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ON A GENERALIZATION OF APPROXIMATIVE ABSOLUTE NEIGHBORHOOD RETRACTS

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Abstract. In this paper we introduce a new class of metric spaces for which the Lefschetz fixed point theorem holds true.
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1. Introduction

In the paper [19] we consider two classes of metric spaces. The first class encompasses the spaces of AMR type (absolute multi-retract), and constitutes a broader class than the class of AR (absolute retract) spaces. We prove that every compact space of AMR type has a fixed point property (also for the multi-valued admissible mappings). The second class of metric spaces is a space of $ANMR$ (absolute neighborhood multi-retract) type, and constitutes a broader class than the class of ANR (absolute neighborhood retract) spaces. We show that if $X \in ANMR$ then every admissible and compact mapping $\varphi : X \multimap X$ is a Lefschetz mapping. In this paper I present even more general class of $AANMR$ (approximative absolute neighborhood multi-retract) compact spaces for which the Lefschetz theorem holds true. All classes of spaces mentioned above are considered in a broader context, namely in the context of locally convex topological vector spaces. Thanks to that, an evident classification of them is obtained. Moreover, we show that every admissible mapping $\varphi : X \multimap X$ is a Lefschetz mapping, where $X \in AANMR$ is of finite type.

2. Preliminaries

Throughout this paper all topological spaces are assumed to be metric. Let H_* be the *Čech* homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional *Čech* homology group with compact carriers of X. For a continuous map $f : X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$ where $f_q : H_q(X) \to H_q(Y)$ (see [2] and [8]). A space X is acyclic if:

(i) X is non-empty,
(ii) $H_q(X) = 0$ for every $q \geq 1$ and
(iii) $H_0(X) \approx \mathbb{Q}$.

A continuous mapping $f : X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is non-empty and compact. A proper map $p : X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. Let X and Y be two spaces and assume that for every $x \in X$ a non-empty closed subset $\varphi(x)$ of Y is given. In such a case we say that $\varphi : X \multimap Y$ is a multi-valued mapping. For a multi-valued mapping $\varphi : X \multimap Y$ and a subset $U \subset Y$, we let:

$$\varphi^{-1}(U) = \{x \in X;\ \varphi(x) \subset U\}.$$

If for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open, then φ is called an upper semi-continuous mapping; we shall write φ is u.s.c.

Proposition 2.1. *(see [2, 8]). Assume that $\varphi : X \multimap Y$ and $\psi : Y \multimap T$ are u.s.c. mappings with compact values and $p : Z \to X$ is a Vietoris mapping. Then:*

(2.1.1) *for any compact $A \subset X$, the image $\varphi(A) = \bigcup_{x \in A} \varphi(x)$ of the set A under φ is a compact set;*

(2.1.2) *the composition $\psi \circ \varphi : X \multimap T$, $(\psi \circ \varphi)(x) = \bigcup_{y \in \varphi(x)} \psi(y)$, is an u.s.c. mapping;*

(2.1.3) *the mapping* $\varphi_p : X \multimap Z$, *given by the formula* $\varphi_p(x) = p^{-1}(x)$, *is u.s.c.*.

Let $\varphi : X \multimap Y$ be a multivalued map. A pair (p, q) of single-valued, continuous map of the form is called a selected pair of φ (written $(p, q) \subset \varphi$) if the following two conditions are satisfied:

(i) p is a Vietoris map,
(ii) $q(p^{-1}(x)) \subset \varphi(x)$ for any $x \in X$.

Definition 2.2. *A multivalued mapping* $\varphi \colon X \multimap Y$ *is called admissible provided there exists a selected pair* (p, q) *of* φ.

Theorem 2.3. *(see* [8]*) Let* $\varphi \colon X \multimap Y$ *and* $\psi \colon Y \multimap Z$ *be two admissible maps. Then the composition* $\psi \circ \varphi \colon X \multimap Z$ *is an admissible map.*

Lemma 2.4. *(see* [8]*) If* $\varphi \colon X \multimap Y$ *is an admissible map,* $Y_0 \subset Y$ *and* $X_0 = \varphi^{-1}(Y_0)$, *then the contraction* $\varphi_0 \colon X_0 \multimap Y_0$ *of* φ *to the pair* (X_0, Y_0) *is an admissible map.*

Theorem 2.5. *(see [2]) If* $p : X \to Y$ *is a Vietoris map, then an induced mapping*

$$p_* : H_*(X) \to H_*(Y)$$

is a linear isomorphism.

Let $u : E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E : u^n(x) = 0 \quad for\ some\ n\}$, where u^n is the nth iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call u admissible provided $dim\widetilde{E} < \infty$.

Let $u = \{u_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a Leray endomorphism if

(i) all u_q are admissible,
(ii) almost all $\widetilde{E_q}$ are trivial.

For such an u, we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q tr(\widetilde{u_q}),$$

where $tr(\widetilde{u_q})$ is the ordinary trace of $\widetilde{u_q}$ (comp. [2]). The following important property of a Leray endomorphism is a consequence of the well-known formula $tr(u \circ v) = tr(v \circ u)$ for the ordinary trace.

