end{aligned}$$

so that $\|x_t - x^*\| \le \|f(x_t) - x^*\| \le \alpha\|x_t - x^*\| + \|f(x^*) - x^*\|$. Thus, we get that $\|x_t - x^*\| \le \frac{1}{1-\alpha}\|f(x^*) - x^*\|$, proving that $\{x_t\}$ is bounded.

(iii) Let $x^* \in F(\mathcal{T})$, and let $j(x_t - x^*) \in J(x_t - x^*)$. Then

$$\begin{aligned}\langle x_t - f(x_t), j(x_t - x^*)\rangle &= \alpha_t\langle T_t x_t - f(x_t), j(x_t - x^*)\rangle \\ &= \alpha_t\langle T_t x_t - x^*, j(x_t - x^*)\rangle + \alpha_t\langle x^* - f(x_t), j(x_t - x^*)\rangle \\ &\le \alpha_t\|x_t - x^*\|^2 + \alpha_t\langle x^* - x_t, j(x_t - x^*)\rangle + \alpha_t\langle x_t - f(x_t), j(x_t - x^*)\rangle \\ &\le \alpha_t\langle x_t - f(x_t), j(x_t - x^*)\rangle.\end{aligned}$$

Thus, $\langle x_t - f(x_t), j(x_t - x^*)\rangle \le 0$.

(iv) Let $0 \le \alpha_s \le \alpha_t < 1$. Then, similarly to the proof of Proposition 3.1, we can derive

$$\|x_t - x_s\| \le \frac{\alpha_t - \alpha_s}{(1-\alpha_t)(1-\alpha)}\|T_s x_s - f(x_s)\| + \frac{\alpha_t}{(1-\alpha_t)(1-\alpha)}\|T_t x_s - T_s x_s\|.$$

Note that

$$\|x_s - f(x_s)\| = \frac{\alpha_s}{1-\alpha_s}\|x_s - T_s x_s\|.$$

Hence

$$\begin{aligned}\|x_t - T_t x_t\| &= \frac{1-\alpha_t}{\alpha_t}\|x_t - f(x_t)\| \\ &\le \frac{1-\alpha_t}{\alpha_t}[\|x_t - x_s\| + \|x_s - f(x_s)\| + \|f(x_s) - f(x_t)\|] \\ &\le \frac{1-\alpha_t}{\alpha_t}[(1+\alpha)\|x_t - x_s\| + \frac{\alpha_s}{1-\alpha_s}\|x_s - T_s x_s\|] \\ &\le \frac{1-\alpha_t}{\alpha_t}\{(1+\alpha)[\frac{\alpha_t - \alpha_s}{(1-\alpha_t)(1-\alpha)}\|T_s x_s - f(x_s)\| + \frac{\alpha_t}{(1-\alpha_t)(1-\alpha)}\|T_t x_s - T_s x_s\|] \\ &\quad + \frac{\alpha_s}{1-\alpha_s}\|x_s - T_s x_s\|\} \\ &\le \frac{1-\alpha_t}{\alpha_t}\{(1+\alpha)[\frac{\alpha_t - \alpha_s}{(1-\alpha_t)(1-\alpha)}(\|T_s x_s - x_s\| + \|x_s - f(x_s)\|)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha_t}{(1-\alpha_t)(1-\alpha)}\|T_tx_s-T_sx_s\|]+\frac{\alpha_s}{1-\alpha_s}\|x_s-T_sx_s\|\}\\
\le\ &\frac{1-\alpha_t}{\alpha_t}\{(1+\alpha)[\frac{\alpha_t-\alpha_s}{(1-\alpha_t)(1-\alpha)}(\|T_sx_s-x_s\|+\frac{\alpha_s}{1-\alpha_s}\|x_s-T_sx_s\|)\\
&+\frac{\alpha_t}{(1-\alpha_t)(1-\alpha)}\|T_tx_s-T_sx_s\|]+\frac{\alpha_s}{1-\alpha_s}\|x_s-T_sx_s\|\}\\
=\ &\frac{1-\alpha_t}{\alpha_t}\{\frac{(1+\alpha)(\alpha_t-\alpha_s)}{(1-\alpha)(1-\alpha_t)(1-\alpha_s)}\|x_s-T_sx_s\|+\frac{\alpha_s}{1-\alpha_s}\|x_s-T_sx_s\|\\
&+\frac{(1+\alpha)\alpha_t}{(1-\alpha)(1-\alpha_t)}\|T_tx_s-T_sx_s\|\}\\
=\ &\frac{1-\alpha_t}{\alpha_t}\{(\frac{(1+\alpha)(\alpha_t-\alpha_s)}{(1-\alpha)(1-\alpha_t)(1-\alpha_s)}\\
&+\frac{\alpha_s}{1-\alpha_s})\|x_s-T_sx_s\|+\frac{(1+\alpha)\alpha_t}{(1-\alpha)(1-\alpha_t)}\|T_tx_s-T_sx_s\|\}\\
\le\ &\frac{(1+\alpha)(1-\alpha_t)}{(1-\alpha)\alpha_t}\{(\frac{\alpha_t-\alpha_s}{(1-\alpha_t)(1-\alpha_s)}+\frac{\alpha_s}{1-\alpha_s})\|x_s-T_sx_s\|+\frac{\alpha_t}{1-\alpha_t}\|T_tx_s-T_sx_s\|\}\\
=\ &\frac{(1+\alpha)}{(1-\alpha)}[\|x_s-T_sx_s\|+\|T_tx_s-T_sx_s\|].
\end{aligned}
$$

This completes the proof. □

Theorem 3.1. *Let X be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of X and let $\mathcal{T}=\{T_t : t\in G\}$ be a family of Lipschitz pseudocontractions such that each $T_t : C\to X$ satisfies the weakly inward condition. Suppose that for each contraction $f\in\Pi_C$, $\{x_t\}_{t\in G}$ is the path generated by (3.1) where $\alpha_t\uparrow 1$ as $t\to\infty$, and that $\mathcal{T}$ satisfies the uniformly left asymptotically regular condition on bounded subsets of C, i.e., for each bounded subset $\widetilde{C}$ of C, there holds*

$$\lim_{s\in G, s\to\infty}\sup_{x\in\widetilde{C}}\|T_r\Gamma T_sx-T_sx\|=0,\quad r\in G,\qquad\text{(ULARC)}$$

where Γ is a nonexpansive retraction of X onto C. If there exist $t_0\in G$ and $u_0\in C$ such that the sets $\{T_tx_{t_0} : t\in G \text{ with } t\ge t_0\}$ and

$$B=\{x\in C: T_tx=u_0+\lambda(x-u_0)\text{ for some }t\in G\text{ and some }\lambda>1\}$$

are bounded, then the path $\{x_t\}_{t\in G}$ converges strongly as $t\to\infty$ to a common fixed point of $\mathcal{T}$. If we define $Q:\Pi_C\to F$ by

