Fixed Point Theory, 10(2009), No. 2, 305-320
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

REMARKS ON ULAM STABILITY OF THE OPERATORIAL EQUATIONS

IOAN A. RUS

Babeş-Bolyai University
Department of Applied Mathematics
Kogălniceanu Nr. 1
400084 Cluj-Napoca, Romania.
E-mail: iarus@math.ubbcluj.ro

Abstract. In this paper we present four types of Ulam stability for operatorial equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. The relations of Ulam stability with the c-weakly Picard operators are also studied. Some examples and counterexamples are given.
Key Words and Phrases: Ulam-Hyers stability, Ulam-Hyers-Rassias stability, fixed point equation, coincidence point equation, integral equation, differential equation, difference equation, operatorial inclusions.
2000 Mathematics Subject Classification: 47H10, 54H25, 45N05, 39A11.

1. Introduction

The Ulam stability (Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin,...) of various functional equations have been investigated by many authors (see [14], [15], [7], [9], [2], [1], [10], [13], [21], [25], [26],...). There are some results for differential equations ([1], [16], [18], [19], [20], [31],...), integral equations ([17], [30],...) and for difference equations ([3], [4], [5], [23], [24],...).

The aim of this paper is to present four types of Ulam stability for the operatorial equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. The relations of Ulam stability with the c-weakly Picard operators ([28], [33], [22], [32]) are studied. Some examples and counterexamples are also given. The plan of the paper is the following:

1. Introduction
2. Ulam-Hyers stability via weakly Picard operators
3. Ulam-Hyers stability of difference equations
4. Generalized Ulam-Hyers stability of a fixed point equation with non-self operator
5. Ulam-Hyers-Rassias stability of the fixed point equations in a space of functions
6. Ulam stability of the coincidence point equations
7. Ulam stability of the operatorial inclusions.

2. Ulam-Hyers stability via weakly Picard operators

We begin our considerations with some notions and results from weakly Picard operator theory (see [28]; see also [32], pp. 119-126).

Let (X, d) be a metric space and $f : X \to X$ an operator. We denote by $F_f := \{x \in X \mid f(x) = x\}$, the fixed point set of the operator f. By definition f is weakly Picard operator if the sequence of successive approximations, $f^n(x)$, converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

If f is weakly Picard operator then we consider the operator $f^\infty : X \to X$ defined by $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$. It is clear that $f^\infty(X) = F_f$. Moreover, f^∞ is a set retraction of X to F_f.

If f is weakly Picard operator and $F_f = \{x^*\}$, then by definition f is a Picard operator. In this case f^∞ is the constant operator, $f^\infty(x) = x^*$, $\forall\, x \in X$.

The following class of weakly Picard operators is very important in our considerations.

Definition 2.1. Let $f : X \to X$ be an weakly Picard operator and $c > 0$ a real number. By definition the operator f is c-weakly Picard operator if

$$d(x, f^\infty(x)) \leq cd(x, f(x)),\ \forall\, x \in X.$$

Example 2.1. Let (X, d) be a complete metric space and $f : X \to X$ an operator with closed graphic. We suppose that f is graphic α-contraction, i.e.,

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)),\ \forall\, x \in X.$$

Then f is a c-weakly Picard operator, with $c = (1-\alpha)^{-1}$.

Example 2.2. Let (X, d) be a complete metric space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function and $f : X \to X$ an operator with closed graphic. We suppose that:

(i) f is a φ-Caristi operator, i.e.,

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),\ \forall\, x \in X;$$

(ii) there exists $c > 0$ such that

$$\varphi(x) \leq cd(x, f(x)),\ \forall\, x \in X.$$

Then f is a c-weakly Picard operator.

Example 2.3. (generic example). Let (X, d) be a metric space, $f : X \to X$ an operator and $X = \bigcup\limits_{i \in I} X_i$ a partition of X. We suppose that:

(i) $f(X_i) \subset X_i,\ \forall\, i \in I$;

(ii) the restriction of f to X_i, $f|_{X_i} : X_i \to X_i$, is c-Picard operator, for all $i \in I$.

