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ERGODIC PROPERTIES OF A PARTICULAR AMENABLE SEMIGROUP OF MAPPINGS IN A BANACH SPACE

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Abstract. We prove that if S is an amenable semigroup and $\varphi = \{T_t : t \in S\}$ is a semigroup of mappings on a nonempty weakly compact, convex subset C of a Banach space E, generated by $\{T_t : t \in A \subseteq S\}$, such that for each $t \in A$, T_t is of type (γ) and $D(\overline{co}F_{1/n}(T_t), F(T_t)) \to 0$, as $n \to \infty$, then $F(\varphi)$ of common fixed points of φ is nonempty and there exists a retraction P of type (γ) from C onto $F(\varphi)$, such that $PT_t = T_tP = P$ for each $t \in S$, and $Px \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$. The compactness of C concludes such imposed conditions.
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1. Introduction

The first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space was established by Baillon [2]: Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n} T^k x$$

converge weakly to some $y \in F(T)$. In Baillon's theorem, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that

$PT^n = T^nP = P$ for all positive integers n and $Px \in \overline{co}\{T^nx : n = 1, 2, \dots\}$ for each $x \in C$.

Takahashi [17] proved the existence of such retractions for noncommutative semigroups of nonexpansive mappings in a Hilbert space H: If S is an amenable semigroup, C is a closed, convex subset of H and $\varphi = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that the set $F(\varphi)$ of common fixed points of φ is nonempty, then there exists a nonexpansive retraction R from C onto $F(\varphi)$ such that $RT_t = T_tR = R$ for each $t \in S$ and $Rx \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$. Rodé [11] found a sequence of means on the semigroups, generalizing the Cesàro means on the positive integers, such that the corresponding sequence of mappings converges to an ergodic retraction onto the set of common fixed points. These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable for commutative semigroups by Hirano, Kido and Takahashi [8] and for amenable semigroups by Lau, Shioji and Takahashi [9]. The existence of an ergodic retraction for amenable nonexpansive semigroups in strictly convex Banach space was studied in [13]. For some related results, we refer to the works in [12, 15, 16].

In this paper, we prove the existence of an ergodic retraction of type (γ) for an amenable semigroup $\varphi = \{T_t : t \in S\}$ of mappings on a weakly compact, convex subset C of a general Banach space, generated by $\{T_t : t \in A \subseteq S\}$, such that for each $t \in A$, T_t is of type (γ) and $D(\overline{co}F_{1/n}(T_t), F(T_t)) \to 0$, as $n \to \infty$, where D is the well known Hausdorff metric. The compactness of C concludes such imposed conditions (see [10]).

2. Preliminaries

Let E be a real Banach space and let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx-Ty\| \le \|x-y\|$ for each $x, y \in C$. We denote by $F_\varepsilon(T)$ the ε-approximate fixed points of T, i.e. $F_\varepsilon(T) = \{x \in C : \|x - Tx\| \le \varepsilon\}$. If C is bounded, then $F_\varepsilon(T) \neq \phi$ for each $\varepsilon > 0$ (see [3]). Let E^* be the topological dual of E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$ or $x^*(x)$. For a subset A of E, we denote by coA the convex hull of A; we also denote by $\mathrm{dis}(x, A)$ the distance from x to A for each x in E. We denote by Γ the set of strictly increasing, continuous convex functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. For each $\gamma \in \Gamma$, a mapping $T : C \to C$ is said to be of type (γ), if for every $x, y \in C$ and