$$Q(f)=\lim_{t\to\infty}x_t,\quad f\in\Pi_C,\tag{3.2}$$

then $Q(f)$ is the unique solution of the variational inequality

$$\langle(I-f)Q(f),J(Q(f)-p)\rangle\le 0,\quad f\in\Pi_C,\ p\in F.$$

In particular, if $f=u\in C$ is a constant, then the limit (3.2) defines the sunny nonexpansive retraction Q from C to F,

$$\langle Q(u)-u,J(Q(u)-p)\rangle\le 0,\quad u\in C,\ p\in F.$$

Proof. It follows from Proposition 3.1 that for each contraction $f \in \Pi_C$, there exists a unique path $t \mapsto x_t \in C,\ t \in G$ satisfying (3.1). Let there exist $u_0 \in C$ such that the set $\{y_t : y_t = \alpha_t T_t y_t + (1-\alpha_t)u_0,\ t \in G\}$ is bounded. Then by Proposition 3.2 (i), the path $\{x_t\}_{t\in G}$ described by (3.1) is bounded. Hence it is easy to see that the net $\{f(x_t)\}_{t\in G}$ is bounded. Note that $\{T_t x_{t_0} : t \in G\}$ is bounded for some $t_0 \in G$ and that $\alpha_t \uparrow 1$ as $t \to \infty$. Now by Proposition 3.2 (iv) we know that for all $t \in G$ with $t \geq t_0$

$$\|x_t - T_t x_t\| \leq \frac{1+\alpha}{1-\alpha}[\|x_{t_0} - T_{t_0}x_{t_0}\| + \|T_t x_{t_0} - T_{t_0}x_{t_0}\|].$$

Thus the set $\{T_t x_t : t \in G \text{ with } t \geq t_0\}$ is bounded. Let $\sup_{t\in G}\|x_t\| \leq M$. Then $\|x_t - x_s\| \leq 2M$ for any $t, s \in G$. Let $\{t_n\}$ be a sequence in G such that $t_n \geq t_0$ and $t_n \uparrow \infty$. Define a function $\psi : C \to [0,\infty)$ by

$$\psi(x) := \mu_n\|x_{t_n} - x\|^2, \quad x \in C,$$

where μ is a Banach limit. Since X is reflexive, ψ is convex, continuous and $\psi(x) \to \infty$ as $\|x\| \to \infty$, we have that the set

$$K := \{y \in C : \psi(y) = \inf_{x\in C}\psi(x)\}$$

is nonempty, closed and convex. Let us show that K is bounded. Let $y \in K$. Then $\psi(y) \leq \mu_n\|x_{t_n} - x_{t_0}\|^2 \leq 4M^2$. Applying the convexity of the functional $\frac{1}{2}\|\cdot\|^2 : C \to [0,\infty)$, we deduce that

$$\begin{aligned}\|y\|^2 &\leq \mu_n\|x_{t_n} - y\|^2 + 2\mu_n\|x_n\|^2 \\ &\leq 2\psi(y) + 2M^2 \leq 10M^2,\end{aligned}$$

i.e., $\|y\| \leq \sqrt{10}M$, for all $y \in K$. Thus, K is bounded. For each $r \in G$, the mapping $J_r = (2I - T_r)^{-1}$ is a nonexpansive self-mapping of C (see [12] Theorem 6). We claim that K is invariant under J_r. Indeed, let $y \in K$ and L_r denote a Lipschitz constant of T_r. Note that both $\{x_t\}_{t\in G}$ and $\{f(x_t)\}_{t\in G}$ are bounded and so is $\{T_t x_t : t \in G \text{ with } t \geq t_0\}$. Then, from $\alpha_t \uparrow 1$ $(t \to \infty)$ it follows that

$$\lim_{t\to\infty}\|x_t - T_t x_t\| = (1-\alpha_t)\|f(x_t) - T_t x_t\| = 0. \tag{3.3}$$

Utilizing (ULARC) we conclude that

$$\begin{gathered}\psi(J_r(y)) = \mu_n\|x_{t_n} - J_r(y)\|^2 \\ \leq \mu_n(\|x_{t_n} - J_r(x_{t_n})\| + \|J_r(x_{t_n}) - J_r(y)\|)^2 \\ \leq \mu_n(\|x_{t_n} - J_r(x_{t_n})\| + \|x_{t_n} - y\|)^2 \\ \leq \mu_n(\|x_{t_n} - T_r x_{t_n}\| + \|x_{t_n} - y\|)^2 \\ \leq \mu_n\{\|x_{t_n} - T_{t_n}x_{t_n}\| + \|T_{t_n}x_{t_n} - T_r\Gamma T_{t_n}x_{t_n}\| + \|T_r\Gamma T_{t_n}x_{t_n} - T_r x_{t_n}\| + \|x_{t_n} - y\|\}^2 \\ \leq \mu_n\{(1+L_r)\|x_{t_n} - T_{t_n}x_{t_n}\| + \|T_{t_n}x_{t_n} - T_r\Gamma T_{t_n}x_{t_n}\| + \|x_{t_n} - y\|\}^2 \\ = \mu_n\|x_{t_n} - y\|^2 = \psi(y).\end{gathered}$$

This implies that K is invariant under J_r for each $r \in G$.

First, let us show that K consists of one point. Indeed, let $w, z \in K$ with $w \neq z$. Then, by [16] Theorem 1, there exists a positive number $k > 0$ such that

$$\langle x_{t_n} - z - (x_{t_n} - w), J(x_{t_n} - z) - J(x_{t_n} - w)\rangle \geq k > 0$$

for every n. Thus we get

$$\mu_n\langle w - z, J(x_{t_n} - z) - J(x_{t_n} - w)\rangle \geq k > 0.$$

Furthermore, since $z, w \in K$, from Lemma 2.3 we have

$$\mu_n\langle w - z, J(x_{t_n} - z)\rangle \leq 0 \quad \text{and} \quad \mu_n\langle z - w, J(x_{t_n} - w)\rangle \leq 0.$$

Hence we have

$$\mu_n\langle w - z, J(x_{t_n} - z) - J(x_{t_n} - w)\rangle \leq 0.$$

This leads to a contradiction. Therefore $z = w$ and so K consists of one point, i.e., $K = \{z\}$. Since K is invariant under J_r for each $r \in G$, z is a common fixed point of $\mathcal{T}$ in C.

