Fixed Point Theory, 10(2009), No. 2, 347-363
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings in a Hilbert space. We prove a strong convergence theorem under mild assumptions on parameters.
Key Words and Phrases: Nonexpansive mapping, equilibrium problem, fixed point problem, Hilbert spaces.
2000 Mathematics Subject Classification: 47H05, 47J05, 47J25.

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[*]The first author was partially supposed by National Natural Science Foundation of China Grant 10771050.

[**]The second author was partially supposed by the grant National Science Council 97-2221-E-230-017.

[***]The third author was partially supported by a grant from the National Science Council.

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $h : C \times C \to R$ be an equilibrium bifunction, i.e., $h(u,u) = 0$ for every $u \in C$. Then, one can define the equilibrium problem that is to find an element $u \in C$ such that

$$\text{EP(h):} \qquad h(u,v) \geq 0 \;\text{ for all }\; v \in C.$$

Denote the set of solutions of EP(h) by SEP(h). This problem contains fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems as special cases; see [1]. Some methods have been proposed to solve the equilibrium problem, please consult [2-4].

Recently, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $SEP(h) \neq \emptyset$ and proved a strong convergence theorem. Motivated by the idea of Combettes and Hirstoaga, very recently Takahashi and Takahashi [4] introduced a new iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Their results extend and improve the corresponding results announced by Combettes and Hirstoaga [2], Moudafi [8], Wittmann [9] and Tada and Takahashi [10].

In this paper, motivated and inspired by Combettes and Hirstoaga [2] and Takahashi and Takahashi [4], we introduce an iterative scheme for finding a common element of the set of solutions of EP(h) and the set of fixed points of finitely many nonexpansive mappings in a Hilbert space. A strong convergence theorem was established.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|,$$

for all $y \in C$. Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \Leftrightarrow \langle x - x^*, x^* - y \rangle \geq 0 \text{ for all } y \in C.$$

Recall that a mapping $T : C \to H$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. Denote the set of fixed points of T by $F(T)$. It is well known that if C is bounded closed convex and $T : C \to C$ is nonexpansive, then $F(T) \neq \emptyset$. We call a mapping $f : H \to H$ is contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \quad \text{for all } x, y \in H.$$

For an equilibrium bifunction $h : C \times C \to R$, we call h satisfying condition **(A)** if h satisfies the following three conditions:

- h is monotone, i.e., $h(x, y) + h(y, x) \leq 0$ for all $x, y \in C$;
- for each $x, y, z \in C$, $\lim_{t \downarrow 0} h(tz + (1 - t)x, y) \leq h(x, y)$;
- for each $x \in C$, $y \mapsto h(x, y)$ is convex and lower semicontinuous.

If an equilibrium bifunction $h : C \times C \to R$ satisfies condition (A), then we have the following two important results. You can find the first lemma in [1] and the second one in [2].

Lemma 2.1. *Let C be a nonempty closed convex subset of H and let h be an equilibrium bifunction of $C \times C$ into R satisfies condition (A). Let $r > 0$ and $x \in H$. Then, there exists $y \in C$ such that*

$$h(y, z) + \frac{1}{r} \langle z - y, y - x \rangle \geq 0 \text{ for all } z \in C.$$

Lemma 2.2. *Assume that h satisfies the same assumptions as Lemma 2.1. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \{y \in C : h(y, z) + \frac{1}{r} \langle z - y, y - x \rangle \geq 0, \forall z \in C\}$$

for all $y \in H$. Then, the following hold:

(1) *S_r is single-valued;*

(2) *S_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y \rangle;$$

(3) *$F(S_r) = SEP(h)$;*

(4) *$SEP(h)$ is closed and convex.*

We also need the following lemmas for proving our main results.