Proposition 2.6. *Assume that, in the category of graded vector spaces, the following diagram commutes*

$$\begin{tikzcd} E' \arrow[r, "u"] & E'' \\ E' \arrow[u, "u'"] \arrow[r, "u"] & E'' \arrow[u, "u''"'] \arrow[ul, "v"'] \end{tikzcd}$$

If one of u', u'' is a Leray endomorphism, then so is the other; and $\Lambda(u') = \Lambda(u'')$.

Let $\varphi : X \multimap X$ be an admissible map. Let $(p, q) \subset \varphi$, where $p : Z \to X$ is a Vietoris mapping and $q : Z \to X$ a continuous map. Assume that $q_* \circ p_*^{-1} : H_*(X) \to H_*(X)$ is a Leray endomorphism for all pairs $(p, q) \subset \varphi$. For such a φ, we define the Lefschetz number $\Lambda(\varphi)$ of φ by putting $\Lambda(\varphi) = \{\Lambda(q_*p_*^{-1});\ (p, q) \subset \varphi\}$. Let us observe that if X is an acyclic or, in particular, contractible space, then for every admissible map $\varphi : X \multimap X$ and for any pair $(p, q) \subset \varphi$ the endomorphism $q_*p_*^{-1} : H_*(X) \to H_*(X)$ is a Leray endomorphism and $\Lambda(q_*p_*^{-1}) = 1$.

Theorem 2.7. *(see [8]) If $\varphi : X \multimap Y$ and $\psi : Y \multimap T$ are admissible, then the composition $\psi \circ \varphi : X \multimap T$ is admissible and for every $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$ there exists a pair $(p, q) \subset \psi \circ \varphi$ such that $q_{2*}p_{2*}^{-1} \circ q_{1*}p_{1*}^{-1} = q_*p_*^{-1}$.*

Definition 2.8. *An admissible map $\varphi : X \multimap X$ is called a Lefschetz map provided the generalized Lefschetz number $\Lambda(\varphi)$ of φ is well defined and $\Lambda(\varphi) \neq \{0\}$ implies that the set $Fix(\varphi) = \{x \in X : x \in \varphi(x)\}$ is non-empty.*

Theorem 2.9. *(see [17]) Let U be an open subset of a normed space E and let X be a compact subset U. Then for each sufficiently small $\varepsilon > 0$ there exists a finite polyhedron $K_\varepsilon \subset U$ and a mapping $p_\varepsilon : X \to U$ such that:*
(2.2.1) *$\|x - p_\varepsilon(x)\| < \varepsilon$ for all $x \in X$,*
(2.2.2) *$p_\varepsilon(X) \subset K_\varepsilon$,*
(2.2.3) *p_ε is homotopic to i, where $i : X \to U$ is an inclusion.*

Let Y be a metric space and let $Id_Y : Y \to Y$ be a map given by formula $Id_Y(y) = y$ for each $y \in Y$.

Definition 2.10. *A map $r\colon X \to Y$ of a space X onto a space Y is said to be an r-map if there is a map $s\colon Y \to X$ such that $r \circ s = Id_Y$.*

Definition 2.11. *A metric space X is called an absolute neighborhood retract (notation: $X \in ANR$) provided there exists an open subset U of some normed space E and an r-map $r\colon U \to X$ from U onto X.*

Definition 2.12. *A metric space X is called an absolute retract (notation: $X \in AR$) provided there exists a normed space E and an r-map $r\colon E \to X$ from E onto X.*

Let $A \subset X$ be a nonempty set. We shall say that A is a retract of X if there exists a continuous map $r : X \to A$ such that for each $x \in A$ $r(x) = x$. A nonempty set $B \subset X$ is a neighborhood retract in X if there exists an open set $U \subset X$ such that $B \subset U$ and B is a retract of U.

Theorem 2.13. *(see [8]) $X \in ANR$ if and only if for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, the set $h(X)$ is a neighborhood retract of Y.*

Theorem 2.14. *(see [8]) $X \in AR$ if and only if for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, the set $h(X)$ is a retract in Y.*

Now we shall recall a generalization of the concept of absolute neighborhood retracts, which was introduced by Clapp.

Definition 2.15. *We shall say that a compact metric space X is an approximative absolute neighborhood retract in the sense of Clapp (notation: $X \in AANR_C$) provided for every $\varepsilon > 0$ there exists an open subset U_ε of some normed linear space E_ε and two maps $r_\varepsilon\colon U_\varepsilon \to X$, $s_\varepsilon : X \to U_\varepsilon$ such that $d(x, r_\varepsilon(s_\varepsilon(x))) < \varepsilon$ for any $x \in X$.*

Theorem 2.16. *(see [8]) $X \in AANR_C$ if and only if for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, for each $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset Y$ such that $h(X) \subset U_\varepsilon$ and a continuous map*

$r_\varepsilon : U_\varepsilon \to h(X)$ such that for each $y \in h(X)$

$$d(r_\varepsilon(y), y) < \varepsilon.$$

Definition 2.17. *Let E be a topological vector space. We shall say that E is a Klee admissible space provided for any compact subset $K \subset E$ and for any open neighborhood V of $0 \in E$ there exists a map:*

$$\pi_V : K \to E$$

such that the following two conditions are satisfied:
(2.17.1) $\pi_V(x) \in (x + V)$, for any $x \in K$,
(2.17.2) there exists a natural number $n = n_K$ such that $\pi_V(K) \subset E^n$, where E^n is an n-dimensional subspace of E.

