REMARKS ON ULAM STABILITY OF THE OPERATORIAL EQUATIONS

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Abstract. In this paper we present four types of Ulam stability for operatorial equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. The relations of Ulam stability with the $c$-weakly Picard operators are also studied. Some examples and counterexamples are given.

Key Words and Phrases: Ulam-Hyers stability, Ulam-Hyers-Rassias stability, fixed point equation, coincidence point equation, integral equation, differential equation, difference equation, operatorial inclusions.

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1. Introduction

The Ulam stability (Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin,...) of various functional equations have been investigated by many authors (see [14], [15], [7], [9], [2], [1], [10], [13], [21], [25], [26],...). There are some results for differential equations ([1], [16], [18], [19], [20], [31],...), integral equations ([17], [30],...) and for difference equations ([3], [4], [5], [23], [24],...).

The aim of this paper is to present four types of Ulam stability for the operatorial equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. The relations of Ulam stability with the $c$-weakly Picard operators ([28], [33], [22], [32]) are studied. Some examples and counterexamples are also given. The plan of the paper is the following:
1. Introduction
2. Ulam-Hyers stability via weakly Picard operators
3. Ulam-Hyers stability of difference equations
4. Generalized Ulam-Hyers stability of a fixed point equation with non-self operator
5. Ulam-Hyers-Rassias stability of the fixed point equations in a space of functions
6. Ulam stability of the coincidence point equations
7. Ulam stability of the operatorial inclusions.

2. ULAM-HYERS STABILITY VIA WEAKLY PICARD OPERATORS

We begin our considerations with some notions and results from weakly Picard operator theory (see [28]; see also [32], pp. 119-126).

Let \((X, d)\) be a metric space and \(f : X \to X\) an operator. We denote by \(F_f := \{x \in X \mid f(x) = x\}\), the fixed point set of the operator \(f\). By definition \(f\) is weakly Picard operator if the sequence of successive approximations, \(f^n(x)\), converges for all \(x \in X\) and the limit (which may depend on \(x\)) is a fixed point of \(f\).

If \(f\) is weakly Picard operator then we consider the operator \(f^\infty : X \to X\) defined by \(f^\infty(x) := \lim_{n \to \infty} f^n(x)\). It is clear that \(f^\infty(X) = F_f\). Moreover, \(f^\infty\) is a set retraction of \(X\) to \(F_f\).

If \(f\) is weakly Picard operator and \(F_f = \{x^*\}\), then by definition \(f\) is a Picard operator. In this case \(f^\infty\) is the constant operator, \(f^\infty(x) = x^*, \forall x \in X\).

The following class of weakly Picard operators is very important in our considerations.

**Definition 2.1.** Let \(f : X \to X\) be an weakly Picard operator and \(c > 0\) a real number. By definition the operator \(f\) is c-weakly Picard operator if

\[d(x, f^\infty(x)) \leq cd(x, f(x)), \forall x \in X.\]

**Example 2.1.** Let \((X, d)\) be a complete metric space and \(f : X \to X\) an operator with closed graphic. We suppose that \(f\) is graphic \(\alpha\)-contraction, i.e.,

\[d(f^2(x), f(x)) \leq \alpha d(x, f(x)), \forall x \in X.\]
Then \( f \) is a \( c \)-weakly Picard operator, with \( c = (1 - \alpha)^{-1} \).

**Example 2.2.** Let \((X, d)\) be a complete metric space, \(\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) a function and \(f : X \rightarrow X\) an operator with closed graphic. We suppose that:

(i) \(f\) is a \(\varphi\)-Caristi operator, i.e.,
\[
d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \quad \forall \ x \in X;
\]

(ii) there exists \(c > 0\) such that
\[
\varphi(x) \leq cd(x, f(x)), \quad \forall \ x \in X.
\]

Then \( f \) is a \( c \)-weakly Picard operator.

**Example 2.3.** (generic example). Let \((X, d)\) be a metric space, \(f : X \rightarrow X\) an operator and \(X = \bigcup_{i \in I} X_i\) a partition of \(X\). We suppose that:

(i) \(f(X_i) \subset X_i\), \(\forall \ i \in I\);
(ii) the restriction of \(f\) to \(X_i\), \(f|_{X_i} : X_i \rightarrow X_i\), is \(c\)-Picard operator, for all \(i \in I\).