Then f is c-weakly Picard operator.

On the other hand by the analogy with the notion of the Ulam-Hyers stability in the theory of functional equation (see [14], [15], [7], [2], [9], [10]-[13], [21], [25], [26],...) we have

Definition 2.2. Let (X, d) be a metric space and $f : X \to X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{2.1}$$

is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that: for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{2.2}$$

there exists a solution x^* of the equation (2.1) such that

$$d(y^*, x^*) \leq c_f \varepsilon.$$

Now, we have

Remark 2.1. If f is a c-weakly Picard operator, then the fixed point equation (2.1) is Ulam-Hyers stable.

Indeed, let $\varepsilon > 0$ and y^* a solution of (2.2). Since f is c-weakly Picard operator, we have that

$$d(x, f^\infty(x)) \leq cd(x, f(x)),\ \forall\ x \in X.$$

If we take $x := y^*$ and $x^* := f^\infty(y)$, we have that, $d(y^*, x^*) \leq c\varepsilon$.

Remark 2.2. Let (X, d) be a metric space, $f : X \to X$ an operator and $X = \bigcup_{i \in I} X_i$ a partition of X such that $f(X_i) \subset X_i$, $\forall\ i \in I$. If the equation (2.1) is Ulam-Hyers stable in each (X_i, d), $i \in I$, then it is Ulam-Hyers stable in (X, d).

Remark 2.3. Let d and ρ be two metrics on a set X and $f : X \to X$ an operator. Let d and ρ be metric equivalent, i.e., there exists $c_1, c_2 > 0$ such that

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y),\ \forall\ x, y \in X.$$

Then the following statements are equivalent:

(i) the equation (2.1) is Ulam-Hyers stable in (X, d);

(ii) the equation (2.1) is Ulam-Hyers stable in (X, ρ).

Now we shall give some applications of the above remarks.

Example 2.4. Let us consider the following functional-integral equation

$$x(t) = x(0) + \int_0^t K(t, s, x(s))ds, \quad t \in [0, 1]. \tag{2.3}$$

We suppose that:

(a) $K \in C([0, 1] \times [0, 1] \times \mathbb{R})$;

(b) there exists $L_K > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L_K |u - v|,$$

for all $t, s \in [0, 1]$ and $u, v \in \mathbb{R}$.

We consider on $C[0, 1]$ the Bielecki metric

$$d_\tau(x, y) := \max_{0 \leq t \leq 1} (|x(t) - y(t)| e^{-\tau t}).$$

where $\tau \in \mathbb{R}_+^*$ is such that $\dfrac{L_K}{\tau} < 1$.

In this case the operator $f : C[0,1] \to C[0,1]$ is defined by

$$f(x)(t) := x(0) + \int_0^t K(t,s,x(s))ds, \quad t \in [0,1].$$

We take $X := C[0,1]$ and for $\alpha \in \mathbb{R}$,

$$X_\alpha := \{x \in C[0,1] \mid x(0) = \alpha\}.$$

It is clear that:

(1) $X = \bigcup_{\alpha \in \mathbb{R}} X_\alpha$ is a partition of X;

(2) $X_\alpha \subset (X, d_\tau)$ is a closed subset, for all $\alpha \in \mathbb{R}$;

(3) $f(X_\alpha) \subset X_\alpha$, $\forall\ \alpha \in \mathbb{R}$;

(4) the restriction of f to X_α

$$f|_{X_\alpha} : X_\alpha \to X_\alpha$$

is a $\dfrac{L_K}{\tau}$-contraction, i.e., the operator $f|_{X_\alpha}$ is a Picard operator.