Lemma 2.3. ([5]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$$

for all integers $n \ge 0$ and

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0.$

Lemma 2.4.([6]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0.$

3. Iterative scheme and strong convergence theorems

In this section, we first introduce our iterative scheme. Consequently we will establish strong convergence theorems for this iteration scheme. To be more specific, let $\lambda_{n1},\ \lambda_{n2},\cdots,\lambda_{nN} \in (0,1]$, $n \in \mathbf{N}$. Given the mappings $T_1, T_2,\ \cdots, T_N$, following [7] one can define, for each $n \in \mathbf{N}$, mappings $U_{n1}, U_{n2},\cdots, U_{nN}$ by

$$\begin{aligned} U_{n1} &= \lambda_{n1}T_1 + (1-\lambda_{n1})I, \\ U_{n2} &= \lambda_{n2}T_2U_{n1} + (1-\lambda_{n2})I, \\ &\ \vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})I, \\ W_n : = U_{nN} &= \lambda_{nN}T_NU_{n,N-1} + (1-\lambda_{nN})I. \end{aligned} \tag{1}$$

Such a mapping W_n is called the W-mapping generated by $T_1, \cdots, T_N$ and $\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nN}$.

Now we introduce the following iteration scheme: Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$ and given $x_0 \in H$ arbitrarily. Suppose the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated iteratively by

$$\begin{cases} h(y_n, x) + \dfrac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, \quad \forall x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \end{cases} \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{r_n\}$ is a real sequence in $(0,\infty)$, h is an equilibrium bifunction and W_n is the W-mapping defined by (1).

We have the following crucial conclusion concerning W_n.

Lemma 3.1. ([7]) *Let C be a nonempty closed convex subset of a Banach space E. Let $T_1, T_2, \cdots, T_N$ be finite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i)$ is nonempty, and let $\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nN}$ be real numbers such that $0 < \lambda_{ni} \leq b < 1$ for any $i \in \mathbf{N}$. For any $n \in \mathbf{N}$, let W_n be the W-mapping of C into itself generated by $T_N, T_{N-1}, \cdots, T_1$ and $\lambda_{nN}, \lambda_{n,N-1}, \cdots, \lambda_{n1}$. Then W_n is nonexpansive. Further, if E is strictly convex, then $F(W_n) = \cap_{i=1}^N F(T_i)$.*

Now we state and prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of H. Let $h : C \times C \to R$ be an equilibrium bifunction satisfying condition (A) and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H such that $\cap_{i=1}^N F(T_i) \bigcap SEP(h) \neq \emptyset$. Let $\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nN}$ be real numbers such that $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ for all $i = 1, 2, \cdots, N$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ and $\{r_n\} \subset (0,\infty)$ is a real sequence. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by (2) converge strongly to $x^* \in \cap_{i=1}^N F(T_i) \bigcap SEP(h)$, where $x^* = P_{\cap_{i=1}^N F(T_i) \bigcap SEP(h)} f(x^*)$.

Proof. Let $Q = P_{\cap_{i=1}^N F(T_i)\bigcap SEP(h)}$. Note that f is a contraction mapping with coefficient $\alpha \in (0,1)$. Then $\|Qf(x)-Qf(y)\| \leq \|f(x)-f(y)\| \leq \alpha\|x-y\|$ for all $x, y \in H$. Therefore, Qf is a contraction of H into itself, which implies that there exists a unique element $x^* \in H$ such that $x^* = Qf(x^*)$. At the same time, we note that $x^* \in C$.

Let $p \in \cap_{i=1}^N F(T_i)\bigcap SEP(h)$. From the definition of S_r, we note that $y_n = S_{r_n}x_n$. It follows that

$$\begin{aligned} \|y_n - p\| &= \|S_{r_n}x_n - S_{r_n}p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Next we prove that $\{x_n\}$ and $\{y_n\}$ are bounded. Indeed, from Lemma 3.1 we have $p \in W_n$. Then from (1) and (2), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n - p\| \\ &\leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|W_n y_n - p\| \\ &\leq \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|x_n - p\| \\ &\quad + \gamma_n\|y_n - p\| \\ &\leq \alpha_n(\alpha\|x_n - p\| + \|f(p) - p\|) + (1-\alpha_n)\|x_n - p\| \\ &\leq \max\{\|x_0 - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\}. \end{aligned}$$

Therefore $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{W_n x_n\}$ and $\{f(x_n)\}$ are all bounded. We shall use M to denote the possible different constants appearing in the following reasoning.

