Fixed Point Theory, 10(2009), No. 2, 289-303
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON THE GLOBAL BIFURCATION FOR SOLUTIONS OF LINEAR FREDHOLM INCLUSIONS WITH CONVEX-VALUED PERTURBATIONS

NGUYEN VAN LOI * AND VALERI OBUKHOVSKII **

* Faculty of Mathematics
Voronezh State Pedagogical University
(394043) Voronezh, Russia
E-mail: loinv14@yahoo.com

** Faculty of Mathematics
Voronezh State University
(394006) Voronezh, Russia
E-mail: valerio@math.vsu.ru

Abstract. We apply the topological degree theory for compact multivalued operators to study the global structure of solutions for an one-parameter family of inclusions containing a linear Fredholm operator, a nonlinear map and a convex-valued multimap.
Key Words and Phrases: Global bifurcation, Fredholm operator, multivalued map, topological degree, bifurcation index, discontinuous nonlinearity.
2000 Mathematics Subject Classification: 47J15, 47H04, 47H11, 47J05, 58B15.

1. Introduction

The bifurcation problem for inclusions with convex-valued multimaps was studied by J.C. Alexander and P.M. Fitzpatrick [1]. The authors of this work presented the sufficient conditions under which the set of all non-trivial solutions near the origin $(0, 0)$ admits a bifurcation to infinity, either bifurcation to the border of the considered domain, or bifurcation to some trivial solution of the inclusion.

In the present paper, by using the results of [1] and applying the topological degree theory for compact multivalued operators we consider the global bifurcation problem for a class of inclusions containing an abstract linear Fredholm operator with one-dimensional kernel and a convex-valued multioperator. The paper is organized as follows. In Section 2 we give the necessary preliminaries from the fields of multivalued maps, linear Fredholm operators and global bifurcation theorem of J.C. Alexander and P.M. Fitzpatrick. In Section 3 we describe the problem and present some examples arising from the study of equations with discontinuous nonlinearities . The main results are given in Theorems 7 and 8.

2. Preliminaries

Recall some notions and notation from the multivalued maps theory and the theory of linear Fredholm operators (see, e.g. [2, 5, 6, 7]).

Let X and Y be Banach spaces. Denote by $P(Y)$ $[C(Y), Cv(Y), Kv(Y)]$ the collection of all nonempty [respectively: nonempty closed, nonempty closed convex, nonempty compact convex] subsets of Y.

Definition 1. *A multimap $\mathcal{F}\colon X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) if for every open subset $V \subset Y$ the set*

$$\mathcal{F}_+^{-1}(V) = \{x \in X \colon \mathcal{F}(x) \subset V\}$$

is open in X. A multimap $\mathcal{F}\colon X \to C(Y)$ is closed if its graph

$$\Gamma_{\mathcal{F}} = \{(x, y) \in X \times Y : y \in \mathcal{F}(x)\}$$

is a closed subset of $X \times Y$. A multimap $\mathcal{F}$ is called compact if the set $\mathcal{F}(\tilde{X})$ is relatively compact for every bounded subset $\tilde{X} \subset X$.

Let $U \subset X$ be an open bounded subset and $\mathcal{F}\colon \overline{U} \to Kv(X)$ be a compact u.s.c. multimap. Denote by i the inclusion map. If $\mathcal{F}$ has no fixed points ($x \notin \mathcal{F}(x)$) on the boundary ∂U, then the topological degree $deg(i - \mathcal{F}, \overline{U})$ is well defined and has all usual properties (see, e.g. [2, 6, 7]).