Definition 2.18. *We shall say that a topological vector space E is locally convex provided that for each $x \in E$ and for each open set $U \subset E$ such that $x \in U$ there exists an open and convex set $V \subset E$ such that $x \in V \subset U$.*

It is clear that if E is a normed space then E is locally convex.

Theorem 2.19. *(see* [2, 7]*) Let E be locally convex. Then E is a Klee admissible space.*

Theorem 2.20. *(see* [9]*) Let E be a Klee admissible space. For each compact subset $K \subset E$ and for any open set $U \subset E$ such that $K \subset U$ there exists a continuous map $\pi_K : K \to U$ such that the following conditions are satisfied:*
2.20.1 $\pi_K(K) \subset E^n$, where E^n is an n-dimensional subspace of E,
2.20.2 $\pi_K : K \to U$ and $i : K \to U$ are homotopic, where $i : K \to U$ is an inclusion.

The following theorem is obvious.

Theorem 2.21. *Let E_s be a locally convex space for every $s \in S$. Then the space $E = \prod_{s \in S} E_s$ is a locally convex space.*

Theorem 2.22. *(see* [9]*) Let U be an open subset in a Klee admissible space E and $\varphi : U \multimap U$ be an admissible and compact map, then φ is a Lefschetz map.*

Definition 2.23. *A metric space X is of finite type provided that for almost every $q \in N$ $H_q(X) = \{0\}$ and for any $q \in N$ $dim H_q(X) < \infty$.*

Theorem 2.24. *(see* [8]*) Let X and Y be a compact of finite type spaces. Then $X \times Y$ is a compact of finite type space.*

Theorem 2.25. *(see* [8]*) Let X and Y be acyclic and compact spaces. Then $X \times Y$ is a compact and acyclic space.*

Definition 2.26. *Let X be an ANR and let $X_0 \subset X$ be a closed subset. We say that X_0 is movable in X provided every neighborhood U of X_0 admits a neighborhood U' of X_0, $U' \subset U$, such that for every neighborhood U'' of X_0, $U'' \subset U$, there exists a homotopy $H\colon U' \times [0,1] \to U$ with $H(x,0) = x$ and $H(x,1) \in U''$, for any $x \in U'$.*

Definition 2.27. *Let X be a compact metric space. We say that X is movable provided there exists $Z \in ANR$ and an embedding $e : X \to Z$ such that $e(X)$ is movable in Z.*

Let us notice that the property of being movable is an absolute property, that is if A is a movable set in some $ANR\ X$ and $j : A \to X'$ is an embedding into an $ANR\ X'$, then $j(A)$ is movable in X' (see [3] or [4]). We shall make use of the following result from [3].

Lemma 2.28. *Let X be an ANR and let $X_0 \subset X$ be a compact absolute approximative neighborhood retract in the sense of Clapp. Then X_0 is movable in X.*

Lemma 2.29. *(see* [3], [4]*) Let X and Y be compact metric spaces. If X or Y is not movable, then $X \times Y$ is also not movable.*

Theorem 2.30. *(*[8]*) Let X be a compact metric space of finite type. Then there exists $\varepsilon > 0$ such that for every two maps $f, g : Y \to X$, where Y is a Hausdorff space, the condition*

$$d(f(y), g(y)) < \varepsilon \text{ for each } y \in Y$$

implies $f_* = g_*$.

Let Q be a Hilbert cube and let $\{A_n\}_{n\in\mathbb{N}}$ be a family of compact sets such that:

(1) for any n $A_n \subset Q$,
(2) for any n $A_{n+1} \subset A_n$.

We observe that the family $R = \{H_*(A_n), (j^n_{n+1})_*\}_{n\in\mathbb{N}}$ is an inverse system, where $j^n_{n+1} : A_{n+1} \to A_n$ is an inclusion for any n. Let $\lim\limits_{\leftarrow} R$ be a limit of the inverse system R.

Theorem 2.31. *(see [6]) Let $i_n : \bigcap\limits_{n=1}^{\infty} A_n \to A_n$ be an inclusion for any n. A map*

$$i_* : H_*(\bigcap_{n=1}^{\infty} A_n) \to \lim_{\leftarrow} R \ \text{ given by}$$

$$i_*(a) = (i_{1*}(a), i_{2*}(a), ..., i_{n*}(a), ...) \ \text{ for each } \ a \in H_*(\bigcap_{n=1}^{\infty} A_n)$$

is a linear isomorphism.

Theorem 2.32. *Let Y_n be a compact and acyclic metric space for every n. Then $\prod\limits_{n=1}^{\infty} Y_n$ is compact and acyclic.*

Proof. Let us recall that every compact metric space can be embedded in the Hilbert cube. Because of that we can for each n identify a compact metric space Y_n with some closed subset of the Hilbert cube Q. Let $Y = \prod\limits_{n=1}^{\infty} Y_n$. It is clear that Y is a compact metric space. For each n we define the set $B_n \subset (Q \times Q \times ... \times Q \times ...) \approx Q$ given by

$$B_1 = Y_1 \times Q \times Q \times ... \times Q \times ...,$$

$$B_2 = Y_1 \times Y_2 \times Q \times ... \times Q \times ...,$$

$$...$$

$$B_n = Y_1 \times ... \times Y_n \times Q \times ... \times Q \times ...,$$

$$...$$

We observe that

(1) $B_{n+1} \subset B_n$ for each n,
(2) B_n is compact and acyclic (see 2.25) for each n,
(3) $\bigcap\limits_{n=1}^{\infty} B_n = \prod\limits_{n=1}^{\infty} Y_n$.