Setting $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ for all $n \geq 0$. It follows that

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}W_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n W_n y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})f(x_n) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(W_{n+1}y_{n+1} - W_n y_n) + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})W_n y_n. \end{aligned}$$

So, we have

$$\begin{aligned}\|z_{n+1}-z_n\| \le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|f(x_n)\| \\ + \|W_n y_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|W_{n+1}y_{n+1} - W_n y_n\|.\end{aligned} \tag{3}$$

From (1), since T_N and $U_{n,N}$ are nonexpansive,

$$\begin{aligned}&\|W_{n+1}y_n - W_n y_n\| \\ &= \|\lambda_{n+1,N}T_N U_{n+1,N-1}y_n + (1-\lambda_{n+1,N})y_n \\ &\quad - \lambda_{n,N}T_N U_{n,N-1}y_n - (1-\lambda_{n,N})y_n\| \\ &\le |\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\| + \|\lambda_{n+1,N}T_N U_{n+1,N-1}y_n - \lambda_{n,N}T_N U_{n,N-1}y_n\| \\ &\le |\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\| + \|\lambda_{n+1,N}(T_N U_{n+1,N-1}y_n - T_N U_{n,N-1}y_n)\| \\ &\quad + |\lambda_{n+1,N}-\lambda_{n,N}|\|T_N U_{n,N-1}y_n\| \\ &\le 2M|\lambda_{n+1,N}-\lambda_{n,N}| + \lambda_{n+1,N}\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\|.\end{aligned} \tag{4}$$

Again, from (1), we have

$$\begin{aligned}&\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| \\ &= \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n + (1-\lambda_{n+1,N-1})y_n \\ &\quad - \lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n - (1-\lambda_{n,N-1})y_n\| \\ &\le |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|y_n\| \\ &\quad + \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n\| \\ &\le |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|y_n\| \\ &+ \lambda_{n+1,N-1}\|T_{N-1}U_{n+1,N-2}y_n - T_{N-1}U_{n,N-2}y_n\| \\ &+ |\lambda_{n+1,N-1}-\lambda_{n,N-1}|M \\ &\le 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}| + \lambda_{n+1,N-1}\|U_{n+1,N-2}y_n - U_{n,N-2}y_n\| \\ &\le 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}| + \|U_{n+1,N-2}y_n - U_{n,N-2}y_n\|.\end{aligned} \tag{5}$$

Therefore, we have

$$\begin{aligned}\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| &\le 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}| + 2M|\lambda_{n+1,N-2}-\lambda_{n,N-2}| \\ +\|U_{n+1,N-3}y_n - U_{n,N-3}y_n\| &\le 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| + \|U_{n+1,1}y_n - U_{n,1}y_n\|\end{aligned}$$

$$= \|\lambda_{n+1,1}T_1y_n+(1-\lambda_{n+1,1})y_n-\lambda_{n,1}T_1y_n-(1-\lambda_{n,1})y_n\|+2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|,$$

then

$$\begin{aligned}
&\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| \\
&\leq |\lambda_{n+1,1}-\lambda_{n,1}|\|y_n\| + \|\lambda_{n+1,1}T_1y_n - \lambda_{n,1}T_1y_n\| \\
&\quad + 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| \\
&\leq 2M\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned} \tag{6}$$

Substituting (6) into (4), we have

$$\begin{aligned}
\|W_{n+1}y_n - W_ny_n\| &\leq 2M|\lambda_{n+1,N}-\lambda_{n,N}| \\
&\quad + 2\lambda_{n+1,N}M\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| \\
&\leq 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|W_{n+1}y_{n+1} - W_ny_n\| &\leq \|W_{n+1}y_{n+1} - W_{n+1}y_n\| + \|W_{n+1}y_n - W_ny_n\| \\
&\leq \|y_{n+1}-y_n\| + \|W_{n+1}y_n - W_ny_n\| \\
&\leq \|y_{n+1}-y_n\| + 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned} \tag{7}$$

Substituting (7) into (3), we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|f(x_n)\| \\
&\quad + \|W_ny_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\| \\
&\quad + \frac{2M\gamma_{n+1}}{1-\beta_{n+1}}\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned} \tag{8}$$

On the other hand, from $y_n = S_{r_n}x_n$ and $y_{n+1} = S_{r_{n+1}}x_{n+1}$, we have

$$h(y_n, x) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0 \text{ for all } x \in C \tag{9}$$

and

$$h(y_{n+1}, x) + \frac{1}{r_{n+1}}\langle x - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0 \text{ for all } x \in C. \tag{10}$$