Then \( f \) is \( c \)-weakly Picard operator.

On the other hand by the analogy with the notion of the Ulam-Hyers stability in the theory of functional equation (see [14], [15], [7], [2], [9], [10]-[13], [21], [25], [26],...) we have

**Definition 2.2.** Let \((X, d)\) be a metric space and \(f : X \rightarrow X\) be an operator. By definition, the fixed point equation
\[
x = f(x)
\]
is Ulam-Hyers stable if there exists a real number \(c_f > 0\) such that: for each \(\varepsilon > 0\) and each solution \(y^*\) of the inequation
\[
d(y, f(y)) \leq \varepsilon
\]
there exists a solution \(x^*\) of the equation (2.1) such that
\[
d(y^*, x^*) \leq c_f \varepsilon.
\]

**Remark 2.1.** If \(f\) is a \(c\)-weakly Picard operator, then the fixed point equation (2.1) is Ulam-Hyers stable.
Indeed, let $\varepsilon > 0$ and $y^*$ a solution of (2.2). Since $f$ is $c$-weakly Picard operator, we have that

$$d(x, f^\infty(x)) \leq cd(x, f(x)), \forall \ x \in X.$$  

If we take $x := y^*$ and $x^* := f^\infty(y)$, we have that, $d(y^*, x^*) \leq c\varepsilon$.

**Remark 2.2.** Let $(X, d)$ be a metric space, $f : X \to X$ an operator and $X = \bigcup_{i \in I} X_i$ a partition of $X$ such that $f(X_i) \subset X_i, \forall i \in I$. If the equation (2.1) is Ulam-Hyers stable in each $(X_i, d), i \in I$, then it is Ulam-Hyers stable in $(X, d)$.

**Remark 2.3.** Let $d$ and $\rho$ be two metrics on a set $X$ and $f : X \to X$ an operator. Let $d$ and $\rho$ be metric equivalent, i.e., there exists $c_1, c_2 > 0$ such that

$$c_1d(x, y) \leq \rho(x, y) \leq c_2d(x, y), \forall \ x, y \in X.$$  

Then the following statements are equivalent:

(i) the equation (2.1) is Ulam-Hyers stable in $(X, d)$;

(ii) the equation (2.1) is Ulam-Hyers stable in $(X, \rho)$.

Now we shall give some applications of the above remarks.

**Example 2.4.** Let us consider the following functional-integral equation

$$x(t) = x(0) + \int_0^t K(t, s, x(s))ds, \ t \in [0, 1]. \quad (2.3)$$  

We suppose that:

(a) $K \in C([0, 1] \times [0, 1] \times \mathbb{R});$

(b) there exists $L_K > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L_K|u - v|,$$

for all $t, s \in [0, 1]$ and $u, v \in \mathbb{R}$.

We consider on $C[0, 1]$ the Bielecki metric

$$d_\tau(x, y) := \max_{0 \leq t \leq 1} (|x(t) - y(t)|e^{-\tau t}),$$

where $\tau \in \mathbb{R}_+^*$ is such that $\frac{L_K}{\tau} < 1$. 

In this case the operator \( f : C[0, 1] \to C[0, 1] \) is defined by
\[
f(x)(t) := x(0) + \int_0^t K(t, s, x(s))ds, \quad t \in [0, 1].
\]
We take \( X := C[0, 1] \) and for \( \alpha \in \mathbb{R} \),
\[
X_\alpha := \{ x \in C[0, 1] \mid x(0) = \alpha \}.
\]
It is clear that:
(1) \( X = \bigcup_{\alpha \in \mathbb{R}} X_\alpha \) is a partition of \( X \);
(2) \( X_\alpha \subset (X, d_\tau) \) is a closed subset, for all \( \alpha \in \mathbb{R} \);
(3) \( f(X_\alpha) \subset X_\alpha, \forall \alpha \in \mathbb{R} \);
(4) the restriction of \( f \) to \( X_\alpha \)
\[
f|_{X_\alpha} : X_\alpha \to X_\alpha
\]
is a \( \frac{LK}{\tau} \)-contraction, i.e., the operator \( f|_{X_\alpha} \) is a Picard operator.