$\lambda \in [0,1]$, $\gamma\left(\|\lambda Tx+(1-\lambda)Ty-T(\lambda x+(1-\lambda)y)\|\right) \le \|x-y\|-\|Tx-Ty\|$. Obviously, if T is of type (γ) for some $\gamma \in \Gamma$, then T is nonexpansive. Moreover if C is also weakly compact, then $F(T)$ is a nonempty closed convex set (see [7]). If C is compact and E is a strictly convex Banach space, then every nonexpansive mapping $T : C \to C$ is of type (γ) (see [1, 4, 5]). Throughout this paper, S is a semigroup and $B(S)$ is the space of all bounded real-valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$ respectively. Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be l_s-and r_s-invariant, i.e. $l_s(X) \subset X$ and $r_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be right (resp. left) invariant if $\mu(r_s f) = \mu(f)$ (resp. $\mu(l_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be invariant if it is both left and right invariant. X is said to be (right) amenable if there is an (right) invariant mean on X. As is well known, $B(S)$ is amenable when S is a commutative semigroup [6]. A net $\{\mu_\alpha\}$ of means on X is said to be asymptotically invariant if $\lim_\alpha(\mu_\alpha(l_s f) - \mu_\alpha(f)) = 0$ and $\lim_\alpha(\mu_\alpha(r_s f) - \mu_\alpha(f)) = 0$ for each $f \in X$ and $s \in S$, and it is said to be strongly regular if, $\lim_\alpha \|l_s^*\mu_\alpha - \mu_\alpha\| = 0$ and $\lim_\alpha \|r_s^*\mu_\alpha - \mu_\alpha\| = 0$ for each $s \in S$, where l_s^* and r_s^* are the adjoint operators of l_s and r_s, respectively. A family $\varphi = \{T_s : s \in S\}$ of mappings from C into itself is said to be a semigroup on C if $T_{ts} = T_t T_s$ for each $t, s \in S$. We denote by $F(\varphi)$ the set of common fixed points of φ.

3. Some Lemmas

In [10], it is proved that if T is a nonexpansive mapping on a nonempty compact convex subset C of a Banach space, then

$$D(\overline{co}F_{1/n}(T), F(T)) \to 0, \text{ as } n \to \infty;$$

as well as, are given examples in l^∞ of mappings with noncompact domains that enjoy the above convergence property (D is the Hausdorff metric).
The following two lemmas which we need in the sequel, are proved in [10].

Lemma 3.1. [10] *Let C be a nonempty closed, convex subset of a Banach space E and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \varnothing$ and*

$D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$.

The class of mappings of type (γ) has proved to be very useful in connection with weak approximation of fixed points.

Lemma 3.2. [10] *Let C be a nonempty closed, convex bounded subset of a Banach space E and $T : C \to C$ be a mapping of type (γ) such that $F(T) \neq \varnothing$ and $D(\overline{co}F_{\frac{1}{n}}(T), F(T)) \to 0$, as $n \to \infty$. Then $\lim_n \|S_n(y) - TS_n(y)\| = 0$, uniformly in $y \in C$, where $S_n = \frac{1}{n}(I + T + \cdots + T^{n-1})$.*

The following lemma is proved in [14, Corollary 2.4]:

Lemma 3.3. *Assume that C is a nonempty convex subset of a normed vector space and $T_1, T_2, ..., T_n : C \to C$ are mappings of type (γ). Then $T_1...T_n$ is a mapping of type (γ_0), where $\gamma_0(t) = n\gamma(\frac{t}{n})$.*
If, in addition, for some $k > 0$, $\gamma(t) = kt$, $\forall t \geq 0$, then $T_1...T_n$ is a mapping of type (γ).

We denote by Γ_1 the subset of Γ consisting of all members γ of Γ for which $\gamma(t) = kt$, $\forall t \geq 0$, for some $k > 0$. By considering Lemma 3.3, it is easy to see that the family of mappings on C of type (γ), with $\gamma \in \Gamma_1$ is a semigroup. We refer to [14] to see more classes of this kind of semigroups.

The following result which we need is well known (see [17, 8]).

Lemma 3.4. *Let f be a function of S into E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of $B(S)$ containing all the functions $t \to < f(t), x^* >$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$< f_\mu, x^* >= \mu_t < f(t),\ x^* >$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X then

$$\int f(t)d\mu(t) \in \overline{co}\{f(t) : t \in S\}.$$

We can write f_μ by $\int f(t)d\mu(t)$.

As a direct consequence of Lemma 3.4, we have the following lemma (see [9, Lemma 1] or [13, Lemma 3.2]).