From Remark 2.2 the operator f is $\left(1 - \dfrac{L_K}{\tau}\right)^{-1}$-weakly Picard operator and from Remark 2.1, the equation (2.3) is Ulam-Hyers stable. In a more precise manner we have

Theorem 2.1. *We consider the equation (2.3) in the conditions (a) and (b). Let ε be a positive real number. If $y^* \in C[0,1]$ is a solution of the inequation*

$$\left| y(t) - y(0) - \int_0^t K(t,s,y(s))ds \right| \leq \varepsilon,\ \forall\ t \in [0,1], \tag{2.4}$$

then there exists a solution $x^* \in C[0,1]$ of the equation (2.3) such that

$$|y^*(t) - x^*(t)| \leq \left(1 - \frac{L_K}{\tau}\right)^{-1} e^{2\tau}\varepsilon,\ \forall\ t \in [0,1].$$

Proof. The inequality (2.4) implies that

$$\left\| y(\cdot) - y(0) - \int_0^{(\cdot)} K((\cdot),s,y(s))ds \right\|_\tau \leq \varepsilon e^\tau.$$

This inequality implies that

$$\|y^* - x^*\|_\tau \leq \left(1 - \frac{L_K}{\tau}\right)^{-1} e^\tau \varepsilon. \tag{2.5}$$

We consider on $C[0,1]$ the metric d_0 defined by

$$d_0(x,y) := \max_{0\le t\le 1} |x(t) - y(t)|.$$

We remark that

$$d_\tau \le d_0 \le d_\tau e^\tau.$$

From (2.5) it follows that

$$\|y^* - x^*\|_0 \le \left(1 - \frac{L_K}{\tau}\right)^{-1} e^{2\tau}\varepsilon.$$

So,

$$|y^*(t) - x^*(t)| \le \left(1 - \frac{L_K}{\tau}\right)^{-1} e^{2\tau}\varepsilon,\ \forall\ t \in [0,1].$$

□

Example 2.5. If we consider a Volterra integral equation on a noncompact interval then, in general, we do not have the Ulam-Hyers stability, as the following example illustrates.

We consider the equation

$$x(t) = \int_0^t x(s)ds, \quad t \in \mathbb{R}_+ \tag{2.6}$$

in $C(\mathbb{R}_+)$ endowed with the generalized metric $(d(x,y) \in \mathbb{R}_+ \cup \{+\infty\})$, (see for example [32], pp. 69-76)

$$d(x,y) := \sup_{t\in\mathbb{R}_+} |x(t) - y(t)|.$$

The equation (2.6) has in $C(\mathbb{R}_+)$ a unique solution $x^* = 0$. On the other hand we remark that $y^*(t) = \varepsilon e^t$ is a solution of the inequation

$$\left| y(t) - \int_0^t y(s)ds \right| \le \varepsilon,\ \forall\ t \in \mathbb{R}_+. \tag{2.7}$$

But,

$$|y^*(t) - x^*(t)| = \varepsilon e^t \to +\infty \text{ as } t \to \infty.$$

So, the equation (2.6) does not have the Ulam-Hyers stability in $(C(\mathbb{R}_+), d)$.

Other applications of the above general remarks will be given in the next section.

3. Ulam-Hyers stability of difference equations

We begin with some notations from the theory of infinite matrices.

Let X be a nonempty set,

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X\} = \{(x_1, \dots, x_n, \dots) \mid x_n \in X,\ n \in \mathbb{N}^*\}$$

and

$$M(X) := \{(x_{ij})_1^\infty \mid x_{ij} \in X,\ i, j \in \mathbb{N}^*\}$$

where

$$(x_{ij})_1^\infty := \begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots \\ x_{21} & x_{22} & x_{23} & \dots \\ x_{31} & x_{32} & x_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots \end{pmatrix}$$

is an infinite matrix.

Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space. On $M(\mathbb{B})$ we consider the following generalized norm

$$A \in M(\mathbb{B}), \quad |A| = |(a_{ij})_1^\infty| := \sup_{1\leq i<+\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

and on $s(\mathbb{B})$ the following vectorial norm

$$x \in s(\mathbb{B}), \quad |x|_v = |(x_n)_1^\infty|_v := (|x_n|)_1^\infty.$$

Let $k \in \mathbb{N}^*$ and $f_n : \mathbb{B}^k \to \mathbb{B}$, $n \in \mathbb{N}^*$ some given operators. We consider the following k-order difference equation

$$x_n = f_n(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), \quad n \in \mathbb{N}^*. \tag{3.1}$$

By a solution of this equation we understand an $x \in \mathbb{B}^k \times s(\mathbb{B})$ which satisfies (3.1).