The family $S = \{H_*(B_n), (i^n_{n+1})_*\}_{n\in\mathbb{N}}$ is an inverse system, where $i^n_{n+1} : B_{n+1} \to B_n$ is an inclusion for any n. From 2.31 we get that Y is acyclic. □

3. Absolute neighborhood multi-retracts

Let us consider the classes of AMR and $ANMR$ spaces in a broader sense, namely in the context of locally convex spaces. To make things simple, we use the same denotations. Let us remind some notions and facts that can be found in the paper [19].

Definition 3.1. *(see* [19]*) A map $r\colon X \to Y$ of a space X onto a space Y is said to be an mr-map if there is an admissible map $\varphi\colon Y \multimap X$ such that $r \circ \varphi = Id_Y$.*

Definition 3.2. *(see* [19]*) A metric space X is called an absolute multi-retract (notation: $X \in AMR$) provided there exists a locally convex space E and an mr-map $r\colon E \to X$ from E onto X.*

Definition 3.3. *(see* [19]*) A metric space X is called an absolute neighborhood multi-retract (notation: $X \in$ANMR) provided there exists an open subset U of some locally convex space E and an mr-map $r\colon U \to X$ from U onto X.*

Definition 3.4. *A metric space X is called an absolute neighborhood retract in a broader sense (notation: $X \in ANR^{LC}$) provided there exists an open subset U of some locally convex space E and an r-map $r\colon U \to X$ from U onto X.*

Definition 3.5. *A metric space X is called an absolute retract in a broader sense (notation : $X \in AR^{LC}$) provided there exists a locally convex space E and an r-map $r\colon E \to X$ from E onto X.*

Theorem 3.6. *$ANR = ANR^{LC}$ and $AR = AR^{LC}$.*

Proof. It is clear that $ANR \subset ANR^{LC}$. Let $X \in ANR^{LC}$, then there exists an open set U of some locally convex space E and two continuous maps $r : U \to X$ and $s : X \to U$ such that $r \circ s = Id_X$. Consider a homeomorphism h mapping X onto a closed subset $h(X)$ of a metric space Y. Then $g = s \circ h^{-1}$ maps $h(X)$ into $U \subset E$ and so, by the theorem Dugundji there is a continuous extension $\widetilde{g}$ of g mapping Y into E. Let U' be the counter-image of U under $\widetilde{g}$. Then U' is a neighborhood of $h(X)$ in Y. Setting $r'(y) = h(r(\widetilde{g}(y)))$ for all $y \in U'$, we obtain a retraction map r' and from 2.13 $X \in ANR$. We prove the second part of the theorem in a similar way. $\square$

Theorem 3.7. *(see* [19]*) A space X is an $ANMR$ if and only if there exists a metric space Z and a Vietoris map $p : Z \to X$ which factors through an open subset U of some locally convex space E, i.e. there are two continuous maps α and β such that the following diagram*

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[dr, "\alpha"'] & X \\ & U \arrow[u, "\beta"'] \end{tikzcd}$$

is commutative.

Theorem 3.8. *(see* [19]*) Let $X \in AMR$, then X is acyclic.*

Proof. Let $r : E \to X$ be a mr-map, where E is a locally convex space, then there exists an admissible map $\varphi : X \multimap E$ such that $r \circ \varphi = Id_X$. Let $(p, q) \subset \varphi$. We observe that from 2.7 a linear map $q_* p_*^{-1} : H_*(X) \to H_*(E)$ is a monomorphism. Hence X is an acyclic space. $\square$

Theorem 3.9. *Let X be a compact metric space and $X \in ANMR$, then X is of finite type.*

Proof. From 3.7 we get: a space Z, maps $r : U \to X$, $q : Z \to U$ and a Vietoris map $p : Z \to X$ such that $r \circ q = p$, where U is an open subset in some locally convex space E. Hence $q_* : H_*(Z) \to H_*(U)$ is a monomorphism. We observe, that the space Z is compact and $H_*(Z) \approx H_*(X)$. Theorem 2.20 implies that for a compact set $K = q(Z) \subset U \subset E$ there exists a map $\pi_K : K \to U$ such that $\pi_K(K) \subset E^n$ and maps $\pi_K, i : K \to U$ are homotopic, where $E^n \subset E$ is an n-dimensional subspace of E and $i : K \to U$ is an inclusion. Let $i_1 : U \cap E^n \to U$ be an inclusion, $\widehat{q} : Z \to K$ given by $\widehat{q}(z) = q(z)$ for each $z \in Z$ and

$$t : Z \to U \cap E^n, \text{ given by } t(z) = \pi_K(\widehat{q}(z)) \text{ for each } z \in Z$$

then we have the following commutative diagram:

$$\begin{tikzcd} H_*(Z) \arrow[r, "q_*"] \arrow[dr, "t_*"'] & H_*(U) \\ & H_*(U \cap E^n). \arrow[u, "i_{1*}"'] \end{tikzcd}$$