Definition 2. *A bounded linear operator $L\colon domL \subseteq X \to Y$ is called Fredholm of index zero if*

(1i) *ImL is closed in Y;*

(2i) *$KerL$ and $CokerL = Y/ImL$ have the finite dimension and*

$$dim\, KerL = dim\, CokerL.$$

Let $L\colon domL \subseteq X \to Y$ be a Fredholm operator of index zero, then there exist projectors $P\colon X \to X$ and $Q\colon Y \to Y$ such that $ImP = KerL$ and $KerQ = ImL$. If the operator

$$L_P\colon domL \cap KerP \to ImL$$

is defined as the restriction of L on $domL \cap KerP$ then it is clear that L_P is an algebraic isomorphism and we may define $K_P\colon ImL \to domL$ as $K_P = L_P^{-1}$. Now if $\Pi\colon Y \to CokerL$ is the canonical surjection:

$$\Pi(z) = z + ImL$$

and $\Lambda\colon CokerL \to KerL$ is a one-to-one linear mapping, then the equation

$$Lx = y,\ y \in Y$$

is equivalent to the equation

$$(i - P)x = (\Lambda\Pi + K_{P,Q})y,$$

where $K_{P,Q}\colon Y \to X$ be defined as

$$K_{P,Q} = K_P(i - Q).$$

Now let $\mathcal{O} \subset X \times \mathbb{R}$ be an open subset containing the closed neighborhood $\mathcal{B}_X(0, r_1) \times [-r_2, r_2]$ of origin $(0, 0)$, where

$$\mathcal{B}_X(0, r_1) = \{x \in X\colon \|x\| \leq r_1\}.$$

Consider the following one-parameter family of inclusions

$$x \in F(x, \mu) \tag{1}$$

where $F\colon \mathcal{O} \to Kv(X)$ is a multimap.
Assume that

- $(F1)$ F is an u.s.c. and compact multimap and there exists a neighborhood V of 0 in $\mathbb{R}$ such that $0 \in F(0, \mu)$ when $\mu \in V$;
- $(F2)$ for each μ, $0 < |\mu| \leq r_2$, there is $\delta_\mu > 0$ such that $x \notin F(x, \mu)$ provided $0 < \|x\| \leq \delta_\mu$;
- $(F3)$ for each $\varepsilon > 0$ there exists $\eta > 0$ such that

  $$h(F(x, \mu), F(x, \mu')) < \varepsilon$$

  provided $(x, \mu), (x, \mu') \in \mathcal{B}_X(0, r_1) \times [-r_2, r_2]$ and $|\mu - \mu'| < \eta$, where h denotes the Hausdoff metric on the space $Kv(Y)$.

From $(F1)-(F2)$ it follows that for each μ, $0<|\mu|\leq r_2$ the topological degree

$$deg(i-F(\cdot,\mu),\mathcal{B}_X(0,\delta_\mu))$$

is well defined. Then **the bifurcation index of the multimap** F may be defined as

$$\mathcal{B}(F)=\lim_{\mu\to 0^+} deg(i-F(\cdot,\mu),\mathcal{B}_X(0,\delta_\mu))-\lim_{\mu\to 0^-} deg(i-F(\cdot,\mu),\mathcal{B}_X(0,\delta_\mu)).$$

We regard $\{0\}\times V$ as constituting the trivial solutions of inclusion (1) and let us denote by $\mathcal{S}$ the set of all non-trivial solutions to (1), i.e.,

$$\mathcal{S}=\{(x,\mu)\in\mathcal{O}\colon x\in F(x,\mu) \text{ and } (x,\mu)\notin\{0\}\times V\}$$

The following assertion can be found in [1].

Theorem 3. *Let conditions $(F1)-(F3)$ hold. Assume that $\mathcal{B}(F)\neq 0$. Then there exists a connected subset $\mathcal{C}$ of $\mathcal{S}$ with $(0,0)\in\overline{\mathcal{C}}$ and at least one of the following occurs:*

(a) $\mathcal{C}$ is unbounded;
(b) $\overline{\mathcal{C}}\cap\partial\mathcal{O}\neq\emptyset$;
(c) $(0,\mu_*)\in\overline{\mathcal{C}}$ for some $\mu_*\neq 0$.