Putting $x = y_{n+1}$ in (9) and $x = y_n$ in (10), we have

$$h(y_n, y_{n+1}) + \frac{1}{r_n}\langle y_{n+1} - y_n, y_n - x_n\rangle \geq 0, \tag{11}$$

and

$$h(y_{n+1}, y_n) + \frac{1}{r_{n+1}}\langle y_n - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0. \tag{12}$$

From the monotonicity of h, we have

$$h(y_n, y_{n+1}) + h(y_{n+1}, y_n) \leq 0.$$

So, from (11) and (12), we can conclude that

$$\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0$$

and hence

$$\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - x_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1})\rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we may assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbf{N}$. Then, we have

$$\begin{aligned}\|y_{n+1} - y_n\|^2 &\leq \langle y_{n+1} - y_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(y_{n+1} - x_{n+1})\rangle \\ &\leq \|y_{n+1} - y_n\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|y_{n+1} - x_{n+1}\|\}\end{aligned}$$

and hence

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \frac{M}{b}|r_{n+1} - r_n|. \tag{13}$$

Substituting (13) into (8), we have

$$\begin{aligned}\|z_{n+1}-z_n\| \le & \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| \\ & + \|W_n y_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\ & + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \times \frac{M}{b}|r_{n+1}-r_n| + \frac{2M\gamma_{n+1}}{1-\beta_{n+1}}\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}| \\ \le & \|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n) + \|W_n y_n\|) \\ & + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \times \frac{M}{b}|r_{n+1}-r_n| + \frac{2M\gamma_{n+1}}{1-\beta_{n+1}}\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|,\end{aligned}$$

this together with $\alpha_n \to 0$, $r_{n+1}-r_n \to 0$ and $\lambda_{n+1,i}-\lambda_{n,i} \to 0$ imply that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Hence by Lemma 2.3, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

From (13) and $\lim_{n\to\infty}(r_{n+1}-r_n) = 0$, we have

$$\lim_{n\to\infty}\|y_{n+1}-y_n\| = 0.$$

Since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n$, we have

$$\begin{aligned}\|x_n - W_n y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n y_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - W_n y_n\| + \beta_n\|x_n - W_n y_n\|,\end{aligned}$$

that is

$$\|x_n - W_n y_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - W_n y_n\|.$$

It follows that

$$\lim_{n\to\infty}\|x_n - W_n y_n\| = 0.$$

For $p \in \cap_{i=1}^N F(T_i) \bigcap SEP(h)$, note that S_r is firmly nonexpansive, then we have

$$\begin{aligned}\|y_n - p\|^2 &= \|S_{r_n}x_n - S_{r_n}p\|^2\\ &\leq \langle S_{r_n}x_n - S_{r_n}p, x_n - p\rangle\\ &= \langle y_n - p, x_n - p\rangle\\ &= \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2)\end{aligned}$$

and hence

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2.$$

Therefore, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n - p\|^2\\ &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|W_n y_n - p\|^2\\ &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2\\ &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2\\ &\quad + \gamma_n(\|x_n - p\|^2 - \|x_n - y_n\|^2)\\ &\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n\|x_n - y_n\|^2.\end{aligned}$$

Then we have

$$\begin{aligned}\gamma_n\|x_n - y_n\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\leq \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\\ &\quad \times (\|x_n - p\| - \|x_{n+1} - p\|)\\ &\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).\end{aligned}$$

It is easily seen that $\liminf_{n\to\infty}\gamma_n > 0$. So, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0.$$

From

$$\|W_n y_n - y_n\| \leq \|W_n y_n - x_n\| + \|x_n - y_n\|.$$

We also have $\|W_n y_n - y_n\| \to 0$.

Next, we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \leq 0,$$

where $x^* = P_{\bigcap_{i=1}^N F(T_i) \bigcap SEP(h)} f(x^*)$. First we can choose a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j\to\infty} \langle f(x^*) - x^*, y_{n_j} - x^* \rangle = \limsup_{n\to\infty} \langle f(x^*) - x^*, y_n - x^* \rangle.$$

Since $\{y_{n_j}\}$ is bounded, there exists a subsequence $\{y_{n_{ji}}\}$ of $\{y_{n_j}\}$ which converges weakly to w. Without loss of generality, we can assume that $y_{n_j} \to w$ weakly. From $\|W_n y_n - y_n\| \to 0$, we obtain $W_n y_{n_j} \to w$ weakly. Now we show $w \in SEP(h)$. By $y_n = S_{r_n} x_n$, we have

$$h(y_n, x) + \frac{1}{r_n} \langle x - y_n, y_n - x_n \rangle \geq 0, \quad \forall x \in C.$$