In the above diagram we get that $i_{1*} \circ t_* = q_*$ and hence t_* is a monomorphism. From 2.9, for a compact set $K_1 = \pi_K(K) \subset U \cap E^n = V$ and for sufficiently small $\varepsilon > 0$ there exists a projection $p_\varepsilon : K_1 \to V$ such that $p_\varepsilon(K_1) \subset K_\varepsilon$ and maps $p_\varepsilon, i_2 : K_1 \to V$ are homotopic, where $i_2 : K_1 \to V$ is an inclusion and K_ε is a polyhedron of finite type such that $K_\varepsilon \subset V$. We have the following commutative diagram:

$$\begin{tikzcd} H_*(Z) \arrow[r, "t_*"] \arrow[dr, "r_*"'] & H_*(V) \\ & H_*(K_\varepsilon), \arrow[u, "i_{3*}"'] \end{tikzcd}$$

where $i_3 : K_\varepsilon \to V$ is an inclusion and $r : Z \to K_1$ given by $r(z) = p_\varepsilon(t(z))$ for each $z \in Z$. It is clear, that r_* is a monomorphism. Hence Z is a space of finite type. Since $H_*(Z) \approx H_*(X)$, therefore X is a space of finite type. □

4. Approximative absolute neighborhood multi-retracts

Definition 4.1. *A compact metric space X is called an approximative absolute neighborhood retract in a broader sense (notation: $X \in AANR^{LC}$) provided for each $\varepsilon > 0$ there exists an open subset U_ε of some locally convex space E_ε and maps $r_\varepsilon : U_\varepsilon \to X$, $s_\varepsilon : X \to U_\varepsilon$ such that for each $x \in X$*

$$d(r_\varepsilon(s_\varepsilon(x)), x) < \varepsilon.$$

Theorem 4.2. *$AANR_C = AANR^{LC}$.*

Proof. It is obvious that $AANR_C \subset AANR^{LC}$. Let $X \in AANR^{LC}$. Then for each $\varepsilon' > 0$ there exists an open subset $U_{\varepsilon'}$ of some locally convex space $E_{\varepsilon'}$ and maps $r_{\varepsilon'} : U_{\varepsilon'} \to X$, $s_{\varepsilon'} : X \to U_{\varepsilon'}$ such that for each $x \in X$

$$d(r_{\varepsilon'}(s_{\varepsilon'}(x)), x) < \varepsilon'. \tag{1}$$

Let $\varepsilon > 0$ and let $h : X \to Y$ be a homeomorphism such that $h(X)$ is a closed subset of a metric space Y. A metric space X is compact, hence there exists $\delta > 0$ such that for any $x, z \in X$ we have

$$(d(x,z) < \delta) \Rightarrow (d(h(x), h(z)) < \varepsilon). \tag{2}$$

For $\varepsilon' = \delta$ we define $g : h(X) \to U_\delta$ given by $g = s_\delta \circ h^{-1}$. From theorem Dugundji there is a continuous extension $\widetilde{g}$ of g mapping Y into E_δ. Let U'_ε be the counter-image of U_δ under $\widetilde{g}$ and let

$$r'_\varepsilon : U'_\varepsilon \to h(X) \quad r'_\varepsilon = h \circ r_\delta \circ \widetilde{g}.$$

From (1) and (2) for any $y \in h(X)$ we have

$$d(r'_\varepsilon(y), y) = d(h(r_\delta(s_\delta(x)), h(x)) < \varepsilon,$$

and from 2.16 $X \in AANR_C$. □

Definition 4.3. *Let X be a compact space. We shall say that X is an approximative ANMR (we write $X \in AANMR$) provided that for any $\varepsilon > 0$ there exists a locally convex space E_ε and an open set $U_\varepsilon \subset E_\varepsilon$, a map $r_\varepsilon : U_\varepsilon \to X$ and an admissible map $\varphi_\varepsilon : X \multimap U_\varepsilon$ such that for any $x \in X$*

$$r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon),$$

where $B(x,\varepsilon)$ is an open ball in X.

Definition 4.4. *Let X be a compact space. We shall say that X is an approximative AMR (we write $X \in AAMR$) provided that for any $\varepsilon > 0$ there exists a locally convex space E_ε, a map $r_\varepsilon : E_\varepsilon \to X$ and an admissible map $\varphi_\varepsilon : X \multimap E_\varepsilon$ such that for any $x \in X$*

$$r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon),$$

where $B(x,\varepsilon)$ is an open ball in X.