3. Main results

3.1. **The statement of the problem.** Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary; for $p\geq 1$, let $L_p(\Omega)$ denote the Banach space of $p-$integrable functions on Ω with the norm

$$\|u\|_p=\|u\|_{L_p}=(\int_\Omega |u|^p dx)^{1/p}.$$

For two functions $f_*,f^*\in L_p(\Omega)$ with $f_*(x)\leq f^*(x)$ for a.e. $x\in\Omega$, denote by $[f_*,f^*]\subset L_p(\Omega)$ the interval

$$[f_*,f^*]=\{f\in L_p(\Omega): f_*(x)\leq f(x)\leq f^*(x) \text{ for a.e. } x\in\Omega\}.$$

For an integer $k>0$ we denote by $W_p^k(\Omega)$ the *Sobolev space* of functions

$$W_p^k(\Omega)=\{u\in L_p(\Omega)\colon D^\alpha u\in L_p(\Omega) \text{ for all } |\alpha|\leq k\},$$

where $D^\alpha u$ denotes the distributional derivative of u of order α. We will assume that $W_p^k(\Omega)$ is equipped with the norm

$$\|u\|_{p,k} = \sum_{\|\alpha\|\leq k} \|D^\alpha u\|_p.$$

By $\overset{\circ}{W}{}_p^k(\Omega)$ we will denote the subset of $W_p^k(\Omega)$ consisting of all functions vanishing on the boundary $\partial\Omega$; by $C(\Omega)$ denote the space of all continuous functions on Ω. The ball of radius r in $C(\Omega)$ will be denoted by $\mathcal{B}_C(0,r)$.

Let us recall (see, e.g. [4]) that according to Sobolev embedding theorem in case $pk > n$ the space $W_p^k(\Omega)$ is compactly embedded into $C(\Omega)$.

We will study the global structure of solutions of the following family of inclusions

$$Au + g(u,\mu) \in \Phi(u,\mu). \tag{2}$$

We assume the following hypotheses:

($A1$) the operator $A\colon domA := W_p^2(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega) \to L_p(\Omega)$ is a linear Fredholm operator of index zero with $p \geq 2$ and $2p > n$;

($A2$) A is selfajoint in the sence that

$$< Au, v >_{L_2} = < v, Au >_{L_2}$$

for all $u, v \in domA$, where $< u, v >_{L_2} = \int_\Omega uvdx$;

($A3$) $dimKerA = 1$ and $\omega \in domA$, $\|\omega\|_{L_2} = 1$, is the basic element of $KerA$;

(g1) the map $g\colon C(\Omega)\times\mathbb{R} \to L_p(\Omega)$ is continuous and bounded on bounded subsets and $g(0,\mu) = 0$ for all $\mu \in \mathbb{R}$;

(g2) there is $\varepsilon_0 > 0$ such that for every $\kappa > 0$ there exists a number $\delta_\kappa^{(1)} > 0$ such that

$$\|g(u,\mu) - g(u,\mu')\|_{L_p} < \kappa$$

provided $|\mu - \mu'| < \delta_\kappa^{(1)}$ and $(u,\mu), (u,\mu') \in \mathcal{B}_C(0,\varepsilon_0) \times [-\varepsilon_0, \varepsilon_0]$;

(Φ1) the multioperator $\Phi\colon C(\Omega) \times \mathbb{R} \to Cv(L_p(\Omega))$ is u.s.c. and bounded on bounded subsets and $0 \in \Phi(0,\mu)$ for all $\mu \in \mathbb{R}$;

(Φ2) for every $\kappa > 0$ there exists $\delta_\kappa^{(2)} > 0$ such that

$$h(\Phi(u,\mu), \Phi(u,\mu')) < \kappa$$

provided $|\mu-\mu'| < \delta_\kappa^{(2)}$ and $(u,\mu),(u,\mu') \in \mathcal{B}_C(0,\varepsilon_0)\times[-\varepsilon_0,\varepsilon_0]$, where ε_0 is the same as in condition $(g2)$ and h is the Hausdoff metric on the space of closed bounded subsets of $L_p(\Omega)$;