For $\varepsilon \in (\mathbb{R}_+^*)^k \times s(\mathbb{R}_+^*)$ we consider the following difference inequation:

$$|y_n - f_n(y_{n-k}, y_{n-k+1}, \dots, y_{n-1})| \leq \varepsilon_n, \quad n \in \mathbb{N}^*. \tag{3.2}$$

Definition 3.1. By definition the equation (3.1) is Ulam-Hyers stable if there exists an infinite matrix $(c_{ij})_{-k+1}^\infty$ with the elements $c_{ij} \in \mathbb{R}_+$,

$$i, j \in \{-k+1, -k+2, \dots, 0, 1, \dots, n, \dots\}$$

such that for each $\varepsilon \in (\mathbb{R}_+^*)^k \times s(\mathbb{R}_+^*)$ and each solution y^* of (3.2) there exists a solution x^* of (3.1) having the property:

$$|y^* - x^*|_v \leq ((c_{ij})_{-k+1}^{\infty})\varepsilon.$$

Let us consider the operator

$$F : \mathbb{B}^k \times s(\mathbb{B}) \to \mathbb{B}^k \times s(\mathbb{B})$$

defined by

$$F(x_{-k+1}, \ldots, x_0, x_1, \ldots, x_n, \ldots)$$
$$= (x_{-k+1}, \ldots, x_0, f_1(x_{-k+1}, \ldots, x_0), \ldots, f_n(x_{n-k}, \ldots, x_{n-1}), \ldots).$$

In terms of the operator F the equation (3.1) takes the following form:

$$x = F(x) \tag{3.1'}$$

while the inequation (3.2) takes the form

$$|y - F(y)|_v \leq \varepsilon. \tag{3.2'}$$

Definition 3.2. Let $C = (c_{ij})_{-k+1}^{\infty}$ be an infinite matrix with $c_{ij} \in \mathbb{R}_+$. The operator F is C-weakly Picard operator if it is weakly Picard operator with respect to $|\cdot|_v$ on $\mathbb{B}^k \times s(\mathbb{B})$ and

$$|x - F^{\infty}(x)|_v \leq C|x - F(x)|_v, \ \forall\ x \in \mathbb{B}^k \times s(\mathbb{B}).$$

Remark 3.1. If the operator F is C-weakly Picard operator then the corresponding equation (3.1) is Ulam-Hyers stable.

Following Example 2.3 we take $X := \mathbb{B}^k \times s(\mathbb{B})$ and for $\alpha \in \mathbb{B}^k$,

$$X_\alpha := \{x \in \mathbb{B}^k \times s(\mathbb{B}) \mid (x_{k+1}, \ldots, x_0) = \alpha\}.$$

We remark that:

(1) $X = \bigcup_{\alpha \in \mathbb{B}^k} X_\alpha$ is a partition of X;

(2) $X_\alpha \subset (X, |\cdot|_v)$ is a closed subset for all $\alpha \in \mathbb{B}^k$;

(3) $F(X_\alpha) \subset X_\alpha,\ \forall\ \alpha \in \mathbb{B}^k$.

(4) If the restriction of F to X_α, $F|_{X_\alpha} : X_\alpha \to X_\alpha$ is a C-Picard operator for all $\alpha \in \mathbb{B}^k$, then the operator F is C-weakly Picard operator.

Let us suppose that the operators f_n, $n \in \mathbb{N}$ satisfy the conditions:

$$|f_n(u_1, \ldots, u_k) - f_n(v_1, \ldots, v_k)| \leq \sum_{j=1}^{k} a_{nj}|u_j - v_j|, \quad n \in \mathbb{N}^* \tag{3.3}$$

for all $u_i, v_i \in \mathbb{B}$.

Let us take $s_{ii} := 1$ for $i \in \{-k+1, -k+2, \ldots, 1, 0\}$, $s_{n\ n+j} := a_{nj}$, $n \in \mathbb{N}^*$ and $j \in \{1, 2, \ldots, k\}$ and in rest, $s_{ij} = 0$. Thus we obtain a matrix S.