Theorem 4.5. *A space X is an $AANMR$ if and only if for any $\varepsilon > 0$ there exists a space Z_ε, a Vietoris map $p_\varepsilon : Z_\varepsilon \to X$, a locally convex space E_ε, an open set $U_\varepsilon \subset E_\varepsilon$, and maps $r_\varepsilon : U_\varepsilon \to X$, $q_\varepsilon : Z_\varepsilon \to U_\varepsilon$ such that for any $z \in Z_\varepsilon$*

$$d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon.$$

Proof. Assume that $X \in AANMR$. We have a map $r'_\varepsilon : U'_\varepsilon \to X$, an admissible map $\varphi'_\varepsilon : X \multimap U'_\varepsilon$ such that for any $x \in X$ $r'_\varepsilon(\varphi'_\varepsilon(x)) \subset B(x, \varepsilon)$, where the set U'_ε is open in some locally convex space E'_ε. Let $(p'_\varepsilon, q'_\varepsilon) \subset \varphi'_\varepsilon$. From the definition of an admissible multivalued map we get a space Z'_ε such that:

$$X \xleftarrow{p'_\varepsilon} Z'_\varepsilon \xrightarrow{q'_\varepsilon} U'_\varepsilon$$

and for any $x \in X$ $q'_\varepsilon((p'_\varepsilon)^{-1}(x)) \subset \varphi'_\varepsilon(x)$. We define $Z_\varepsilon = Z'_\varepsilon$, $U_\varepsilon = U'_\varepsilon$, $p_\varepsilon = p'_\varepsilon$, $r_\varepsilon = r'_\varepsilon$ and $q_\varepsilon = q'_\varepsilon$. We will show that for any $z \in Z_\varepsilon$

$$d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon.$$

Let $z \in Z_\varepsilon$, then there exists $x \in X$ such that $z \in p_\varepsilon^{-1}(x)$. We observe that the assumption $d(r_\varepsilon(q_\varepsilon(z)), x) < \varepsilon$ implies $d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon$. Assume now that there exists a metric space Z'_ε, maps $r'_\varepsilon : U'_\varepsilon \to X$, $q'_\varepsilon : Z'_\varepsilon \to U'_\varepsilon$ and a Vietoris map $p'_\varepsilon : Z'_\varepsilon \to X$ such that for any $z \in Z'_\varepsilon$ $d(r'_\varepsilon(q'_\varepsilon(z)), p'_\varepsilon(z)) < \varepsilon$. Let $Z_\varepsilon = Z'_\varepsilon$, $U_\varepsilon = U'_\varepsilon$, $p_\varepsilon = p'_\varepsilon$, $q_\varepsilon = q'_\varepsilon$ and $r_\varepsilon = r'_\varepsilon$. We define an admissible map $\varphi_\varepsilon : X \multimap U_\varepsilon$ given by $\varphi_\varepsilon(x) = q_\varepsilon(p_\varepsilon^{-1}(x))$ for each $x \in X$. We will show that for any $x \in X$

$$r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon).$$

Let $x \in X$ and $z \in p_\varepsilon^{-1}(x)$, then from the assumption we have

$$d(r_\varepsilon(q_\varepsilon(z)), x) = d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon.$$

Hence, $r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon)$ and the proof is complete. $\square$

Similarly, we can prove the following theorem.

Theorem 4.6. *A space X is an $AAMR$ if and only if for any $\varepsilon > 0$ there exists a space Z_ε, a Vietoris map $p_\varepsilon : Z_\varepsilon \to X$, a locally convex space E_ε and maps $r_\varepsilon : E_\varepsilon \to X$, $q_\varepsilon : Z_\varepsilon \to E_\varepsilon$ such that for any $z \in Z_\varepsilon$*

$$d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon.$$

The next theorem is the conclusion of the theorems 2.30 and 4.6.

Theorem 4.7. *Let $X \in AAMR$ and let X be of finite type. Then X is acyclic.*

It is clear that $ANMR \subset AANMR$ and $AANR_C \subset AANMR$. The example below shows that these inclusions cannot be reversed.

Example 4.8. *Let $C \in AANR_C$ such that C is not of finite type (see* [5]*) , Y be a compact space such that $Y \in ANMR$ and Y is not movable (see* [19]*). We show that We show that $(C \times Y) \notin ANMR$, $(C \times Y) \notin AANR_C$, but $(C \times Y) \in AANMR$. It is clear that $C \times Y$ is a metric space $(C \times Y, d)$, where d is a product metric. From 2.29 and 2.28 we get that $(C \times Y) \notin AANR_C$. We prove now that $(C \times Y) \notin ANMR$. Assume on the contrary that $(C \times Y) \in ANMR$, then there exists a map $r : U \to C \times Y$, an admissible map $\varphi : C \times Y \multimap U$ such that $r \circ \varphi = Id_{C \times Y}$, where U is an open set in some locally convex space E. Let $\pi : C \times Y \to C$ be a map given by $\pi(x, y) = x$ for any $(x, y) \in (C \times Y)$ and let $s : C \to C \times Y$ be a map given by $s(x) = (x, y_0)$ for any $x \in C$, where $y_0 \in Y$ is a stationary point. We define a map $r' : U \to C$ $r' = \pi \circ r$ and an admissible map $\varphi' : C \multimap U$ $\varphi' = \varphi \circ s$. We observe that*

$$r' \circ \varphi' = (\pi \circ r) \circ (\varphi \circ s) = \pi \circ (r \circ \varphi) \circ s = \pi \circ s = Id_C,$$