Definition 4. *A pair $(u,\mu) \in domA \times \mathbb{R}$ is called a solution to family* (2) *if there exists a function $f \in \Phi(u,\mu)$ such that*

$$Au + g(u,\mu) = f.$$

Let us denote by $\mathcal{S}$ the set of all non-trivial solutions of family (2), i.e.,

$$\mathcal{S} = \{(u,\mu) \in domA \times \mathbb{R} \colon u \neq 0 \text{ and } Au + g(u,\mu) \in \Phi(u,\mu)\}.$$

Remark 5. *Family of inclusions* (2) *naturally arises while the study of equations with discontinuous nonlinearities. For example, we can consider the following family of equations*

$$(Au)(x) + g(u,\mu)(x) = \mu\varphi(x,u(x)), \tag{3}$$

where the function $\varphi\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:

($\varphi 1$) for a.e. $x\in\Omega$ there exist finite limits

$$\underline{\varphi}(x,\xi) = \liminf_{\xi'\to\xi}\varphi(x,\xi'); \qquad \overline{\varphi}(x,\xi) = \limsup_{\xi'\to\xi}\varphi(x,\xi')$$

and the functions $\underline{\varphi}$, $\overline{\varphi}$ are superpositionally measurable;

($\varphi 2$) there exist functions $f_*, f^* \in L_p(\Omega)$ such that

$$f_*(x) \le \varphi(x,\xi) \le f^*(x)$$

for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}$.

Let us recall (see, e.g. [8]*) that Carathéodory functions, pointwise limits of continuous functions, and Borel measurable functions belong to the class of superpositionally measurable functions.*

Define the multimap $\Phi : C(\Omega) \to Cv(L_p(\Omega))$ by the rule

$$\Phi(u) = [\underline{\varphi}(x,u(x)), \overline{\varphi}(x,u(x))]. \tag{4}$$

According to [3]*, Theorem 1.1 let us mention the following property.*

Proposition 6. *The multimap Φ is u.s.c.*

So, we may substitute the family of equations (3) *by the following family of operator inclusions*

$$Au + g(u,\mu) \in \mu\Phi(u),$$

whose solutions are called the generalized solutions to (3).

In turn, let us mention that equations of type (3) *appear in many problems of mathematical physics. For example, Lavrentiev's problem on detachable currents at the presence of resonance and nonlinear perturbations may be described by the following equation (cf.* [9]*):*

$$-\triangle u(x) + \lambda u(x) + g(u(x),\mu) = \mu\, sign(u(x)), \tag{5}$$

$$u(x)|_{\partial\Omega} = 0,$$

where $\mu > 0$.

Here

$$\underline{\varphi}(x,\xi) = \begin{cases} 1, & \xi > 0, \\ -1, & \xi \leq 0; \end{cases}$$

and

$$\overline{\varphi}(x,\xi) = \begin{cases} 1, & \xi \geq 0, \\ -1, & \xi < 0. \end{cases}$$

Now defining the multimap $\Phi : C(\Omega) \times (0,+\infty) \to Cv(L_p(\Omega))$ *as* (4)*, we may substitute equation* (5) *by the following operator inclusion*

$$Au + g(u,\mu) \in \Phi(u,\mu),$$

where $Au = -\triangle u + \lambda u$.

3.2. Main results: the global structure of $\mathcal{S}$.

Theorem 7. *Let conditions* $(A1)-(A3)$, $(g1)-(g2)$ *and* $(\Phi1)-(\Phi2)$ *hold. In addition, assume that*

$(g3)$ *for* $\varepsilon_0 > 0$ *same as in* $(g2)$ *and* $(\Phi2)$ *there exist number* $\beta > 0$ *and a function* $h\colon [-\varepsilon_0, 0) \cup (0, \varepsilon_0] \to (0,+\infty)$ *such that*

$$| < g(u,\mu), \omega >_{L_2} | \geq h(\mu)\|u\|^{\beta}_{L_2},$$

for $(u,\mu) \in \mathcal{B}_C(0,\varepsilon_0) \times [-\varepsilon_0, \varepsilon_0]$, $\mu \neq 0$;