Lemma 3.5. *Let S be a semigroup, C be a nonempty closed convex subset of a Banach space E, $\varphi = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that weak closure of $\{T_t x : t \in S\}$ is weakly compact for each $x \in C$, X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \mapsto \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$, and μ be a mean on X. If we write $T_\mu x$ instead of $\int T_t x d\mu(t)$, then*
(i) *T_μ is a nonexpansive mapping from C into itself;*
(ii) *$T_\mu x = x$ for each $x \in F(\varphi)$, the common fixed points of φ;*
(iii) *$T_\mu x \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$;*
(iv) *If X is r_s-invariant for each $s \in S$ and μ is right invariant, then $T_\mu T_t = T_\mu$ for each $t \in S$.*

4. Ergodic theorems

The following theorem is our main result.

Theorem 4.1. *Let C be a nonempty weakly compact convex subset of a Banach space E, $\varphi = \{T_t : t \in S\}$ be a semigroup of mappings on C generated by $\{T_t : t \in A \subseteq S\}$, and let for each $t \in A$, T_t is of type (γ) and $D(\overline{co}F_{\frac{1}{n}}(T_t), F(T_t)) \to 0$, as $n \to \infty$. Let X be an invariant subspace of $B(S)$ such that $1 \in X$ and the mapping $t \mapsto \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. If X is amenable, then $F(\varphi) \neq \phi$ and there exists a nonexpansive retraction P from C onto $F(\varphi)$, such that $PT_t = T_t P = P$ for each $t \in S$, and $Px \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$. Moreover, if for a fixed $\gamma_1 \in \Gamma_1$ and every $t \in A$, T_t is of type (γ_1), then P is of type (γ_1).*

Proof. Assume that μ is an invariant mean on X. Consider $t \in A$, $x \in C$ and $\varepsilon > 0$. By Lemma 3.1, there exists $\delta > 0$ such that

$$\overline{co}F_\delta(T_t) \subset F_\varepsilon(T_t).$$

By Lemma 3.2, there also exists a natural number N such that

$$\|\frac{1}{N+1}\sum_{i=0}^{N}(T_t)^i(T_s x) - T_t\Big(\frac{1}{N+1}\sum_{i=0}^{N}(T_t)^i(T_s x)\Big)\| \leq \delta$$

for all $s \in S$. Since μ is left invariant, we get

$$T_\mu x = \int T_s x d\mu(s) = \frac{1}{N+1}\sum_{i=0}^{N}\int T_{t^i s} x \, d\mu(s)$$

$$= \int \frac{1}{N+1} \sum_{i=0}^{N} (T_t)^i (T_s x) d\mu(s)$$

$$\in \overline{co}\{\frac{1}{N+1} \sum_{i=0}^{N} (T_t)^i (T_s x) : s \in S\} \subset \overline{co} F_\delta(T_t) \subset F_\varepsilon(T_t).$$

Since $\varepsilon > 0$ was arbitrary, $T_t T_\mu x = T_\mu x$, $\forall t \in A$. Therefore $T_t T_\mu = T_\mu T_t = T_\mu$ for each $t \in S = \langle A \rangle$ and $T_\mu x \in \overline{co}\{T_t x : t \in S\}$ from Lemma 3.5. Now, by taking $P = T_\mu$, the proof of the first part of the theorem is completed.

If for a fixed $\gamma_1 \in \Gamma_1$ and all $t \in A$, T_t is of type (γ_1), then by applying Lemma 3.3, φ is a semigroup of mappings of type (γ_1). We prove that T_μ is also of type (γ_1). Consider $x, y \in C$, $0 < \lambda < 1$ and $x^* \in J(\lambda T_\mu x + (1-\lambda) T_\mu y - T_\mu(\lambda x + (1-\lambda) y))$. Then

$$\|\lambda T_\mu x + (1-\lambda) T_\mu y - T_\mu(\lambda x + (1-\lambda) y)\|^2$$

$$= \langle \lambda T_\mu x + (1-\lambda) T_\mu y - T_\mu(\lambda x + (1-\lambda) y), x^* \rangle$$

$$= \mu_t \langle \lambda T_t x + (1-\lambda) T_t y - T_t(\lambda x + (1-\lambda) y), x^* \rangle$$

$$\leq \|\lambda T_\mu x + (1-\lambda) T_\mu y - T_\mu(\lambda x + (1-\lambda) y)\|$$

$$\times \sup_t \|\lambda T_t x + (1-\lambda) T_t y - T_t(\lambda x + (1-\lambda) y)\|.$$