hence $C \in ANMR$, but it contradicts theorem 3.9 since C is not of finite type. We prove that $(C \times Y) \in AANMR$. The space $C \in AANR_C$, therefore for each $\varepsilon > 0$ there exists an open subset U'_ε in some locally convex space E_ε and maps $r'_\varepsilon : U'_\varepsilon \to C$, $s_\varepsilon : C \to U'_\varepsilon$ such that for each $c \in C$ $d(r'_\varepsilon(s_\varepsilon(c)), c) < \varepsilon$. The space $Y \in ANMR$, so there exists an open set V in some locally convex space E, a metric space Z_1, and maps $r' : V \to Y$, $q' : Z_1 \to V$ such that $r' \circ q' = p'$ is a Vietoris map. Let $U_\varepsilon = U'_\varepsilon \times V \subset E_\varepsilon \times E$ and let $Z_\varepsilon = C \times Z_1$. We define maps:

$$r_\varepsilon : U_\varepsilon \to C \times Y, \quad q_\varepsilon : Z_\varepsilon \to U_\varepsilon, \quad \text{a Vietoris map } p_\varepsilon : Z_\varepsilon \to C \times Y$$

given by

$$r_\varepsilon(x, y) = (r'_\varepsilon(x), r'(y)) \text{ for each } (x, y) \in U_\varepsilon,$$

$$q_\varepsilon(c, z) = (s_\varepsilon(c), q'(z)) \text{ for each } (c, z) \in (C \times Z_1) = Z_\varepsilon,$$

$$p_\varepsilon(c, z) = (c, p'(z)) \text{ for each } (c, z) \in (C \times Z_1) = Z_\varepsilon.$$

It is clear that maps r_ε, q_ε and p_ε satisfy the assumptions of 4.5.

Theorem 4.9. *Let $X_n \in AANMR$ for any $n \in \mathbb{N}$, then a space $X = X_1 \times X_2 \times ... \times X_n \times ... = \prod_{n=1}^{\infty} X_n$ is $AANMR$.*

Proof. Let (X_n, d_n) be a metric space and $X_n \in AANMR$ for any $n \in \mathbb{N}$. Assume that for any n and for all $x_n, y_n \in X_n$ $d_n(x_n, y_n) \leq 1$. We define the metric in a space X given by:

$$d(x,y) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n},$$

where $x = (x_1, x_2, ..., x_n, ...)$, $y = (y_1, y_2, ..., y_n, ...)$. Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2}$. From the definition of $AANMR$ for any n we get $r_\delta^n : U_\delta^n \to X_n$, $q_\delta^n : Z_\delta^n \to U_\delta^n$ and a Vietoris map $p_\delta^n : Z_\delta^n \to X_n$ such that for all $z_n \in Z_\delta^n$ $d_n(r_\delta^n(q_\delta^n(z_n)), p_\delta^n(z_n)) < \delta$, where $U_\delta^n \subset E_\delta^n$ is an open subset in some locally convex space. Let $E_\varepsilon = \prod_{n=1}^{\infty} E_\delta^n$ (from 2.21 E_ε is a locally convex space) and let $Z_\varepsilon = \prod_{n=1}^{\infty} Z_\delta^n$. We observe that the space Z_ε is compact. There exists a natural number n_0 such that for any $n \geq n_0$

$$\sum_{n=n_0+1}^{\infty} \frac{d_n(x_n, y_n)}{2^n} < \delta = \frac{\varepsilon}{2}.$$

We define an open set in the space E_ε given by:

$$U_\varepsilon = \prod_{i=1}^{n_0} U_\delta^i \times \prod_{n=n_0+1}^{\infty} E_\delta^n.$$

Let $r_\varepsilon : U_\varepsilon \to X$ be given by:

$$r_\varepsilon(x_1, x_2, ..., x_n, ...) = (r_\delta^1(x_1), r_\delta^2(x_2), ..., r_\delta^{n_0}(x_{n_0}), y_{n_0+1}, ..., y_m, ...)$$

for each $(x_1, x_2, ..., x_n, ...) \in U_\varepsilon$, where $y_m \in X_m$ for all $m > n_0$ are stationary points and let $q_\varepsilon : Z_\varepsilon \to U_\varepsilon$ be given by:

$$q_\varepsilon(z_1, z_2, ..., z_n, ...) = (q_\delta^1(z_1), q_\delta^2(z_2), ..., q_\delta^n(z_n), ...)$$

for each $(z_1, z_2, ..., z_n, ...) \in Z_\varepsilon$. A Čech homology theory is continuous, therefore a map $p_\varepsilon : Z_\varepsilon \to X$ given by

$$p_\varepsilon(z_1, z_2, ..., z_n, ...) = (p_\delta^1(z_1), p_\delta^2(z_2), ..., p_\delta^n(z_n), ...)$$

for each $(z_1, z_2, ..., z_n, ...) \in Z_\varepsilon$ is a Vietoris map (see theorem 2.32) since

$$p_\varepsilon^{-1}(x_1, x_2, ..., x_n, ...) = \prod_{n=1}^{\infty} (p_\delta^n)^{-1}(x_n)$$

for any $x = (x_1, x_2, ..., x_n, ...) \in X$. It is clear that the maps r_ε, q_ε and p_ε satisfy the assumptions of 4.5. $\square$

We know that (see [19]) the Cartesian product of the finite number of compact metric spaces of $ANMR$ type is also of the $ANMR$ type. From 3.9 and 4.9 it follows that the Cartesian product of the infinite number of compact metric spaces does not have to be of $ANMR$ type (Cartesian product of the infinite number of compact metric spaces out of which every space is of finite type, does not have to be of finite type), but it certainly is of $AANMR$ type. This shows that the class of spaces of $AANMR$ type is considerably larger than the class of compact spaces of $ANMR$ type.