$(g4)$ *if* $a \neq 0$ *and* $\mu \neq 0$ *are such that* $|a\mu|$ *is sufficiently small, then*

$$a\mu < g(a\omega,\mu), \omega >_{L_2} > 0;$$

($\Phi 3$) *there exist* $c > 0$ *and* $\alpha > \beta$ *such that*

$$\|\Phi(u,\mu)\|_{L_2} \leq c\,\|u\|^{\alpha}_{L_2},$$

for $(u,\mu) \in \mathcal{B}_C(0,\varepsilon_0) \times [-\varepsilon_0,\varepsilon_0]$, $\mu \neq 0$, *where*

$$\|\Phi(u,\mu)\|_{L_2} = \sup\{\|f\|_{L_2} \colon f \in \Phi(u,\mu)\}.$$

Then there is a connected subset $\mathcal{C} \subset \mathcal{S}$ *with* $(0,0) \in \overline{\mathcal{C}}$ *and at least one of the following occurs:*

(1^0) $\mathcal{C}$ *is unbounded;*

(2^0) $(0,\mu_*) \in \overline{\mathcal{C}}$ *for some* $\mu_* \neq 0$.

Proof. Let us mention that for every $(u,\mu) \in C(\Omega) \times \mathbb{R}$ the value

$$\Phi(u,\mu) - g(u,\mu)$$

is a bounded convex closed subset of $L_p(\Omega)$. Hence, the set

$$(\Lambda\Pi + K_{P,Q}) \circ (\Phi(u,\mu) - g(u,\mu))$$

is bounded convex closed in $domA$. From the Sobolev embedding theorem it follows that

$$(\Lambda\Pi + K_{P,Q}) \circ (\Phi(u,\mu) - g(u,\mu))$$

is a compact convex subset of $C(\Omega)$.

Now define the multimap $F \colon C(\Omega) \times \mathbb{R} \to Kv(C(\Omega))$ as

$$F(u,\mu) = Pu + (\Lambda\Pi + K_{P,Q}) \circ (\Phi(u,\mu) - g(u,\mu)).$$

Then the problem of bifurcation of solutions of family (2) can be reduced to the problem of bifurcation of solutions of the following inclusion

$$u \in F(u,\mu). \tag{6}$$

We will show that the multimap F satisfies conditions $(F1)-(F3)$ of Theorem 3.

At first, notice that from the properties of the operator A it follows that the spaces $E = domA$ and $Z = L_p(\Omega)$ may be decomposed as

$$E = E_0 \oplus E_1,$$

where $E_0 = KerA$ and

$$Z = Z_0 \oplus Z_1,$$

where $Z_0 = KerA$ and $Z_1 = ImA$. The corresponding decompositions of elements $u \in E$ and $f \in Z$ will be denoted by

$$u = u_0 + u_1,$$

and

$$f = f_0 + f_1.$$

STEP 1. From conditions $(g1)$ and $(\Phi 1)$ we have $0 \in F(0, \mu)$ when $\mu \in \mathbb{R}$, i.e., $\{0\} \times \mathbb{R}$ is the set of all trivial solutions to (2). Following [[10], Lemma 5] let us demonstrate that the multimap F is u.s.c. and compact.

In fact, denote by Z_w the space Z endowed with the weak topology. From the reflexivity of space $L_p(\Omega)$ and conditions $(g1)$ and $(\Phi 1)$ it follows that the multimap

$$\Psi \colon C(\Omega) \times \mathbb{R} \to P(Z_w),$$
$$\Psi(u, \mu) = \Phi(u, \mu) - g(u, \mu),$$

has $w-$compact values and is $\omega-$compact. Moreover, from the same conditions it follows that Ψ is u.s.c.