From Remark 2.2 the operator \( f \) is \( \left( 1 - \frac{LK}{\tau} \right)^{-1} \)-weakly Picard operator and from Remark 2.1, the equation (2.3) is Ulam-Hyers stable. In a more precise manner we have

**Theorem 2.1.** We consider the equation (2.3) in the conditions (a) and (b). Let \( \varepsilon \) be a positive real number. If \( y^* \in C[0, 1] \) is a solution of the inequation
\[
\left| y(t) - y(0) - \int_0^t K(t, s, y(s))ds \right| \leq \varepsilon, \quad \forall \ t \in [0, 1],
\]
then there exists a solution \( x^* \in C[0, 1] \) of the equation (2.3) such that
\[
|y^*(t) - x^*(t)| \leq \left( 1 - \frac{LK}{\tau} \right)^{-1} e^{2\tau \varepsilon}, \quad \forall \ t \in [0, 1].
\]

**Proof.** The inequality (2.4) implies that
\[
\left\| y(\cdot) - y(0) - \int_0^\cdot K(( \cdot ), s, y(s))ds \right\|_\tau \leq \varepsilon e^\tau.
\]
This inequality implies that
\[
\|y^* - x^*\|_\tau \leq \left( 1 - \frac{LK}{\tau} \right)^{-1} e^\tau \varepsilon.
\]

(2.5)
We consider on \( C[0, 1] \) the metric \( d_0 \) defined by

\[
d_0(x, y) := \max_{0 \leq t \leq 1} |x(t) - y(t)|.
\]

We remark that

\[
d_\tau \leq d_0 \leq d_\tau e^\tau.
\]

From (2.5) it follows that

\[
\|y^* - x^*\|_0 \leq \left( 1 - \frac{L_K}{\tau} \right)^{-1} e^{2\tau \varepsilon}.
\]

So,

\[
|y^*(t) - x^*(t)| \leq \left( 1 - \frac{L_K}{\tau} \right)^{-1} e^{2\tau \varepsilon}, \quad \forall \ t \in [0, 1].
\]

\[\square\]

**Example 2.5.** If we consider a Volterra integral equation on a noncompact interval then, in general, we do not have the Ulam-Hyers stability, as the following example illustrates.

We consider the equation

\[
x(t) = \int_0^t x(s)ds, \quad t \in \mathbb{R}_+ \quad (2.6)
\]

in \( C(\mathbb{R}_+) \) endowed with the generalized metric \( d(x, y) \in \mathbb{R}_+ \cup \{+\infty\} \), (see for example [32], pp. 69-76)

\[
d(x, y) := \sup_{t \in \mathbb{R}_+} |x(t) - y(t)|.
\]

The equation (2.6) has in \( C(\mathbb{R}_+) \) a unique solution \( x^* = 0 \). On the other hand we remark that \( y^*(t) = \varepsilon e^t \) is a solution of the inequation

\[
\left| y(t) - \int_0^t y(s)ds \right| \leq \varepsilon, \quad \forall \ t \in \mathbb{R}_+. \quad (2.7)
\]

But,

\[
|y^*(t) - x^*(t)| = \varepsilon e^t \rightarrow +\infty \text{ as } t \rightarrow \infty.
\]

So, the equation (2.6) does not have the Ulam-Hyers stability in \( (C(\mathbb{R}_+), d) \).

Other applications of the above general remarks will be given in the next section.
3. Ulam-Hyers stability of difference equations

We begin with some notations from the theory of infinite matrices.
Let $X$ be a nonempty set,
$$s(X) := \{(x_n)_{n \in \mathbb{N}} \mid x_n \in X\} = \{(x_1, \ldots, x_n, \ldots) \mid x_n \in X, n \in \mathbb{N}^*\}$$
and
$$M(X) := \{(x_{ij})_1^\infty \mid x_{ij} \in X, i, j \in \mathbb{N}^*\}$$
where
$$(x_{ij})_1^\infty := \begin{pmatrix}
  x_{11} & x_{12} & x_{13} & \ldots \\
  x_{21} & x_{22} & x_{23} & \ldots \\
  x_{31} & x_{32} & x_{33} & \ldots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$
is an infinite matrix.

Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space. On $M(\mathbb{B})$ we consider the following generalized norm
$$A \in M(\mathbb{B}), \quad |A| = |(a_{ij})_1^\infty| := \sup_{1 \leq i < +\infty} \sum_{j=1}^{\infty} |a_{ij}|$$
and on $s(\mathbb{B})$ the following vectorial norm
$$x \in s(\mathbb{B}), \quad |x|_v = |(x_n)_1^\infty|_v := (|x_n|)_1^\infty.$$

Let $k \in \mathbb{N}^*$ and $f_n : \mathbb{B}^k \to \mathbb{B}, n \in \mathbb{N}^*$ some given operators. We consider the following $k$-order difference equation
$$x_n = f_n(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), \quad n \in \mathbb{N}^*. \quad (3.1)$$

By a solution of this equation we understand an $x \in \mathbb{B}^k \times s(\mathbb{B})$ which satisfies (3.1).

For $\varepsilon \in (\mathbb{R}^*_+)^k \times s(\mathbb{R}^*_+)$ we consider the following difference inequation:
$$|y_n - f_n(y_{n-k}, y_{n-k+1}, \ldots, y_{n-1})| \leq \varepsilon_n, \quad n \in \mathbb{N}^*. \quad (3.2)$$

**Definition 3.1.** By definition the equation (3.1) is Ulam-Hyers stable if there exists an infinite matrix $(c_{ij})_{-k+1}^\infty$ with the elements $c_{ij} \in \mathbb{R}_+$,
$$i, j \in \{-k + 1, -k + 2, \ldots, 0, 1, \ldots, n, \ldots\}$$
such that for each \( \epsilon \in (\mathbb{R}^*_+)^k \times s(\mathbb{R}^*_+) \) and each solution \( y^* \) of (3.2) there exists a solution \( x^* \) of (3.1) having the property:

\[
|y^* - x^*|_v \leq ((c_{ij})^\infty_{k+1})\epsilon.
\]

Let us consider the operator

\[
F : \mathbb{B}^k \times s(\mathbb{B}) \to \mathbb{B}^k \times s(\mathbb{B})
\]

defined by

\[
F(x_{-k+1}, \ldots, x_0, x_1, \ldots, x_n, \ldots) = (x_{-k+1}, \ldots, x_0, f_1(x_{-k+1}, \ldots, x_0), \ldots, f_n(x_{n-k}, \ldots, x_{n-1}), \ldots).
\]

In terms of the operator \( F \) the equation (3.1) takes the following form:

\[
x = F(x) \quad (3.1')
\]

while the inequation (3.2) takes the form

\[
|y - F(y)|_v \leq \epsilon. \quad (3.2')
\]

**Definition 3.2.** Let \( C = (c_{ij})_{k+1}^\infty \) be an infinite matrix with \( c_{ij} \in \mathbb{R}_+ \). The operator \( F \) is C-weakly Picard operator if it is weakly Picard operator with respect to \(| \cdot |_v\) on \( \mathbb{B}^k \times s(\mathbb{B}) \) and

\[
|x - F^\infty(x)|_v \leq C|x - F(x)|_v, \quad \forall \ x \in \mathbb{B}^k \times s(\mathbb{B}).
\]

**Remark 3.1.** If the operator \( F \) is C-weakly Picard operator then the corresponding equation (3.1) is Ulam-Hyers stable.

Following Example 2.3 we take \( X := \mathbb{B}^k \times s(\mathbb{B}) \) and for \( \alpha \in \mathbb{B}^k \),

\[
X_\alpha := \{ x \in \mathbb{B}^k \times s(\mathbb{B}) \mid (x_{k+1}, \ldots, x_0) = \alpha \}.
\]

We remark that:

1. \( X = \bigcup_{\alpha \in \mathbb{B}^k} X_\alpha \) is a partition of \( X \);
2. \( X_\alpha \subset (X, | \cdot |_v) \) is a closed subset for all \( \alpha \in \mathbb{B}^k \);
3. \( F(X_\alpha) \subset X_\alpha, \ \forall \ \alpha \in \mathbb{B}^k \);
4. If the restriction of \( F \) to \( X_\alpha, F|_{X_\alpha} : X_\alpha \to X_\alpha \) is a C-Picard operator for all \( \alpha \in \mathbb{B}^k \), then the operator \( F \) is C-weakly Picard operator.
Let us suppose that the operators $f_n, n \in \mathbb{N}$ satisfy the conditions:

$$|f_n(u_1, \ldots, u_k) - f_n(v_1, \ldots, v_k)| \leq \sum_{j=1}^{k} a_{nj}|u_j - v_j|, \quad n \in \mathbb{N}^* \quad (3.3)$$

for all $u_i, v_i \in \mathbb{B}$.