From the monotonicity of h, we have

$$\frac{1}{r_n} \langle x - y_n, y_n - x_n \rangle \geq -h(y_n, x) \geq h(x, y_n),$$

and hence

$$\langle x - y_{n_j}, \frac{y_{n_j} - x_{n_j}}{r_{n_j}} \rangle \geq h(x, y_{n_j}).$$

Since $\frac{y_{n_j} - x_{n_j}}{r_{n_j}} \to 0$ and $y_{n_j} \to w$ weakly, from the lower semi-continuity of $h(x, y)$ on the second variable y, we have

$$h(x, w) \leq 0$$

for all $x \in C$. For t with $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)w$. Since $x \in C$ and $w \in C$, we have $x_t \in C$ and hence $h(x_t, w) \leq 0$. So, from the convexity of equilibrium bifunction $h(x, y)$ on the second variable y , we have

$$\begin{aligned} 0 &= h(x_t, x_t) \\ &\leq th(x_t, x) + (1-t)h(x_t, w) \\ &\leq th(x_t, x). \end{aligned}$$

and hence $h(x_t, x) \geq 0$. Then, we have

$$h(w, x) \geq 0$$

for all $x \in C$ and hence $w \in SEP(h)$.

We shall show $w \in F(W_n)$. Assume $w \notin F(W_n)$. Since $y_{n_j} \to w$ weakly and $w \neq W_n w$, from Opial's condition, we have

$$\begin{aligned}\liminf_{j\to\infty} \|y_{n_j} - w\| &< \liminf_{j\to\infty} \|y_{n_j} - W_n w\| \\ &\leq \liminf_{j\to\infty} (\|y_{n_j} - W_n y_{n_j}\| + \|W_n y_{n_j} - W_n w\|) \\ &\leq \liminf_{j\to\infty} \|y_{n_j} - w\|.\end{aligned}$$

This is a contradiction. So, we get $w \in F(W_n) = \cap_{i=1}^N F(T_i)$. Therefore $w \in \cap_{i=1}^N F(T_i) \bigcap SEP(h)$. Since $x^* = P_{\cap_{i=1}^N F(T_i) \bigcap SEP(h)} f(x^*)$, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{j\to\infty} \langle f(x^*) - x^*, x_{n_j} - x^*\rangle \\ &= \lim_{j\to\infty} \langle f(x^*) - x^*, y_{n_j} - x^*\rangle \qquad (14) \\ &= \langle f(x^*) - x^*, w - x^*\rangle \leq 0.\end{aligned}$$

Finally, we prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to x^*. From (2), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(W_n y_n - x^*)\|^2 \\ &\leq \|\beta_n(x_n - x^*) + \gamma_n(W_n y_n - x^*)\|^2 \\ &\quad + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\ &\leq \{\beta_n\|x_n - x^*\| + \gamma_n\|W_n y_n - x^*\|\}^2 \\ &\quad + 2\alpha_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\ &\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq \{\beta_n\|x_n - x^*\| + \gamma_n\|y_n - x^*\|\}^2 \\ &\quad + 2\alpha\alpha_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\ &\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + \alpha\alpha_n(\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2) \\ &\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle,\end{aligned}$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{(1-\alpha_n)^2+\alpha\alpha_n}{1-\alpha\alpha_n}\|x_n-x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \\
&= \frac{1-2\alpha_n+\alpha\alpha_n}{1-\alpha\alpha_n}\|x_n-x^*\|^2 + \frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle \qquad (15)\\
&\le \{1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}\}\|x_n-x^*\|^2 + \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \\
&\quad \times \{\frac{M\alpha_n}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\} \\
&= (1-\delta_n)\|x_n-x^*\|^2 + \delta_n\sigma_n,
\end{aligned}
$$

where $\delta_n = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$ and $\sigma_n = \frac{M\alpha_n}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle$. It is easily seen that $\sum_{n=0}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Now applying Lemma 2.4 and (14) to (15) concludes that $x_n \to x^*(n\to\infty)$. This completes the proof. □

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $h: C\times C\to R$ be an equilibrium bifunction satisfying condition (A) such that $SEP(h)\neq\emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ and $\{r_n\}\subset(0,\infty)$ is a real sequence. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}(r_{n+1}-r_n) = 0$.