Now, let us demonstrate that the multimap $\Theta \circ \Psi \colon C(\Omega) \times \mathbb{R} \to Kv(C(\Omega))$ is closed, where $\Theta = \Lambda\Pi + K_{P,Q}$. In fact, let $\{(u_n, \mu_n)\} \subset C(\Omega) \times \mathbb{R}$, $(u_n, \mu_n) \to (u_0, \mu_0)$, $\{y_n\} \subset C(\Omega)$, $y_n \in \Theta \circ \Psi(u_n, \mu_n)$, and $y_n \to y_0$. Take a sequence $z_n \in \Psi(u_n, \mu_n)$ such that $y_n = \Theta(z_n)$. We may assume w.l.o.g. that $z_n \underset{w}{\to} z_0$. Since Θ is the continuous linear operator, we have that $y_0 = \Theta(z_0)$. From the other side, the multimap Ψ is closed with respect to the weak topology of Z (see, e.g. [2], [7]) and hence $z_0 \in \Psi(u_0, \lambda_0)$. So

$$y_0 \in \Theta \circ \Psi(u_0, \lambda_0).$$

Further, for every bounded subset $U \subset C(\Omega) \times \mathbb{R}$, $\Psi(U)$ is a bounded subset of Z. But then $\Theta \circ \Psi(U)$ is a bounded subset of E, and by the Sobolev embedding theorem it is relatively compact subset of $C(\Omega)$. Closed and compact multimap $\Theta \circ \Psi$ is u.s.c. (see, e.g. [2], [7]) and now the assertion follows from the fact that P is continuous and has a finite-dimensional range. So condition $(F1)$ holds.

STEP 2. We are going to show that for each μ, $0 < |\mu| \leq \varepsilon_0$ (where ε_0 is constant in $(g2)$), there exists a number $\pi_\mu = \pi(\mu) \in (0, \varepsilon_0)$ such that

$$u \notin F(u, \mu)$$

provided $u \in \mathcal{B}_C(0, \pi_\mu) \setminus \{0\}$.

In fact, let $(u, \mu) \in \mathcal{B}_C(0, \varepsilon_0) \times [-\varepsilon_0, \varepsilon_0]$, $u \neq 0$, $\mu \neq 0$ be a solution of family (6), then there exists a function $f \in \Phi(u, \mu)$ such that

$$Au + g(u, \mu) = f.$$

By multiplying this equality by ω, we obtain

$$< g(u, \mu), \omega >_{L_2} = < f, \omega >_{L_2} .$$

By virtue of $(g3)$ we have

$$h(\mu) \|u\|_{L_2}^{\beta} \leq | < f, \omega >_{L_2} |.$$

On the other hand

$$| < f, \omega >_{L_2} | \leq \|f\|_{L_2}.$$

By virtue of $(\Phi 3)$ we have

$$| < f, \omega >_{L_2} | \leq c \, \|u\|_{L_2}^{\alpha}.$$

Therefore

$$h(\mu) \|u\|_{L_2}^{\beta} \leq c \, \|u\|_{L_2}^{\alpha}.$$

So

$$\|u\|_{L_2}^{\alpha - \beta} \geq \frac{h(\mu)}{c}.$$

Notice that

$$\|u\|_{L_2}^{\alpha - \beta} \leq \|u\|_{C}^{\alpha - \beta} \, (mes(\Omega))^{\frac{\alpha - \beta}{2}},$$

where $mes(\Omega)$ is measure of Ω.

Hence we obtain the following estimate for u:

$$\|u\|_C \geq \frac{h^{1/(\alpha - \beta)}(\mu)}{c^{1/(\alpha - \beta)} \, (mes(\Omega))^{1/2}}.$$

Set

$$r(\mu) = \frac{h^{1/(\alpha - \beta)}(\mu)}{c^{1/(\alpha - \beta)} \, (mes(\Omega))^{1/2}}.$$

and choose $0 < \pi_\mu < \min\{\varepsilon_0, r(\mu)\}$. Then we obtain that family (6) has only trivial solution on $\mathcal{B}_C(0, \pi_\mu)$. So condition $(F2)$ holds true.