Let us take $s_{ii} := 1$ for $i \in \{-k + 1, \ldots, 0\}$, $s_{n+n} := a_{nj}, n \in \mathbb{N}^*$ and $j \in \{1, \ldots, k\}$ and in rest, $s_{ij} = 0$. Thus we obtain a matrix $S$.

We have that

$$|F(x) - F(y)|_v \leq S|x - y|_v, \quad \forall x, y \in X,$$

and

$$|F(x) - F(y)|_v \leq \tilde{S}|x - y|_v, \quad \forall x, y \in X_\alpha, \quad \alpha \in \mathbb{B}^k$$

where $\tilde{s}_{ii} = 0$ for $i \in \{-k + 1, \ldots, 0\}$ and $\tilde{s}_{ij} = s_{ij}$ in the rest of the cases.

If $\sup_{n \in \mathbb{N}^*} \sum_{j=1}^{k} a_{nj} < 1$, then from Theorem 4.1 in [29] we have that $F|_{X_\alpha}$ is a C-Picard operator with $C = (E - \tilde{S})^{-1}$. So, we have

**Theorem 3.1.** We suppose that

1. $|f_n(y_1, \ldots, y_k) - f_n(v_1, \ldots, v_k)| \leq \sum_{j=1}^{k} a_{nj}|y_j - v_j|, \quad \text{for all } u, v \in \mathbb{B}^k$;

2. $\sup_{n \in \mathbb{N}^*} \sum_{j=1}^{k} a_{nj} < 1$.

Then the equation (3.1) is Ulam-Hyers stable.

**Remark 3.2.** For other results on stability of difference equations see [3], [4], [5], [23], [24],...
(AB)_f(x^*) := \{ x \in Y \mid f^n(x) \xrightarrow{d} x^* \in F_f \}
- the attraction basin of the fixed point x^* with respect to f

(AB)_f := \bigcup_{x^* \in F_f} (AB)_f(x^*) - the attraction basin of f.

**Definition 4.1.** By definition an operator \( f : Y \to X \) is weakly Picard operator if \( F_f \neq \emptyset \) and \( (MI)_f = (AB)_f \). If \( F_f = \{ x^* \} \) then an weakly Picard operator is said to be Picard operator.

**Definition 4.2.** For each weakly Picard operator \( f : Y \to X \) we define the operator \( f^\infty : (AB)_f \to (AB)_f \) by \( f^\infty(x) = \lim_{n \to \infty} f^n(x) \).

**Definition 4.3.** Let \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be an increasing function which is continuous in 0 and \( \psi(0) = 0 \). An operator \( f : Y \to X \) is said to be a \( \psi \)-weakly Picard operator if it is weakly Picard operator and
\[
d(x, f^\infty(x)) \leq \psi(d(x, \psi(x))), \quad \forall x \in (MI)_f.
\]

In the case that \( \psi(t) = ct \) with \( c > 0 \), we say that \( f \) is \( c \)-weakly Picard operator.

For some examples of nonself weakly Picard operators and \( \psi \)-weakly Picard operators see [8].

Now, let us consider the fixed point equation
\[
x = f(x)
\]
and the inequation
\[
d(y, f(y)) \leq \varepsilon.
\]

**Definition 4.4.** The equation (4.1) is generalized Ulam-Hyers stable if there exists \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) increasing and continuous in 0 with \( \psi(0) = 0 \) such that: for each \( \varepsilon > 0 \) and for each solution \( y^* \in (AB)_f \) of (4.2) there exists a solution \( x^* \) of (4.1) such that
\[
d(y^*, x^*) \leq \psi(\varepsilon).
\]

In the case that \( \psi(t) = ct \), \( c > 0 \), the equation (4.1) is said to be Ulam-Hyers stable (see Definition 2.2 for the case of self operators).