5. Fixed point result

Theorem 5.1. *Let $X \in AANMR$ and let X be of finite type, then an admissible map $\psi : X \multimap X$ is a Lefschetz map.*

Proof. From 4.5, there exists a locally convex space E_ε, an open set $U_\varepsilon \subset E_\varepsilon$, a metric space Z_ε, maps $r_\varepsilon : U_\varepsilon \to X$, $q_\varepsilon : Z_\varepsilon \to U_\varepsilon$ and a Vietoris map $p_\varepsilon : Z_\varepsilon \to X$ such that for each $z \in Z_\varepsilon$ $d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon$. Let $(p, q) \subset \psi$ and let and let $\varphi_\varepsilon : X \multimap U_\varepsilon$ given by $\varphi_\varepsilon(x) = q_\varepsilon(p_\varepsilon^{-1}(x))$ for each $x \in X$. From 2.7 there exists a pair $(\widetilde{p}, \widetilde{q}) \subset \psi \circ r_\varepsilon$ such that

$$\widetilde{q}_* \widetilde{p}_*^{-1} = q_* p_*^{-1} \circ r_{\varepsilon *}$$

and there exists a pair $(\widehat{p}, \widehat{q}) \subset \varphi_\varepsilon \circ \psi \circ r_\varepsilon$ such that

$$\widehat{q}_* \widehat{p}_*^{-1} = q_{\varepsilon *} p_{\varepsilon *}^{-1} \circ q_* p_*^{-1} \circ r_{\varepsilon *}.$$

A space X is of finite type, therefore from 2.30 there exists a real number $\varepsilon_1 > 0$ such that for each $0 < \varepsilon \le \varepsilon_1$ we get

$$r_{\varepsilon *} \circ q_{\varepsilon *} = p_{\varepsilon *}.$$

We have the following commutative diagram:

$$\begin{tikzcd}
H_*(X) \arrow[r, "q_{\varepsilon *} p_{\varepsilon *}^{-1}"] & H_*(U_\varepsilon) \\
H_*(X) \arrow[u, "q_* p_*^{-1}"] \arrow[r, "q_{\varepsilon *} p_{\varepsilon *}^{-1}"] & H_*(U_\varepsilon). \arrow[ul, "\widetilde{q}_* \widetilde{p}_*^{-1}"'] \arrow[u, "\widehat{q}_* \widehat{p}_*^{-1}"']
\end{tikzcd}$$

A map $q_* p_*^{-1}$ is a Leray endomorphism, since a space X is of finite type. Assume that $\lambda(\psi) \neq \{0\}$, then from the above diagram $\Lambda(\varphi_\varepsilon \circ \psi \circ r_\varepsilon) \neq \{0\}$. From 2.22 we get $x_\varepsilon \in U_\varepsilon$ such that $x_\varepsilon \in (\varphi_\varepsilon \circ \psi \circ r_\varepsilon)(x_\varepsilon)$. Hence, there exists $z_\varepsilon \in p_\varepsilon^{-1}(\psi(r_\varepsilon(x_\varepsilon)))$ such that

$$r_\varepsilon(x_\varepsilon) = r_\varepsilon(q_\varepsilon(z_\varepsilon)).$$

Let $y_\varepsilon = p_\varepsilon(z_\varepsilon) \in \psi(r_\varepsilon(x_\varepsilon))$, then

$$d(r_\varepsilon(x_\varepsilon), y_\varepsilon) = d(r_\varepsilon(q_\varepsilon(z_\varepsilon)), p_\varepsilon(z_\varepsilon)) < \varepsilon.$$

We observe that for each $\varepsilon > 0$ $r_\varepsilon(x_\varepsilon)$ is the ε-fixed point of a map ψ. The space X is compact, hence ψ has a fixed point. $\square$

From 5.1 we get the following theorem.

Theorem 5.2. *Assume that $X \in AAMR$. Let X be of finite type and let $\psi : X \multimap X$ be an admissible map. Then $Fix(\psi) \neq \emptyset$.*

Remark 5.3. *From 3.8 we get that the AMR-type spaces are acyclical. And in 3.9 we get that a compact $ANMR$ space is of finite type. We proved in section 4 that a compact $AANMR$ space does not have to be of finite type. According to that, the following diagram shows*

$$\begin{array}{ccccc} AR & \xrightarrow{\subset} & ANR & \xrightarrow{\subset} & AANR_C \\ \big\downarrow \cap & & \big\downarrow \cap & & \big\downarrow \cap \\ AMR & \xrightarrow{\subset} & ANMR & \xrightarrow{\subset} & AANMR, \end{array}$$

that horizontal inclusions cannot be reversed. Notice that vertical inclusions can neither be reversed because metric spaces of $AANR_C$ type are movable (see 2.28) whereas metric spaces of AMR type don't have to be movable (see [19]*).*

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