Step 3. We are going to show that for every $\kappa > 0$ there is $\delta_\kappa > 0$ such that

$$h(F(u, \mu), F(u, \mu')) < \kappa$$

provided $(u, \mu), (u, \mu') \in \mathcal{B}_C(0, \varepsilon_0) \times [-\varepsilon_0, \varepsilon_0]$ and $|\mu - \mu'| < \delta_\kappa$.

In fact, from $(g2)$ and $(\Phi 2)$ it follows that for a given $\kappa > 0$ there exist $\delta_\kappa^{(1)} > 0$ and $\delta_\kappa^{(2)} > 0$ such that

$$\|g(u,\mu) - g(u,\mu')\|_{L_p} < \frac{\kappa}{2T}$$

provided $(u,\mu),(u,\mu') \in \mathcal{B}_C(0,\varepsilon_0) \times [-\varepsilon_0,\varepsilon_0]$ and $|\mu - \mu'| < \delta_\kappa^{(1)}$,
and

$$h(\Phi(u,\mu),\Phi(u,\mu')) < \frac{\kappa}{2T},$$

provided $(u,\mu),(u,\mu') \in \mathcal{B}_C(0,\varepsilon_0) \times [-\varepsilon_0,\varepsilon_0]$ and $|\mu - \mu'| < \delta_\kappa^{(2)}$, where

$$T = \|\Lambda\Pi + K_{P,Q}\|.$$

Choosing $\delta_\kappa = \min\{\delta_\kappa^{(1)}, \delta_\kappa^{(2)}\}$ we obtain that

$$h(F(u,\mu),F(u,\mu')) < \kappa$$

provided $(u,\mu),(u,\mu') \in \mathcal{B}_C(0,\varepsilon_0) \times [-\varepsilon_0,\varepsilon_0]$ and $|\mu - \mu'| < \delta_\kappa$. So condition $(F3)$ holds true.

Step 4. Now, we will evaluate the bifurcation index $\mathcal{B}(F)$. Towards this goal, fix $\mu, 0 < |\mu| < \varepsilon_0$ and choose π_μ as in Step 2. Let $\Sigma_\mu\colon C(\Omega) \times [0,1] \to Kv(C(\Omega))$ be defined as

$$\Sigma_\mu(u,\lambda) = Pu + (\Lambda\Pi + K_{P,Q}) \circ (\alpha(\Phi(u,\mu),\lambda) - g_0(u,\mu) - \lambda g_1(u,\mu)),$$

where $g(u,\mu) = g_0(u,\mu) + g_1(u,\mu) \in Z_0 + Z_1$ and $\alpha\colon L_p(\Omega) \times [0,1] \to L_p(\Omega)$ is defined as

$$\alpha(f_0 + f_1, \lambda) = f_0 + \lambda f_1.$$

The compactness and upper semicontinuity of the multimap Σ_μ may be verified as in Step 1. We will show that

$$u \notin \Sigma_\mu(u,\lambda)$$

provided $(u,\lambda) \in \partial\mathcal{B}_C(0,\pi_\mu) \times [0,1]$.

To the contrary, assume that there are an element $u \in \partial\mathcal{B}_C(0,\pi_\mu)$ and $\lambda \in [0,1]$ such that $u \in \Sigma_\mu(u,\lambda)$. Then there exists a function $f \in \Phi(u,\mu)$ such that

$$\begin{cases} Au_1 + \lambda g_1(u,\mu) = \lambda f_1 \\ g_0(u,\mu) = f_0 \end{cases}$$

where $f_0 \in Z_0$, $f_1 \in Z_1$ and $f_0 + f_1 = f$.
We obtain

$$< g(u,\mu), \omega >_{L_2} = < g_0(u,\mu), \omega >_{L_2} = < f_0, \omega >_{L_2} = < f, \omega >_{L_2} .$$

As in Step 2, these relations imply

$$\|u\|_C \geq r(\mu) > \pi_\mu,$$

giving the contradiction.