**Remark 4.1.** If an operator \( f : Y \to X \) is \( \psi \)-weakly Picard operator, then the fixed point equation (4.1) is generalized Ulam-Hyers stable. If \( f \) is \( c \)-weakly Picard operator, then the equation (4.1) is Ulam-Hyers stable.
Example 4.1. (see [8], p. 76). Let \((X,d)\) be a metric space, \(Y \subset X\) and \(f : Y \to X\) a strict \(\varphi\)-contraction with \(F_f \neq \emptyset\). Then the equation (4.1) is generalized Ulam-Hyers stable.

5. Ulam-Hyers-Rassias stability of the fixed point equations in a space of functions

Let \(K\) be \(\mathbb{R}\) or \(\mathbb{C}\). Let \(\Omega \subset K^m\) be a nonempty subset of \(K^m\), \(X\) a set of functions \(x : \Omega \to K\) and \(f : X \to X\) an operator.

**Definition 5.1.** Let \(\varphi : \Omega \to \mathbb{R}_+\) be a function. The fixed point equation

\[ x = f(x) \tag{5.1} \]

is Ulam-Hyers-Rassias stable with respect to \(\varphi\) if there exists \(c > 0\) such that: for each \(\varepsilon > 0\) and each \(y^* \in X\) a solution of the inequation

\[ |y(t) - f(y)(t)| \leq \varepsilon \varphi(t), \quad \forall \ t \in \Omega \tag{5.2} \]

there exists a solution \(x^* \in X\) of (5.1) such that

\[ |y^*(t) - x^*(t)| \leq c \varepsilon \varphi(t), \quad \forall \ t \in \Omega. \]

**Definition 5.2.** Let \(\varphi : \Omega \to \mathbb{R}_+\) be a function. The equation (5.1) is generalized Ulam-Hyers-Rassias stable with respect to \(\varphi\) if there exists \(c > 0\) such that: for each solution \(y^* \in X\) of the inequation

\[ |y(t) - f(y)(t)| \leq \varphi(t), \quad \forall \ t \in \Omega \tag{5.3} \]

there exists a solution \(x^* \in X\) of (5.1) such that

\[ |y^*(t) - x^*(t)| \leq c \varphi(t), \quad \forall \ t \in \Omega. \]

For some results in this direction see [31]. See also [17], [20].

6. Ulam stability of the coincidence equations

Let \((X,d)\) and \((Y,\rho)\) be two metric spaces. If \(f, g : X \to Y\) are two operators then, we denote by

\[ C(f, g) := \{ x \in X \mid f(x) = g(x) \} \]

the coincidence point set of the pair \(f, g\).
Definition 6.1. Let $c > 0$ be a real number. By definition the pair $f, g : X \to Y$ is $c$-weakly Picard pair if there exists an operator $h : X \to X$ such that:

(i) $h$ is weakly Picard operator;
(ii) $F_h = C(f, g)$;
(iii) $d(x, h^\infty(x)) \leq c\rho(f(x), g(x)), \forall x \in X$.

We remark that $h^\infty(X) = C(f, g)$.

For some examples of $c$-weakly Picard pair see [6], pp. 37-40.

Definition 6.2. The equation

$$f(x) = g(x) \quad (6.1)$$

is Ulam-Hyers stable if there exists $c > 0$ such that: for each $\varepsilon > 0$ and for each solution $y^*$ of the inequation

$$\rho(f(y), g(y)) \leq \varepsilon \quad (6.2)$$

there exists a solution $x^*$ of (6.1) such that

$$d(y^*, x^*) \leq c\varepsilon.$$  

In a similar way we can define the generalized Ulam-Hyers stability of equation (6.1) and in the case of the function spaces $X$ and $Y$ the Ulam-Hyers-Rassias stability and the generalized Ulam-Hyers-Rassias stability of the equation (6.1).

In what follows we shall consider the Ulam-Hyers stability.

Remark 6.1. If a pair $f, g : X \to Y$ is a $c$-weakly Picard pair, then the equation (6.1) is Ulam-Hyers stable.

Indeed, let $y^*$ be a solution of the inequation (6.2). Then we take $x^* = h^\infty(y^*)$. From the condition (iii) in Definition 6.1 we have that

$$d(y^*, x^*) \leq c\rho(f(y^*), g(y^*)) \leq c\varepsilon.$$  

From the Remark 6.1 it follows that from each $c$-weakly Picard pair we have an example of coincidence equation which is Ulam-Hyers stable.