Second, let us show that the path $\{x_t\}_{t\in G}$ converges strongly as $t \to \infty$ to $z \in F \cap K$. Indeed, let $\tau \in (0, 1)$. Then $\psi(z) \leq \psi((1 - \tau)z + \tau x),\ x \in C$, and utilizing Lemma 2.1, we have that

$$0 \leq \frac{\psi((1 - \tau)z + \tau x) - \psi(z)}{\tau} \leq -2\mu_n\langle x - z, J(x_{t_n} - z - \tau(x - z))\rangle.$$

Thus

$$\mu_n\langle x - z, J(x_{t_n} - z - \tau(x - z))\rangle \leq 0.$$

Since, in this setting, J is norm-to-weak* uniformly continuous on bounded subsets of X, letting $\tau \to 0$, we have that

$$\mu_n\langle x - z, J(x_{t_n} - z)\rangle \leq 0, \quad x \in C.$$

In particular,

$$\mu_n\langle f(z) - z, J(x_{t_n} - z)\rangle \leq 0. \tag{3.4}$$

Observe that

$$(1 - \alpha)\|x_{t_n} - z\|^2 \leq \langle x_{t_n} - f(x_{t_n}), J(x_{t_n} - z)\rangle + \langle f(z) - z, J(x_{t_n} - z)\rangle.$$

Utilizing Proposition 3.2 (iii) and (3.4) we know that $\mu_n\|x_{t_n} - z\|^2 = 0$. Therefore, there exists a subsequence $\{x_{t_{n_k}}\}$ of $\{x_{t_n}\}$ such that $x_{t_{n_k}} \to z$ as $k \to \infty$. Suppose that there is another subsequence $\{x_{t_{m_i}}\}$ of $\{x_{t_n}\}$ which converges strongly to (say) $y \in C$. Then y must be a common fixed point of $\mathcal{T}$. In fact, observe that

$$\begin{aligned} \|y - T_r y\| &\leq \|y - x_{t_{m_i}}\| + \|x_{t_{m_i}} - T_{t_{m_i}} x_{t_{m_i}}\| \\ &+ \|T_{t_{m_i}} x_{t_{m_i}} - T_r\Gamma T_{t_{m_i}} x_{t_{m_i}}\| + \|T_r\Gamma T_{t_{m_i}} x_{t_{m_i}} - T_r x_{t_{m_i}}\| + \|T_r x_{t_{m_i}} - T_r y\| \\ &\leq (1 + L_r)\|y - x_{t_{m_i}}\| + (1 + L_r)\|x_{t_{m_i}} - T_{t_{m_i}} x_{t_{m_i}}\| + \|T_{t_{m_i}} x_{t_{m_i}} - T_r\Gamma T_{t_{m_i}} x_{t_{m_i}}\|. \end{aligned}$$

Thus, (ULARC) together with (3.3) implies that $y = T_r y$ for all $r \in G$. That is, $y \in F$. Now putting $x^* = y$, we deduce from $x_{t_{n_k}} \to z$ and Proposition 3.2 (iii) that

$$\langle z - f(z), J(z - y)\rangle \leq 0. \tag{3.5}$$

Also, putting $x^* = z$, we deduce from $x_{t_{m_i}} \to y$ and Proposition 3.2 (iii) that

$$\langle y - f(y), J(y - z)\rangle \leq 0. \tag{3.6}$$

Adding inequalities (3.5) and (3.6) yields that

$$\begin{aligned}(1-\alpha)\|z-y\|^2 &\le \langle z-y, J(z-y)\rangle - \langle f(z)-f(y), J(z-y)\rangle \\ &= \langle z-f(z), J(z-y)\rangle + \langle y-f(y), J(y-z)\rangle \le 0,\end{aligned}$$

and thus $z=y$. Therefore, z_t converges strongly as $s\to\infty$ to a point in F.

Finally, we claim that if we define $Q: \Pi_C \to F$ by $Q(f) = \lim_{t\to\infty} x_t$, then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f\in \Pi_C,\ p\in F.$$

Since $x_t = \alpha_t T_t x_t + (1-\alpha_t) f(x_t)$, we have $(I-f)x_t = -\frac{\alpha_t}{1-\alpha_t}(I-T_t)x_t$. Hence for each $p\in F$,

$$\langle (I-f)x_t, J(x_t-p)\rangle = -\frac{\alpha_t}{1-\alpha_t}\langle (I-T_t)x_t - (I-T_t)p, J(x_t-p)\rangle \le 0.$$

Letting $t\to\infty$ we get that $\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0$. In particular, if $f=u\in C$ is a constant, then

$$\langle Qu-u, J(Qu-p)\rangle \le 0, \quad u\in C,\ p\in F.$$

Therefore Q is a sunny nonexpansive retraction from C to F. □

Note that in the case when $X=H$ a Hilbert space, the nonempty closed convex subset C is a sunny nonexpansive retract of H, the nearest point projection P of C onto F is a sunny nonexpansive retraction and the duality mapping J is the identity mapping I.We also note that the boundedness assumption of the set B in Theorem 3.1 was used in [22].

Corollary 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $\mathcal{T}=\{T_t : t\in G\}$ be a family of Lipschitz pseudocontractions such that each $T_t: C\to X$ satisfies the weakly inward condition. Suppose that for each contraction $f\in \Pi_C$, $\{x_t\}_{t\in G}$ is the path generated by (3.1) where $\alpha_t \uparrow 1$ as $t\to\infty$, and that $\mathcal{T}$ satisfies (ULARC) on bounded subsets of C, i.e., for each bounded subset $\widetilde{C}$ of C, there holds*

$$\lim_{s\in G, s\to\infty} \sup_{x\in\widetilde{C}} \|T_r P_C T_s x - T_s x\| = 0, \quad r\in G, \tag{ULARC}$$

where P_C is the nearest point projection of H onto C. If there exist $t_0\in G$ and $u_0\in C$ such that the sets $\{T_t x_{t_0} : t\in G \text{ with } t\ge t_0\}$ and

$$B = \{x\in C : T_t x = u_0 + \lambda(x-u_0) \text{ for some } t\in G \text{ and some } \lambda > 1\}$$

are bounded, then the path $\{x_t\}_{t\in G}$ converges strongly as $t\to\infty$ to a common fixed point of $\mathcal{T}$. If we define $P: \Pi_C\to F$ by

$$P(f) = \lim_{t\to\infty} x_t, \quad f\in \Pi_C, \tag{3.7}$$

then $P(f)$ is the unique solution of the variational inequality

$$\langle (I-f)P(f), P(f)-p\rangle \le 0, \quad f\in\Pi_C,\ p\in F.$$

In particular, if $f=u\in C$ is a constant, then the limit (3.7) defines the nearest point projection P from C to F,

$$\langle P(u)-u, P(u)-p)\rangle \le 0, \quad u\in C,\ p\in F.$$

Let D be a subset of a Banach space X. Recall that a mapping $T : D \to X$ is said to be firmly nonexpansive if, for each $x, y \in D$, the convex function $\phi : [0, 1] \to [0, \infty)$ defined by

$$\phi(t) = \|(1-t)x + tTx - ((1-t)y + tTy)\|,$$

is nonincreasing. Since ϕ is convex, it is easy to check that a mapping $T : D \to X$ is firmly nonexpansive if and only if

$$\|Tx - Ty\| \leq \|(1-t)(x-y) + t(Tx - Ty)\|,$$

for each $x, y \in D$ and each $t \in [0, 1]$. It is obvious that every firmly nonexpansive mapping is nonexpansive.