We have that

$$|F(x) - F(y)|_v \leq S|x - y|_v,\ \forall\ x, y \in X,$$

and

$$|F(x) - F(y)|_v \leq \widetilde{S}|x - y|_v,\ \forall\ x, y \in X_\alpha,\ \alpha \in \mathbb{B}^k$$

where $\widetilde{s}_{ii} = 0$ for $i \in \{-k+1, \ldots, 0\}$ and $\widetilde{s}_{ij} = s_{ij}$ in the rest of the cases.

If $\sup\limits_{n \in \mathbb{N}^*} \sum\limits_{j=1}^{k} a_{nj} < 1$, then from Theorem 4.1 in [29] we have that $F|_{X_\alpha}$ is a C-Picard operator with $C = (E - \widetilde{S})^{-1}$. So, we have

Theorem 3.1. *We suppose that*

(i) $|f_n(y_1, \ldots, u_k) - f_n(v_1, \ldots, v_k)| \leq \sum\limits_{j=1}^{k} a_{nj}|u_j - v_j|$, *for all* $u, v \in \mathbb{B}^k$;

(ii) $\sup\limits_{n \in \mathbb{N}^*} \sum\limits_{j=1}^{k} a_{nj} < 1$.

Then the equation (3.1) is Ulam-Hyers stable.

Remark 3.2. For other results on stability of difference equations see [3], [4], [5], [23], [24],...

4. Generalized Ulam-Hyers stability of a fixed point equation with non-self operator

Let (X, d) be a metric space, $Y \subset X$ be a nonempty subset of X and $f : Y \to X$ an operator. In this section we shall use the following notations and notions (see [8]):

$I(f) := \{Z \subset Y \mid f(Z) \subset Z,\ Z \neq \emptyset\}$ - the set of all invariant subsets of f

$(MI)_f := UI(f)$ - the maximal invariant subset of f

$(AB)_f(x^*) := \{x \in Y \mid f^n(x) \text{ is defined for all } n \in \mathbb{N} \text{ and } f^n(x) \xrightarrow{d} x^* \in F_f\}$ - the attraction basin of the fixed point x^* with respect to f

$(AB)_f := \bigcup_{x^* \in F_f} (AB)_f(x^*)$ - the attraction basin of f.

Definition 4.1. By definition an operator $f : Y \to X$ is weakly Picard operator if $F_f \neq \emptyset$ and $(MI)_f = (AB)_f$. If $F_f = \{x^*\}$ then an weakly Picard operator is said to be Picard operator.

Definition 4.2. For each weakly Picard operator $f : Y \to X$ we define the operator $f^\infty : (AB)_f \to (AB)_f$ by $f^\infty(x) = \lim_{n\to\infty} f^n(x)$.

Definition 4.3. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. An operator $f : Y \to X$ is said to be a ψ-weakly Picard operator if it is weakly Picard operator and

$$d(x, f^\infty(x)) \leq \psi(d(x, \psi(x))),\ \forall\, x \in (MI)_f.$$

In the case that $\psi(t) = ct$ with $c > 0$, we say that f is c-weakly Picard operator.

For some examples of nonself weakly Picard operators and ψ-weakly Picard operators see [8].

Now, let us consider the fixed point equation

$$x = f(x) \tag{4.1}$$

and the inequation

$$d(y, f(y)) \leq \varepsilon. \tag{4.2}$$

Definition 4.4. The equation (4.1) is generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing and continuous in 0 with $\psi(0) = 0$ such that: for each $\varepsilon > 0$ and for each solution $y^* \in (AB)_f$ of (4.2) there exists a solution x^* of (4.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

In the case that $\psi(t) = ct$, $c > 0$, the equation (4.1) is said to be Ulam-Hyers stable (see Definition 2.2 for the case of self operators).

Remark 4.1. If an operator $f : Y \to X$ is ψ-weakly Picard operator, then the fixed point equation (4.1) is generalized Ulam-Hyers stable. If f is c-weakly Picard operator, then the equation (4.1) is Ulam-Hyers stable.