Hence,

$$\gamma_1(\|\lambda T_\mu x + (1-\lambda) T_\mu y - T_\mu(\lambda x + (1-\lambda) y)\|)$$

$$\leq \sup_t \gamma_1(\|\lambda T_t x + (1-\lambda) T_t y - T_t(\lambda x + (1-\lambda) y)\|) \leq \|x - y\| - \inf_t \|T_t x - T_x y\|.$$

On the other hand, by using Lemma 3.5,

$$\|T_\mu x - T_\mu y\| = \|T_\mu T_t x - T_\mu T_t y\| \leq \|T_t x - T_t y\| \ (\forall t).$$

Hence,

$$\|T_\mu x - T_\mu y\| \leq \inf_t \|T_t x - T_t y\|.$$

Therefore,

$$\gamma_1(\|\lambda T_\mu x + (1-\lambda) T_\mu y - T_\mu(\lambda x + (1-\lambda) y)\|) \leq \|x - y\| - \|T_\mu x - T_\mu y\|,$$

and, we have shown that T_μ is of type (γ_1). $\square$

Corollary 4.2. *Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\varphi = \{T_t : t \in S\}$ be a semigroup of mappings on C. Let X be an invariant subspace of $B(S)$ such that $1 \in X$ and the mapping $t \mapsto \langle T_t x, x^* \rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. If X is amenable, then $F(\varphi) \neq \phi$ and there exists a nonexpansive retraction P from C onto $F(\varphi)$, such that $PT_t = T_t P = P$ for each $t \in S$, and $Px \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$.*

By studying the proofs of lemmas in [9] and Theorems 3, 4, 5 and 6 of [9] we can easily see that they all hold in our assumptions. So by combining them with Theorem 4.1 we get the following results that we give without their proofs:

Corollary 4.3. *Let C, E, S, φ and X be as in Theorem 4.1.*
(i) *If X is amenable and the norm of E is Frechet differentiable, then the retraction P in Theorem 4.1 is unique. Furthermore, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on X, then for each $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to Px.*
(ii) *Let $\{\mu_\alpha\}$ be a strongly regular net of means on X. Then for each $t \in S$, $\lim_\alpha \|T_{\mu_\alpha} T_t x - T_{\mu_\alpha} x\| = 0$ and $\lim_\alpha \|T_t T_{\mu_\alpha} x - T_{\mu_\alpha} x\| = 0$ uniformly for $x \in C$. Moreover, if S is commutative and the norm of E is Frechet differentiable, then the same conclusion in Theorem 4.1 holds, and for each $x \in C$, $\{T_{\mu_\alpha} T_t x\}$ converges weakly to Px uniformly in $t \in S$.*
(iii) *If $B(S)$ is amenable, then there exists a net $\{A_\alpha\}$ of finite averages of φ such that for each $t \in S$, $\lim_\alpha \|A_\alpha T_t x - A_\alpha x\| = 0$ and $\lim_\alpha \|T_t A_\alpha x - A_\alpha x\| = 0$ uniformly for $x \in C$.*

Corollary 4.4. *Let C, E, φ and X be as in Theorem 4.1 and $\{\mu_\alpha\}$ be a strongly regular net of means on X. Assume that the norm of E is Frechet differentiable and X is amenable. If there exists one $t \in A$ such that $F(T_t)$ is compact, then $T_{\mu_\alpha} x$ converges strongly to Px for each $x \in C$.*

Proof. Let $x \in C$. Then from Theorem 4.1 and Corollary 4.3 (ii), $T_{\mu_\alpha} x$ converges weakly to Px and $\lim_\alpha \|T_t T_{\mu_\alpha} x - T_{\mu_\alpha} x\| = 0$ for each $t \in S$. Now, since $F(T_t)$ is compact for one t in A, and since $D(\overline{co} F_{\frac{1}{n}}(T_t), F(T_t)) \to 0$, it is easy to verify that $T_{\mu_\alpha} x$ converges strongly to Px. □

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