Corollary 3.2. *Let X be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty closed convex subset of X and let $\mathcal{T}= \{T_t : t \in G\}$ be a family of firmly nonexpansive mappings on C such that each $T_t : C \to X$ satisfies the weakly inward condition. Suppose that for each contraction $f \in \Pi_C$, $\{x_t\}_{t\in G}$ is the path generated by (3.1) where $\alpha_t \uparrow 1$ as $t \to \infty$, and that $\mathcal{T}$ satisfies (ULARC) on bounded subsets of C, i.e., for each bounded subset $\widetilde{C}$ of C, there holds*

$$\lim_{s\in G, s\to\infty} \sup_{x\in\widetilde{C}} \|T_r \Gamma T_s x - T_s x\| = 0, \quad r \in G, \tag{ULARC}$$

where Γ is a nonexpansive retraction of X onto C. If there exist $t_0 \in G$ and $u_0 \in C$ such that the sets $\{T_t x_{t_0} : t \in G \text{ with } t \geq t_0\}$ and

$$B = \{x \in C : T_t x = u_0 + \lambda(x - u_0) \text{ for some } t \in G \text{ and some } \lambda > 1\}$$

are bounded, then $\{z_t\}_{t\in G}$ converges strongly to $Q(f) \in F$, which is the unique solution of the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad f \in \Pi_C, \ p \in F.$$

Proof. Since every firmly nonexpansive mapping is nonexpansive, we have that $\langle T_t x - T_t y, J(x-y)\rangle \leq \|x-y\|^2$, for all $x, y \in C$, for all $t \in G$. Thus, it follows that every family of nonexpansive self-mappings on C is a family of (uniformly) Lipschitz pseudocontractive self-mappings on C. Utilizing Theorem 3.1 we obtain the desired conclusion. This completes the proof. □

4. Iterative approximation of common fixed points

In this section, we will establish some convergence results for Algorithm 1.1.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Banach space X. Let $\mathcal{T}= \{T_t : t \in G\}$ be a family of uniformly Lipschitz (i.e., $\|T_t x - T_t y\| \leq L\|x - y\|$, for all $x, y \in C$, for all $t \in G$ for some $L > 0$) pseudocontractive self-mappings on C such that $F \neq \emptyset$ and let $f \in \Pi_C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \beta_n(\alpha_n T_{r_n} x_n + (1-\alpha_n)x_n) + (1-\beta_n)f(x_n), \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $(0,1)$ *and* $\{r_n\}$ *is a sequence in* G*. Assume that:*

(i) $\{\alpha_n\}$ *is decreasing and* $\lim_{n\to\infty} \alpha_n = 0$*;*

(ii) $\lim_{n\to\infty} \beta_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\beta_n) = \infty$*;*

(iii) (a) $\lim_{n\to\infty} \dfrac{1-\beta_n}{\alpha_n} = 0$*, (b)* $\lim_{n\to\infty} \dfrac{\alpha_n^2}{1-\beta_n} = 0$*,*

(c) $\lim_{n\to\infty} \dfrac{|\beta_n - \beta_{n-1}|}{(1-\beta_n)^2} = 0$*, (d)* $\lim_{n\to\infty} \dfrac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}(1-\beta_n)} = 0$*;*

(iv) $r_n \to \infty$ *such that* $\frac{\alpha_n}{(1-\beta_n)^2}(T_{r_n}y_{n-1} - T_{r_{n-1}}y_{n-1}) \to 0$*, for all* $\{y_n\}$ *bounded in* C*;*

(v) $\mathcal{T}$ *satisfies (ULARC) on bounded subsets of* C*, i.e., for each bounded subset* $\widetilde{C}$ *of* C*, there holds*

$$\lim_{s\in G, s\to\infty} \sup_{x\in\widetilde{C}} \|T_rT_sx - T_sx\| = 0, \quad r \in G; \tag{ULARC}$$

(vi) there exist $t_0 \in G$ *and* $u_0 \in C$ *such that the sets* $\{T_tx_{t_0} : t \in G \text{ with } t \geq t_0\}$ *and* $B = \{x \in C : T_tx = u_0 + \lambda(x-u_0) \text{ for some } t \in G \text{ and some } \lambda > 1\}$ *are bounded.*

Then $\|x_n - T_sx_n\| \to 0$ *as* $n \to \infty$*, for each* $s \in G$*.*

Proof. We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, let $f \in \Pi_C$ with contractive constant $\alpha \in [0,1)$ and $L > 0$ denote the uniformly Lipschitz constant of $\mathcal{T}$. Since $1-\beta_n \to 0$, $\alpha_n \to 0$ and $\frac{\alpha_n^2}{1-\beta_n} \to 0$ as $n \to \infty$, there exists $m_0 \geq 1$ large enough such that $1-\beta_n \leq \frac{1-\alpha}{16}$, $\alpha_n \leq \frac{1-\alpha}{16(1+L)}$ and $\frac{\alpha_n^2}{1-\beta_n} \leq \frac{1-\alpha}{8(1+L)^2}$, for all $n \geq m_0$. Take $x^* \in F$ arbitrarily. Choose $\gamma > 0$ sufficiently large such that $\|x_{m_0} - x^*\| \leq \gamma$ and $\|f(x^*) - x^*\| \leq \frac{1-\alpha}{2}\gamma$. We proceed by induction to show that $\|x_n - x^*\| \leq \gamma$, for all $n \geq m_0$. Assume that $\|x_n - x^*\| \leq \gamma$ for some $n > m_0$. Let us show that $\|x_{n+1} - x^*\| \leq \gamma$. Suppose that $\|x_{n+1} - x^*\| > \gamma$. Observe that

$$\|f(x_n) - x^*\| \leq \alpha\|x_n - x^*\| + \|f(x^*) - x^*\| \leq \frac{1+\alpha}{2}\gamma < \gamma.$$

Then, from the iteration process (4.1) and the pseudocontractivity of $\mathcal{T}$ we estimate as follows:

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|x_n - x^* - (1-\beta_n)(x_{n+1} - x^*) + (1-\beta_n)(x_{n+1} - x_n) \\
&\quad -\alpha_n\beta_n(x_n - T_{r_n}x_n) + (1-\beta_n)(f(x_n) - x^*)\|^2 \\
&= \langle x_n - x^*, j(x_{n+1} - x^*)\rangle - (1-\beta_n)\|x_{n+1} - x^*\|^2 \\
&\quad +(1-\beta_n)\langle x_{n+1} - x_n, j(x_{n+1} - x^*)\rangle - \alpha_n\beta_n\langle x_n - T_{r_n}x_n, j(x_{n+1} - x^*)\rangle \\
&\quad +(1-\beta_n)\langle f(x_n) - x^*, j(x_{n+1} - x^*)\rangle \\
&\leq \|x_n - x^*\|\|x_{n+1} - x^*\| - (1-\beta_n)\|x_{n+1} - x^*\|^2 \\
&\quad +(1-\beta_n)\|x_{n+1} - x_n\|\|x_{n+1} - x^*\| - \alpha_n\beta_n\langle x_n - T_{r_n}x_n, j(x_{n+1} - x^*)\rangle \\
&\quad +(1-\beta_n)\|f(x_n) - x^*\|\|x_{n+1} - x^*\| \\
&\leq \|x_n - x^*\|\|x_{n+1} - x^*\| - (1-\beta_n)\|x_{n+1} - x^*\|^2 + (1-\beta_n)\|x_{n+1} - x_n\|\|x_{n+1} - x^*\| \\
&\quad +\alpha_n\beta_n\langle x_{n+1} - T_{r_n}x_{n+1} - (x_n - T_{r_n}x_n), j(x_{n+1} - x^*)\rangle
\end{aligned}$$

$$+(1-\beta_n)\|f(x_n)-x^*\|\|x_{n+1}-x^*\|$$
$$\leq \|x_n-x^*\|\|x_{n+1}-x^*\|-(1-\beta_n)\|x_{n+1}-x^*\|^2+(1-\beta_n)\|x_{n+1}-x_n\|\|x_{n+1}-x^*\|$$
$$+\alpha_n\beta_n(1+L)\|x_{n+1}-x_n\|\|x_{n+1}-x^*\|+(1-\beta_n)\|f(x_n)-x^*\|\|x_{n+1}-x^*\|.$$

Thus,

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq \|x_n-x^*\|-(1-\beta_n)\|x_{n+1}-x^*\| \\ &\quad +[(1-\beta_n)+(1+L)\alpha_n]\|x_{n+1}-x_n\|+(1-\beta_n)\|f(x_n)-x^*\|.\end{aligned} \tag{4.2}$$

Now,

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n\beta_n(T_{r_n}x_n-x_n)+(1-\beta_n)(f(x_n)-x_n)\| \\ &\leq \alpha_n(1+L)\|x_n-x^*\|+(1-\beta_n)\|x_n-x^*\|+(1-\beta_n)\|f(x_n)-x^*\| \\ &= [\alpha_n(1+L)+(1-\beta_n)]\|x_n-x^*\|+(1-\beta_n)\|f(x_n)-x^*\|.\end{aligned}$$

It follows therefore from (4.2) that

$$\|x_{n+1}-x^*\| \leq \|x_n-x^*\|-(1-\beta_n)\|x_{n+1}-x^*\|+[(1+L)\alpha_n+(1-\beta_n)]^2\|x_n-x^*\|$$
$$+(1-\beta_n)[(1+L)\alpha_n+(1-\beta_n)]\|f(x_n)-x^*\|+(1-\beta_n)\|f(x_n)-x^*\|$$
$$= \|x_n-x^*\|-(1-\beta_n)\|x_{n+1}-x^*\|$$
$$+(1-\beta_n)[(1+L)^2\frac{\alpha_n^2}{1-\beta_n}+2(1+L)\alpha_n+(1-\beta_n)]\|x_n-x^*\|$$
$$+(1-\beta_n)[(1+L)\alpha_n+(1-\beta_n)]\|f(x_n)-x^*\|+(1-\beta_n)\|f(x_n)-x^*\|,$$

so that

$$\|x_{n+1}-x^*\| \leq \|x_n-x^*\|-(1-\beta_n)\|x_{n+1}-x^*\|$$
$$+\frac{5}{16}(1-\alpha)(1-\beta_n)\|x_n-x^*\|+\frac{1}{8}(1-\alpha)(1-\beta_n)\|f(x_n)-x^*\|$$
$$+(1-\beta_n)\|f(x_n)-x^*\|$$
$$<\gamma-(1-\beta_n)\gamma+\frac{5}{16}(1-\alpha)(1-\beta_n)\gamma+\frac{1}{8}(1-\alpha)(1-\beta_n)\gamma+\frac{1+\alpha}{2}(1-\beta_n)\gamma$$
$$<\gamma-(1-\beta_n)\gamma+(1-\beta_n)\gamma=\gamma.$$

Therefore $\|x_n-x^*\|\leq\gamma$, for all $n\geq m_0$, and hence $\{x_n\}$ is bounded.

Step 2. $\|x_n-z_{r_n}\|\to 0$ as $n\to\infty$, for some sequence $\{z_{r_n}\}$ satisfying (3.1), i.e.,

$$z_{r_n}=\widetilde{\alpha}_{r_n}T_{r_n}z_{r_n}+(1-\widetilde{\alpha}_{r_n})f(z_{r_n}),\ \text{where } \{\widetilde{\alpha}_{r_n}\}\subset[0,1) \text{ satisfies } \widetilde{\alpha}_{r_n}\to 1.$$

Indeed, choose the sequence $\{r_n\}\subset G$ as above. Set $\widetilde{\alpha}_{r_n}=\frac{\alpha_n}{1-\beta_n+\alpha_n}$ for each $n\geq 0$. Then $\widetilde{\alpha}_{r_n}\in(0,1)$ for each $n\geq 0$. By the given condition (iii) (a), we have that $\widetilde{\alpha}_{r_n}\to 1$ as $n\to\infty$. It follows from Proposition 3.1 that there exists a unique sequence $\{z_{r_n}\}\subset C$ satisfying the following equation:

$$z_{r_n}=\widetilde{\alpha}_{r_n}T_{r_n}z_{r_n}+(1-\widetilde{\alpha}_{r_n})f(z_{r_n}),\quad n\geq 0. \tag{4.3}$$

Equation (4.3) can be rewritten as follows:

$$z_{r_n}=\beta_n(\alpha_nT_{r_n}z_{r_n}+(1-\alpha_n)z_{r_n})+(1-\beta_n)f(z_{r_n})+(1-\beta_n)\alpha_n(T_{r_n}z_{r_n}-z_{r_n}).$$