Example 4.1. (see [8], p. 76). Let (X, d) be a metric space, $Y \subset X$ and $f : Y \to X$ a strict φ-contraction with $F_f \neq \emptyset$. Then the equation (4.1) is generalized Ulam-Hyers stable.

5. Ulam-Hyers-Rassias stability of the fixed point equations in a space of functions

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Let $\Omega \subset \mathbb{K}^m$ be a nonempty subset of $\mathbb{K}^m$, X a set of functions $x : \Omega \to \mathbb{K}$ and $f : X \to X$ an operator.

Definition 5.1. Let $\varphi : \Omega \to \mathbb{R}_+$ be a function. The fixed point equation

$$x = f(x) \tag{5.1}$$

is Ulam-Hyers-Rassias stable with respect to φ if there exists $c > 0$ such that: for each $\varepsilon > 0$ and each $y^* \in X$ a solution of the inequation

$$|y(t) - f(y)(t)| \leq \varepsilon\varphi(t),\ \forall\ t \in \Omega \tag{5.2}$$

there exists a solution $x^* \in X$ of (5.1) such that

$$|y^*(t) - x^*(t)| \leq c\varepsilon\varphi(t),\ \forall\ t \in \Omega.$$

Definition 5.2. Let $\varphi : \Omega \to \mathbb{R}_+$ be a function. The equation (5.1) is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists $c > 0$ such that: for each solution $y^* \in X$ of the inequation

$$|y(t) - f(y)(t)| \leq \varphi(t),\ \forall\ t \in \Omega \tag{5.3}$$

there exists a solution $x^* \in X$ of (5.1) such that

$$|y^*(t) - x^*(t)| \leq c\varphi(t),\ \forall\ t \in \Omega.$$

For some results in this direction see [31]. See also [17], [20].

6. Ulam stability of the coincidence equations

Let (X, d) and (Y, ρ) be two metric spaces. If $f, g : X \to Y$ are two operators then, we denote by

$$C(f, g) := \{x \in X \mid f(x) = g(x)\}$$

the coincidence point set of the pair f, g.

Definition 6.1. Let $c > 0$ be a real number. By definition the pair $f, g : X \to Y$ is c-weakly Picard pair if there exists an operator $h : X \to X$ such that:

(i) h is weakly Picard operator;

(ii) $F_h = C(f, g)$;

(iii) $d(x, h^\infty(x)) \leq c\rho(f(x), g(x)), \ \forall\, x \in X$.

We remark that $h^\infty(X) = C(f, g)$.

For some examples of c-weakly Picard pair see [6], pp. 37-40.

Definition 6.2. The equation

$$f(x) = g(x) \tag{6.1}$$

is Ulam-Hyers stable if there exists $c > 0$ such that: for each $\varepsilon > 0$ and for each solution y^* of the inequation

$$\rho(f(y), g(y)) \leq \varepsilon \tag{6.2}$$

there exists a solution x^* of (6.1) such that

$$d(y^*, x^*) \leq c\varepsilon.$$

In a similar way we can define the generalized Ulam-Hyers stability of equation (6.1) and in the case of the function spaces X and Y the Ulam-Hyers-Rassias stability and the generalized Ulam-Hyers-Rassias stability of the equation (6.1).

In what follows we shall consider the Ulam-Hyers stability.

Remark 6.1. If a pair $f, g : X \to Y$ is a c-weakly Picard pair, then the equation (6.1) is Ulam-Hyers stable.

Indeed, let y^* be a solution of the inequation (6.2). Then we take $x^* = h^\infty(y^*)$. From the condition (iii) in Definition 6.1 we have that

$$d(y^*, x^*) \leq c\rho(f(y^*), g(y^*)) \leq c\varepsilon.$$

From the Remark 6.1 it follows that from each c-weakly Picard pair we have an example of coincidence equation which is Ulam-Hyers stable.

Remark 6.2. For other considerations on stability of operatorial equations see [2], [11], [12], [13], [26], [27].

7. Ulam stability of the operatorial inclusions

In this section we follow the terminologies and the notations from [33]. See also [22] and [32]. For the convenience of the reader we shall present some of them.