Let f be a contraction of H into itself and given $x_0\in H$ arbitrarily. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated iteratively by

$$
\begin{cases}
h(y_n,x) + \dfrac{1}{r_n}\langle x-y_n, y_n-x_n\rangle \ge 0, \quad \forall x\in C, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n,
\end{cases} \qquad (16)
$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (16) converge strongly to $x^*\in SEP(h)$, where $x^* = P_{SEP(h)}f(x^*)$.

Proof. Take $T_i x = x$ for all $i = 1, 2, \cdots, N$ and for all $x \in C$ in (1), then $W_n x = x$ for all $x \in C$. The conclusion follows from Theorem 3.1. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H such that $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nN}$ be real numbers such that $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ for all $i = 1, 2, \cdots, N$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0, 1)$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily. Let $\{x_n\}$ be sequence generated iteratively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C x_n.$$

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \cap_{i=1}^N F(T_i)$, where $x^* = P_{\cap_{i=1}^N F(T_i)} f(x^*)$.

Proof. Set $h(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbf{N}$. Then, we have $y_n = P_C x_n$. From (2), we have

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C x_n.$$

Then the conclusion follows from Theorem 3.1. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0, 1)$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let f be a contraction of C into itself and given $x_0 \in C$ arbitrarily. Let $\{x_n\}$ be sequence generated iteratively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n.$$

Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, where $x^* = P_{F(T)} f(x^*)$ is a unique solution of the following variational inequality in $F(T)$

$$\langle (I - f)x^*, x^* - p \rangle \leq 0, \forall p \in F(T).$$

Example 3.1. Let $T : C \to C$ be a nonexpansive mapping. Take $h(x, y) = 0$ for all $x, y \in C$, $f(x) = u$ for all $x \in C$ and $r_n = 1$ for all $n \geq 1$. Hence, we can take $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{1}{2}$ and $\gamma_n = \frac{1}{2}(1 - \frac{1}{n})$ for all $n \geq 1$. By using the Corollary 3.3, the iterative sequence $\{x_n\}$ defined by

$$x_{n+1} = \frac{1}{2n}u + \frac{1}{2}x_n + \frac{1}{2}(1 - \frac{1}{n})Tx_n,$$

converges strongly to some fixed point of T.

In particular, let $H = R^2$ and define $T : R^2 \to R^2$ by

$$T(re^{i\theta}) = re^{i(\theta + \frac{\pi}{2})},$$

and take $u = e^{i\pi}$. It is obvious that T is a nonexpansive mapping with a unique fixed point $x^* = 0$. In this case, the sequence $\{x_n\}$ becomes that

$$x_{n+1} = \frac{1}{2n}e^{i\pi} + \frac{1}{2}r_n e^{i\theta_n} + \frac{1}{2}(1 - \frac{1}{n})r_n e^{i(\theta_n + \frac{\pi}{2})}.$$

It is clear that the complex number sequence $\{x_n\}$ converges strongly to a fixed point $x^* = 0$.

Remark 3.1. We conclude the paper with the following observations.

(i) Our iterative scheme (2) is a convex combination of $f(x_n)$, x_n and $W_n x_n$ which includes the iterative schemes studied in [4, 6, 8] as special cases. Our iterative methods studied in present paper can be reviewed as a refinement and modification of the iterative methods in [4, 6, 8]. On the other hand, our iterative scheme concerns a finitely many nonexpansive mappings, in this respect, they can be reviewed as an improvement of the iterative methods in [4, 6, 8].

(ii) We note that the authors in [4, 6] have imposed some additional assumptions: $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}(\alpha_{n+1} - \alpha_n)/\alpha_{n+1} = 0$ on parameters $\{\alpha_{n+1}\}$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ on parameters $\{r_n\}$.

(iii) The advantages of these results in this paper are that less restrictions on the parameters $\{\alpha_n\}$ and $\{r_n\}$ are imposed. Our results unify many recent results including the results in [4, 6, 8].

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Received: 13. 11. 2008; Accepted: 15. 05. 2009.