If there exists $u_0 \in C$ such that the set B is bounded then the sequence $\{z_{r_n}\}$ is bounded (see Proposition 3.2 (i)). Utilizing the pseudocontractivity and uniformly Lipschitz property of $\mathcal{T}$, we make the following estimates:

$$\begin{aligned}\|x_{n+1}-z_{r_n}\|^2 &= \alpha_n\beta_n\langle T_{r_n}x_n - T_{r_n}z_{r_n}, j(x_{n+1}-z_{r_n})\rangle + \beta_n(1-\alpha_n)\langle x_n - z_{r_n}, j(x_{n+1}-z_{r_n})\rangle \\ &+(1-\beta_n)\langle f(x_n)-f(z_{r_n}), j(x_{n+1}-z_{r_n})\rangle + (1-\beta_n)\alpha_n\langle z_{r_n} - T_{r_n}z_{r_n}, j(x_{n+1}-z_{r_n})\rangle \\ &= \alpha_n\beta_n\langle T_{r_n}x_{n+1} - T_{r_n}z_{r_n}, j(x_{n+1}-z_{r_n})\rangle + \alpha_n\beta_n\langle T_{r_n}x_n - T_{r_n}x_{n+1}, j(x_{n+1}-z_{r_n})\rangle \\ &+\beta_n(1-\alpha_n)\langle x_n - z_{r_n}), j(x_{n+1}-z_{r_n})\rangle + (1-\beta_n)\langle f(x_n)-f(z_{r_n}), j(x_{n+1}-z_{r_n})\rangle \\ &+(1-\beta_n)\alpha_n\langle z_{r_n} - T_{r_n}z_{r_n}, j(x_{n+1}-z_{r_n})\rangle \\ &\le \alpha_n\beta_n\|x_{n+1}-z_{r_n}\|^2 + \alpha_n\beta_n\|T_{r_n}x_n - T_{r_n}x_{n+1}\|\|x_{n+1}-z_{r_n}\| \\ &+\beta_n(1-\alpha_n)\|x_n - z_{r_n}\|\|x_{n+1}-z_{r_n}\| + (1-\beta_n)\|f(x_n)-f(z_{r_n})\|\|x_{n+1}-z_{r_n}\| \\ &+(1-\beta_n)\alpha_n\|z_{r_n}-T_{r_n}z_{r_n}\|\|x_{n+1}-z_{r_n}\| \\ &\le \alpha_n\beta_n\|x_{n+1}-z_{r_n}\|^2 + \alpha_n\beta_n L\|x_n - x_{n+1}\|\|x_{n+1}-z_{r_n}\| \\ &+\beta_n(1-\alpha_n)\|x_n - z_{r_n}\|\|x_{n+1}-z_{r_n}\| + (1-\beta_n)\alpha\|x_n - z_{r_n}\|\|x_{n+1}-z_{r_n}\| \\ &+(1-\beta_n)\alpha_n\|z_{r_n}-T_{r_n}z_{r_n}\|\|x_{n+1}-z_{r_n}\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{n+1}-z_{r_n}\| &\le \alpha_n\beta_n\|x_{n+1}-z_{r_n}\| + \alpha_n\beta_n L\|x_n - x_{n+1}\| \\ &+[\beta_n(1-\alpha_n)+(1-\beta_n)\alpha]\|x_n - z_{r_n}\| + (1-\beta_n)\alpha_n\|z_{r_n}-T_{r_n}z_{r_n}\|,\end{aligned}$$

so that

$$\begin{aligned}\|x_{n+1}-z_{r_n}\| &\le [1-\tfrac{(1-\alpha)(1-\beta_n)}{1-\alpha_n\beta_n}]\|x_n - z_{r_{n-1}}\| + \|z_{r_{n-1}} - z_{r_n}\| \\ &+\tfrac{\alpha_n L}{1-\alpha_n\beta_n}\|x_n - x_{n+1}\| + \tfrac{(1-\beta_n)\alpha_n}{1-\alpha_n\beta_n}\|z_{r_n}-T_{r_n}z_{r_n}\|.\end{aligned} \tag{4.4}$$

Since the mapping $\widetilde{J}_{r_n} := [I+\frac{\alpha_n}{1-\beta_n}(I-T_{r_n})]^{-1}$ is nonexpansive and $z_{r_n} = \widetilde{J}_{r_n}(f(z_{r_n}))$,

$$\begin{aligned}\|z_{r_n} - z_{r_{n-1}}\| &= \|\widetilde{J}_{r_n}(f(z_{r_n})) - z_{r_{n-1}}\| \\ &= \|\widetilde{J}_{r_n}(f(z_{r_n})) - \widetilde{J}_{r_n}(f(z_{r_{n-1}})) + \widetilde{J}_{r_n}(f(z_{r_{n-1}})) - z_{r_{n-1}}\| \\ &\le \|f(z_{r_n}) - f(z_{r_{n-1}})\| + \|\widetilde{J}_{r_n}(f(z_{r_{n-1}})) - z_{r_{n-1}}\| \\ &\le \alpha\|z_{r_n} - z_{r_{n-1}}\| + \|\widetilde{J}_{r_n}(f(z_{r_{n-1}})) - z_{r_{n-1}}\|,\end{aligned}$$

so that

$$\begin{aligned}\|z_{r_n} - z_{r_{n-1}}\| &\le \frac{1}{1-\alpha}\|\widetilde{J}_{r_n}(f(z_{r_{n-1}})) - z_{r_{n-1}}\| \\ &= \frac{1}{1-\alpha}\|\widetilde{J}_{r_n}(f(z_{r_{n-1}})) - \widetilde{J}_{r_n}[I+\frac{\alpha_n}{1-\beta_n}(I-T_{r_n})]z_{r_{n-1}}\| \\ &\le \frac{1}{1-\alpha}\|f(z_{r_{n-1}}) - [z_{r_{n-1}} + \frac{\alpha_n}{1-\beta_n}(z_{r_{n-1}} - T_{r_n}z_{r_{n-1}})]\| \\ &= \frac{1}{1-\alpha}\|(\frac{\alpha_{n-1}}{1-\beta_{n-1}} - \frac{\alpha_n}{1-\beta_n})(z_{r_{n-1}} - T_{r_{n-1}}z_{r_{n-1}}) + \frac{\alpha_n}{1-\beta_n}(T_{r_n}z_{r_{n-1}} - T_{r_{n-1}}z_{r_{n-1}})\| \\ &\le \frac{1}{1-\alpha}\{|\frac{\alpha_{n-1}}{1-\beta_{n-1}} - \frac{\alpha_n}{1-\beta_n}|\|z_{r_{n-1}} - T_{r_{n-1}}z_{r_{n-1}}\| + \frac{\alpha_n}{1-\beta_n}\|T_{r_n}z_{r_{n-1}} - T_{r_{n-1}}z_{r_{n-1}}\|\} \\ &= \frac{1}{1-\alpha}\{|1 - \frac{\alpha_n}{1-\beta_n}\frac{1-\beta_{n-1}}{\alpha_{n-1}}|\|f(z_{r_{n-1}}) - z_{r_{n-1}}\| + \frac{\alpha_n}{1-\beta_n}\|T_{r_n}z_{r_{n-1}} - T_{r_{n-1}}z_{r_{n-1}}\|\}\end{aligned}$$

$$= \frac{1}{1-\alpha}\{|\frac{(\alpha_{n-1}-\alpha_n)(1-\beta_n)+\alpha_n(\beta_{n-1}-\beta_n)}{\alpha_{n-1}(1-\beta_n)}|\|f(z_{r_{n-1}})-z_{r_{n-1}}\|$$
$$+\frac{\alpha_n}{1-\beta_n}\|T_{r_n}z_{r_{n-1}}-T_{r_{n-1}}z_{r_{n-1}}\|\}$$
$$\le \frac{1}{1-\alpha}\{[\frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}}+\frac{|\beta_{n-1}-\beta_n|}{1-\beta_n}]\|f(z_{r_{n-1}})-z_{r_{n-1}}\|$$
$$+\frac{\alpha_n}{1-\beta_n}\|T_{r_n}z_{r_{n-1}}-T_{r_{n-1}}z_{r_{n-1}}\|\}.$$