Let (X, d) be a metric space, $A, B \in P(X)$, and $T : X \to P(X)$ a multivalued operator. We denote:

$P_{cp}(X) := \{Y \in P(X) \mid Y \text{ a compact subset of } X\}$

$D_d(A, B) := \inf\{d(a, b) \mid a \in A,\ b \in B\}$

$F_T := \{x \in X \mid x \in T(x)\}$ - the fixed point set of the operator T

$G(T) := \{(x, y) \mid x \in X,\ y \in T(x)\}$ - the graphic of T.

The following notions are given in [33].

Definition 7.1. An operator $T : X \to P(X)$ is a multivalued weakly Picard operator iff for each $x \in X$ and each $y \in T(x)$ there exists a sequence of successive approximations, $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$, such that

(i) $x_0 = x$, $x_1 = y$,

(ii) $x_n \xrightarrow{d} x^* \in F_T$.

Definition 7.2. For a multivalued weakly Picard operator T we define the multivalued operator $T^\infty : G(T) \to P(F_T)$ by

$T^\infty(x, y) := \{z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.

Definition 7.3. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0)$. An operator $T : X \to P(X)$ is ψ-multivalued weakly Picard if there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x, y)) \leq \psi(d(x, y)),\ \forall\ (x, y) \in G(T).$$

If $\psi(t) = ct$, $c > 0$, then T is called a c-multivalued weakly Picard operator.

For some examples of ψ-multivalued weakly Picard operator see [22], [33] and [32].

On the other hand we have the following notions of stability of the equation

$$x \in T(x) \tag{7.1}$$

Definition 7.4. The equation (7.1) is Ulam-Hyers stable if there exists $c > 0$ such that: for each $\varepsilon > 0$ and for each solution u^* of the inequation

$$D_d(u, T(u)) \leq \varepsilon \tag{7.2}$$

there exists a solution x^* of (7.1) such that

$$d(u^*, x^*) \leq c\varepsilon.$$

Definition 7.5. The equation (7.1) is generalized Ulam-Hyers stable if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous in 0 with $\psi(0) = 0$ such that: for each $\varepsilon > 0$ and for each solution u^* of the inequation (7.2) there exists a solution x^* of (7.1) such that

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

In a similar way, in the case of function spaces, we can define the Ulam-Hyers-Rassias stability and the generalized Ulam-Hyers-Rassias stability of the equation (7.1).

Remark 7.1. If the operator $T : X \to P_{cp}(X)$ is a c-multivalued weakly Picard operator, then the equation (7.1) is Ulam-Hyers stable. If $T : X \to P_{cp}(X)$ is a ψ-multivalued weakly Picard operator then the equation (7.1) is generalized Ulam-Hyers stable.

Indeed, let us suppose that T is a ψ-multivalued weakly Picard operator. Let u^* be a solution of (7.2). Let $y \in T(u^*)$ be such that $D_d(u^*, T(u^*)) = d(u^*, y)$. We take $x^* := t^\infty(u^*, y)$ and we have

$$d(u^*, x^*) = d(u^*, t^\infty(u^*, y)) \leq \psi(d(u^*, y)) \leq \psi(\varepsilon).$$

From Remark 7.1 it follows that for each example of ψ-multivalued weakly Picard operator we have an example of equation (7.1) which is generalized Ulam-Hyers stable.

References

[1] C. Alesina, R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl., **2**(1998), 373-380.

[2] W.W. Breckner, and T. Trif, *Convex Functions and Related Functional Equations*, Cluj Univ. Press, Cluj-Napoca, 2008.

[3] J. Brzdek, D. Popa, B. Xu, *Note on nonstability of the linear recurrence*, Abh. Math. Sem. Univ. Hamburg, **76**(2006), 183-189.

[4] J. Brzdek, D. Popa, B. Xu, *The Hyers-Ulam stability of nonlinear recurrences*, J. Math. Anal. Appl., **335**(2007), 443-449.

[5] J. Brzdek, D. Popa, B. Xu, *Hyers-Ulam stability for linear equations of higher orders*, Acta Math. Hungar., 2008.