Remark 6.2. For other considerations on stability of operatorial equations see [2], [11], [12], [13], [26], [27].
7. Ulam stability of the operatorial inclusions

In this section we follow the terminologies and the notations from [33]. See also [22] and [32]. For the convenience of the reader we shall present some of them.

Let \((X, d)\) be a metric space, \(A, B \in P(X)\), and \(T : X \rightarrow P(X)\) a multivalued operator. We denote:

\[
P_{\text{cp}}(X) := \{Y \in P(X) \mid Y \text{ a compact subset of } X\}
\]

\[
D_d(A, B) := \inf \{d(a, b) \mid a \in A, b \in B\}
\]

\[
F_T := \{x \in X \mid x \in T(x)\} - \text{the fixed point set of the operator } T
\]

\[
G(T) := \{(x, y) \mid x \in X, y \in T(x)\} - \text{the graphic of } T.
\]

The following notions are given in [33].

**Definition 7.1.** An operator \(T : X \rightarrow P(X)\) is a multivalued weakly Picard operator iff for each \(x \in X\) and each \(y \in T(x)\) there exists a sequence of successive approximations, \((x_n)_{n \in \mathbb{N}}, x_{n+1} \in T(x_n), n \in \mathbb{N}\), such that

(i) \(x_0 = x, x_1 = y\),

(ii) \(x_n \xrightarrow{d} x^* \in F_T\).

**Definition 7.2.** For a multivalued weakly Picard operator \(T\) we define the multivalued operator \(T^\infty : G(T) \rightarrow P(F_T)\) by

\[
T^\infty(x, y) := \{z \in F_T \mid \text{there exists a sequence of successive approximations of } T \text{ starting from } (x, y) \text{ that converges to } z\}.
\]

**Definition 7.3.** Let \(\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) be an increasing function which is continuous in 0 and \(\psi(0)\). An operator \(T : X \rightarrow P(X)\) is \(\psi\)-multivalued weakly Picard if there exists a selection \(t^\infty\) of \(T^\infty\) such that

\[
d(x, t^\infty(x, y)) \leq \psi(d(x, y)), \forall (x, y) \in G(T).
\]

If \(\psi(t) = ct, c > 0\), then \(T\) is called a \(c\)-multivalued weakly Picard operator.

For some examples of \(\psi\)-multivalued weakly Picard operator see [22], [33] and [32].

On the other hand we have the following notions of stability of the equation

\[
x \in T(x)
\]
Definition 7.4. The equation (7.1) is Ulam-Hyers stable if there exists $c > 0$ such that: for each $\varepsilon > 0$ and for each solution $u^*$ of the inequation

$$D_d(u, T(u)) \leq \varepsilon$$

(7.2)

there exists a solution $x^*$ of (7.1) such that

$$d(u^*, x^*) \leq c\varepsilon.$$

Definition 7.5. The equation (7.1) is generalized Ulam-Hyers stable if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous in 0 with $\psi(0) = 0$ such that: for each $\varepsilon > 0$ and for each solution $u^*$ of the inequation (7.2) there exists a solution $x^*$ of (7.1) such that

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

Remark 7.1. If the operator $T : X \to P_{cp}(X)$ is a $c$-multivalued weakly Picard operator, then the equation (7.1) is Ulam-Hyers stable. If $T : X \to P_{cp}(X)$ is a $\psi$-multivalued weakly Picard operator then the equation (7.1) is generalized Ulam-Hyers stable.

Indeed, let us suppose that $T$ is a $\psi$-multivalued weakly Picard operator. Let $u^*$ be a solution of (7.2). Let $y \in T(u^*)$ be such that $D_d(u^*, T(u^*)) = d(u^*, y)$. We take $x^* := t^\infty(u^*, y)$ and we have

$$d(u^*, x^*) = d(u^*, t^\infty(u^*, y)) \leq \psi(d(u^*, y)) \leq \psi(\varepsilon).$$

From Remark 7.1 it follows that for each example of $\psi$-multivalued weakly Picard operator we have an example of equation (7.1) which is generalized Ulam-Hyers stable.

References


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