We estimate $\|x_n - x_{n+1}\|$. Let $c := \sup_{n\ge 0}\{\frac{1-\beta_n}{\alpha_n}\}$. Since the sequences $\{x_n\}$ and $\{z_{r_n}\}$ are bounded, let $\|x_n - T_{r_n}x_n\| \le M$, $\|z_{r_n} - T_{r_n}z_{r_n}\| \le M$, $\|f(x_n) - x_n\| \le M$, and $\|f(z_{r_n}) - z_{r_n}\| \le M$ for all $n \ge 0$ and some constant $M > 0$. Then, for all $n \ge 0$

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\beta_n(\alpha_n(T_{r_n}x_n - x_n)) + (1-\beta_n)(f(x_n)-x_n)\| \\ &\le \alpha_n\|T_{r_n}x_n - x_n\| + (1-\beta_n)\|f(x_n)-x_n\| \\ &\le [\alpha_n + (1-\beta_n)]M \le \alpha_n(1+c)M,\end{aligned}$$

It follows from (4.4) that

$$\|x_{n+1}-z_{r_n}\| \le [1-\frac{(1-\alpha)(1-\beta_n)}{1-\alpha_n\beta_n}]\|x_n-z_{r_{n-1}}\|+\frac{1}{1-\alpha}[\frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}}+\frac{|\beta_n-\beta_{n-1}|}{1-\beta_n}]M$$
$$+\frac{\alpha_n}{(1-\alpha)(1-\beta_n)}\|T_{r_n}z_{r_{n-1}}-T_{r_{n-1}}z_{r_{n-1}}\|+\frac{\alpha_n^2}{1-\alpha_n\beta_n}L(1+c)M+\frac{(1-\beta_n)\alpha_n}{1-\alpha_n\beta_n}M$$
$$=[1-\frac{(1-\alpha)(1-\beta_n)}{1-\alpha_n\beta_n}]\|x_n-z_{r_{n-1}}\|+[\frac{1}{1-\alpha}(\frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}}+\frac{|\beta_n-\beta_{n-1}|}{1-\beta_n})$$
$$+\frac{\alpha_n^2L(1+c)}{1-\alpha_n\beta_n}+\frac{(1-\beta_n)\alpha_n}{1-\alpha_n\beta_n}]M+\frac{\alpha_n}{(1-\alpha)(1-\beta_n)}\|T_{r_n}z_{r_{n-1}}-T_{r_{n-1}}z_{r_{n-1}}\|.$$

Set $\delta_n = \frac{(1-\alpha)(1-\beta_n)}{1-\alpha_n\beta_n}$ and

$$\begin{aligned}\theta_n &= [\tfrac{1}{1-\alpha}(\tfrac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}}+\tfrac{|\beta_n-\beta_{n-1}|}{1-\beta_n})+\tfrac{\alpha_n^2L(1+c)}{1-\alpha_n\beta_n}+\tfrac{(1-\beta_n)\alpha_n}{1-\alpha_n\beta_n}]M \\ &+\tfrac{\alpha_n}{(1-\alpha)(1-\beta_n)}\|T_{r_n}z_{r_{n-1}}-T_{r_{n-1}}z_{r_{n-1}}\|.\end{aligned}$$

Then we have the inequality

$$\|x_{n+1}-z_{r_n}\| \le (1-\delta_n)\|x_n - z_{r_{n-1}}\| + \theta_n. \tag{4.5}$$

By condition (iv) and the assumptions on the sequences of numbers $\{\alpha_n\}$ and $\{\beta_n\}$ we know that $\theta_n = o(\delta_n)$. Thus, by Lemma 2.2, $\|x_{n+1} - z_{r_n}\| \to 0$ as $n \to \infty$, so that

$$\|x_n - z_{r_n}\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_{r_n}\| \to 0 \text{ as } n \to \infty.$$

Step 3. $\|x_n - T_s x_n\| \to 0$ as $n \to \infty$, for each $s \in G$.

Indeed, since

$$\|z_{r_n} - T_{r_n}z_{r_n}\| = \frac{1-\beta_n}{\alpha_n}\|f(z_{r_n}) - z_{r_n}\| \le \frac{1-\beta_n}{\alpha_n}M \to 0 \text{ as } n\to\infty,$$

the uniformly Lipschitz condition of $\mathcal{T}$ together with (ULARC) implies that

$$\|x_n - T_s x_n\| \le \|x_n - z_{r_n}\| + \|z_{r_n} - T_{r_n}z_{r_n}\| + \|T_{r_n}z_{r_n} - T_sT_{r_n}z_{r_n}\|$$
$$+\|T_sT_{r_n}z_{r_n} - T_s z_{r_n}\| + \|T_s z_{r_n} - T_s x_n\|$$

$$\leq (1+L)\|x_n - z_{r_n}\| + (1+L)\|z_{r_n} - T_{r_n} z_{r_n}\| + \|T_{r_n} z_{r_n} - T_s T_{r_n} z_{r_n}\|.$$

This shows that $\|x_n - T_s x_n\| \to 0$ as $n \to \infty$. □

Example 4.1. Let $X = R^2$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ defined by

$$\langle x, y \rangle = ac + bd \quad \text{and} \quad \|x\| = \sqrt{a^2 + b^2}$$

for all $x, y \in R^2$ with $x = (a, b)$ and $y = (c, d)$. Let $C = \{x \in R^2 : \|x\| \leq 1\}$ and $G = \mathbb{N}$. Let A be a 2×2 positively definite matrix and $u \in C$ be a characteristic vector satisfying $Au = u$ (for example, putting $A = \left\{ \begin{matrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{matrix} \right\}$ and $u = (\frac{1}{2}, \frac{1}{2})$, we know that $u \in C$ and $Au = u$). We define a sequence of nonexpansive self-mappings on C as follows

$$\begin{cases} T_0 = I, \\ T_n = (1 - \frac{1}{n})u + \frac{1}{n} A, \quad n = 1, 2, ... \end{cases}$$