[6] A. Buică, *Principii de coincidenţă şi aplicaţii*, Presa Universitară Clujeană, Cluj-Napoca, 2001.

[7] L. Cădariu, *Stabilitate Ulam-Hyers-Bourgin pentru ecuaţii funcţionale*, Ed. Univ. de Vest, Timişoara, 2007.

[8] A. Chiş-Novac, R. Precup, I.A. Rus, *Data dependence of fixed point for non-self generalized contractions*, Fixed Point Theory, **10**(2009), No. 1, 73-87.

[9] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, 2002.

[10] P. Găvruţă, *On a problem of G. Isac and Th. Rassias concerning the stability of mapping*, J. Math. Anal. Appl., **261**(2001), 543-553.

[11] P.M. Gruber, *Stability of isometries*, Trans. Amer. Math. Soc., **245**(1978), 263-277.

[12] G. Hirisawa, T. Miura, *Hyers-Ulam stability of a closed operator in a Hilbert space*, Bull. Korean Math. Soc., **43**(2006), No. 1, 107-117.

[13] D.H. Hyers, *The stability of homomorphism and related topics*, in: Global Analysis - Analysis on Manifolds (Th. M. Rassias, ed.), Teubner, Leipzig, 1983, 140-153.

[14] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[15] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[16] S.-M. Jung, *Hyers-Ulam stability of first order linear differential equations with constant coefficients*, J. Math. Anal. Appl., **320**(2006), 549-561.

[17] S.-M. Jung, *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory and Applications, Vol. 2007, 9 pages.

[18] S.-M. Jung, K.-S. Lee, *Hyers-Ulam stability of first order linear partial differential equations with constant coefficients*, Math. Ineq. Appl., **10**(2007), no. 2, 261-266.

[19] S.-M. Jung, Th.M. Rassias, *Generalized Hyers-Ulam stability of Riccati differential equation*, Math. Ineq. Appl., **11**(2008), No. 4, 777-782.

[20] T. Miura, S.-M. Jung, S.-E. Takahasi, *Hyers-Ulam-Rassias stability of the Banach space valued linear differential equation* $y' = \lambda y$, J. Korean Math. Soc., **41**(2004), 995-1005.

[21] Zs. Páles, *Generalized stability of the Cauchy functional equation*, Aequationes Math., **56**(1998), no. 3, 222-232.

[22] A. Petruşel, *Multivalued weakly Picard operators and applications*, Scientiae Mathematica Japonicae, **59**(2004), 167-202.

[23] D. Popa, *Hyers-Ulam-Rassias stability of linear recurrence*, J. Math. Anal. Appl., **309**(2005), 591-597.

[24] D. Popa, *Hyers-Ulam stability of the linear recurrence with constant coefficients*, Advances in Difference Equations, **2**(2005), 101-107.

[25] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4**(2003), No. 1, 91-96.

[26] Th. M. Rassias, *On the stability of the linear mappings in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[27] S. Reich, A.J. Zaslavski, *A stability result in fixed point theory*, Fixed Point Theory, **6**(2005), No. 1, 113-118.

[28] I.A. Rus, *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58**(2003), No. 1, 191-219.

[29] I.A. Rus, *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9**(2008), No. 2, 541-559.

[30] I.A. Rus, *Gronwall lemma approach to the Hyers-Ulam-Rassias stability of an integral equation*, in: Nonlinear Analysis and Variational Problems, pp. 147-152 (P. Pardalos, Th. M. Rassias and A.A. Khan (Eds.)), Springer, 2009.

[31] I.A. Rus, *Ulam stability of ordinary differential equations*, Studia Univ. Babeş-Bolyai, Math., **54**(2009), No. 4.

[32] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, 2008.

[33] I.A. Rus, A. Petruşel, A. Sîntămărian, *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal., **52**(2003), 1947-1959.

[34] M. Xu, *Hyers-Ulam-Rassias stability of a system of first order linear recurrences*, Bull. Korean Math. Soc., **44**(2007), No. 4, 841-849.

Received: 10. 02. 2009; Accepted: 02. 06. 2009.