Furthermore, let $r_n = n$, for all $n \geq 0$ and define

$$\alpha_n = \frac{1}{n^{1/3}} \quad \text{and} \quad \beta_n = 1 - \frac{1}{n^{2/5}}, \text{ for all } n \geq 1.$$

Then there hold the following:

(i) $\{\alpha_n\}$ is decreasing and $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \beta_n = 1$ and $\sum_{n=0}^{\infty}(1 - \beta_n) = \infty$;

(iii) (a) $\lim_{n\to\infty} \frac{1 - \beta_n}{\alpha_n} = 0$, (b) $\lim_{n\to\infty} \frac{\alpha_n^2}{1 - \beta_n} = 0$,
(c) $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{(1 - \beta_n)^2} = 0$, (d) $\lim_{n\to\infty} \frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}(1 - \beta_n)} = 0$;

(iv) $r_n \to \infty$ such that $\frac{\alpha_n}{(1-\beta_n)^2}(T_{r_n} y_{n-1} - T_{r_{n-1}} y_{n-1}) \to 0$, for all $\{y_n\}$ bounded in C;

(v) $\mathcal{T}$ satisfies (ULARC) on bounded subsets of C, i.e., for each bounded subset $\widetilde{C}$ of C, there holds

$$\lim_{s\in G, s\to\infty} \sup_{x\in\widetilde{C}} \|T_r T_s x - T_s x\| = 0, \quad r \in G. \tag{ULARC}$$

Indeed, it is easy to see that (i), (ii) and (iii) (a), (b) hold. Then utilizing the L'Hospital rule (with the notation $t = 1/n$) we deduce that

$$\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{(1 - \beta_n)^2} = \lim_{n\to\infty} \frac{1}{1 - \beta_n} \left| \frac{1 - \beta_{n-1}}{1 - \beta_n} - 1 \right| = 0,$$

and

$$\lim_{n\to\infty} \frac{\alpha_{n-1} - \alpha_n}{\alpha_{n-1}(1 - \beta_n)} = \lim_{n\to\infty} \frac{1}{1 - \beta_n}(1 - \frac{\alpha_n}{\alpha_{n-1}}) = 0.$$

This shows that (c) and (d) in (iii) are valid. Next, let us verify that (iv) and (v) are valid. Observe that for all bounded $\{y_n\}$ in C

$$\frac{\alpha_n}{(1 - \beta_n)^2} \|T_n y_{n-1} - T_{n-1} y_{n-1}\|$$

$$= \frac{\frac{1}{n^{1/3}}}{\frac{1}{n^{4/5}}} \|(1 - \frac{1}{n})u + \frac{1}{n} A y_{n-1} - (1 - \frac{1}{n-1})u - \frac{1}{n-1} A y_{n-1}\|$$

$$= n^{7/15}\|\frac{1}{n(n-1)}u - \frac{1}{n(n-1)}Ay_{n-1}\| = \frac{1}{n^{8/15}(n-1)}\|u - Ay_{n-1}\|$$
$$\le \frac{1}{n^{8/15}(n-1)}(\|u\| + \|A\|\|y_{n-1}\|) \to 0,$$

and for each $m \ge 0$ and each bounded subset $\widetilde{C}$ of C,

$$\sup_{x\in\widetilde{C}} \|T_m T_n x - T_n x\| = \sup_{x\in\widetilde{C}} \|(1-\frac{1}{m})u + \frac{1}{m}A[(1-\frac{1}{n})u + \frac{1}{n}Ax] - (1-\frac{1}{n})u - \frac{1}{n}Ax\|$$
$$= \sup_{x\in\widetilde{C}} \|(1-\frac{1}{m})u + (\frac{1}{m}-1)(1-\frac{1}{n})u + \frac{1}{n}(\frac{1}{m}A^2x - Ax)\|$$
$$\le \|(1-\frac{1}{m})u + (\frac{1}{m}-1)(1-\frac{1}{n})u\| + \frac{1}{n}\sup_{x\in\widetilde{C}}(\frac{1}{m}\|A\|^2\|x\| + \|A\|\|x\|)$$
$$\le (1-\frac{1}{m})\frac{1}{n}\|u\| + \frac{1}{n}\|A\|(\frac{1}{m}\|A\| + 1) \to 0 \text{ as } n \to \infty.$$

Therefore, (iv) and (v) are also valid. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X with a uniformly Gâteaux differentiable norm. Let $\mathcal{T} = \{T_t : t \in G\}$ be a family of uniformly Lipschitz pseudocontractive self-mappings on C and let $f \in \Pi_C$. Let $\{x_n\}$ be a sequence generated by (4.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ and $\{r_n\}$ is a sequence in G. Assume that the conditions (i)-(vi) in Theorem 4.1 are satisfied. Then $\{x_n\}$ converges strongly to a common fixed point $Q(f) \in F$ of $\mathcal{T}$, which is the unique solution of the variational inequality*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C,\ p \in F.$$

Proof. By Proposition 3.1 and Theorem 3.1, a sequence $\{z_{r_n}\}$ given by $z_{r_n} = \widetilde{\alpha}_{r_n} T_{r_n} z_{r_n} + (1-\widetilde{\alpha}_{r_n}) f(z_{r_n})$, with $\widetilde{\alpha}_{r_n} = \frac{\alpha_n}{1-\beta_n+\alpha_n}$, for all $n \ge 0$, exists and converges strongly to a common fixed point $Q(f) \in F$ of $\mathcal{T}$, which is the unique solution of the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C,\ p \in F.$$

From the proof of Theorem 4.1, $\|x_n - z_{r_n}\| \to 0$ as $n \to \infty$. Hence, $\{x_n\}$ converges strongly to the same fixed point $Q(f) \in F$ of $\mathcal{T}$. This completes the proof. □

Corollary 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X with a uniformly Gâteaux differentiable norm. Let $\mathcal{T} = \{T_t : t \in G\}$ be a family of nonexpansive self-mappings on C and let $f \in \Pi_C$. Let $\{x_n\}$ be a sequence generated by (4.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ and $\{r_n\}$ is a sequence in G. Assume that the conditions (i)-(vi) in Theorem 4.1 are satisfied. Then $\{x_n\}$ converges strongly to a common fixed point $Q(f) \in F$, which is the unique solution of the variational inequality*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C,\ p \in F.$$

Corollary 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\mathcal{T}=\{T_t : t \in G\}$ be a family of uniformly Lipschitz pseudocontractive self-mappings on C and let $f \in \Pi_C$. Let $\{x_n\}$ be a sequence generated by (4.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ and $\{r_n\}$ is a sequence in G. Assume that the conditions (i)-(vi) in Theorem 4.1 are satisfied. Then $\{x_n\}$ converges strongly to a common fixed point $P(f) \in F$, which is the unique solution of the variational inequality*

$$\langle (I-f)P(f), P(f)-p\rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

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