So Σ_μ is a homotopy connecting two multimaps $\Sigma_\mu(\cdot, 0)$ and $\Sigma_\mu(\cdot, 1)$. From the homotopy invariance property of the topological degree we obtain

$$deg(i - \Sigma_\mu(\cdot, 1), \mathcal{B}_C(0, \pi_\mu)) = deg(i - \Sigma_\mu(\cdot, 0), \mathcal{B}_C(0, \pi_\mu)).$$

Notice that the multimap $\Sigma_\mu(\cdot, 0)$ has the form

$$\begin{aligned}\Sigma_\mu(\cdot, 0) &= Pu + (\Lambda\Pi + K_{P,Q}) \circ (\Phi_0(u,\mu) - g_0(u,\mu)) \\ &= Pu + \Lambda\Pi \circ (\Phi_0(u,\mu) - g_0(u,\mu)).\end{aligned}$$

W.l.o.g. we may assume that the maps $\Pi|_{Z_0}$ and Λ are identities. Then the multimap $\Sigma_\mu(\cdot, 0)$ has its range in E_0, and in accordance with the principle of map restriction (see, e.g. [2, 7])

$$deg(i - \Sigma_\mu(\cdot, 0), \mathcal{B}_C(0, \pi_\mu)) = deg(i - \Sigma^0_\mu(\cdot, 0), \mathcal{B}^0_C(0, \pi_\mu)),$$

where $\mathcal{B}^0_C(0, \pi_\mu) = \mathcal{B}_C(0, \pi_\mu) \cap E_0$ and $\Sigma^0_\mu(\cdot, 0)$ is the restriction of $\Sigma_\mu(\cdot, 0)$ to $\mathcal{B}^0_C(0, \pi_\mu)$.
The multifield $i - \Sigma^0_\mu(\cdot, 0)$ has the form

$$g_0(u_0, \mu) - \Phi_0(u_0, \mu).$$

Since 0 is the unique singular point of the field $i - \Sigma^0_\mu(\cdot, 0)$ in the ball $\mathcal{B}^0_C(0, \pi_\mu)$, we have

$$deg(i - \Sigma^0_\mu(\cdot, 0), \mathcal{B}^0_C(0, \pi_\mu)) = deg(i - \Sigma^0_\mu(\cdot, 0), \mathcal{B}^0_C(0, \tau)),$$

where $0 < \tau < \pi_\mu$.

Consider the fields $i - \Sigma^0_\mu(\cdot, 0)$ and μi on the sphere $\partial\mathcal{B}^0_C(0, \tau)$, $0 < \tau < \pi_\mu$. Let $u_0 = a\omega$ and $f_0 \in \Phi_0(a\omega, \mu)$, where $|a| = \frac{\tau}{\|\omega\|_C}$. Denoting

$$\sigma = < g_0(a\omega, \mu) - f_0, \mu a\omega >_{L_2}$$

we have

$$\begin{aligned}\sigma &= \mu a \int_\Omega g_0(a\omega,\mu)\omega\,dx - \mu a\int_\Omega f_0\omega\,dx \\ &= \mu a\left(\int_\Omega g(a\omega,\mu)\omega\,dx - \int_\Omega f\omega\,dx\right)\end{aligned}$$

where $f = f_0 + f_1 \in \Phi(a\omega,\mu)$.

The case $\mu > 0$:

If $a = \frac{\tau}{\|\omega\|_C}$: by virtue of $(g3)$ and $(g4)$ we have

$$< g(a\omega,\mu),\omega >_{L_2} \geq\ h(\mu)a^\beta.$$

Therefore,

$$\sigma \geq \mu a\left(h(\mu)a^\beta - \int_\Omega |f|\,|\omega|\,dx\right) \geq \mu a\left(h(\mu)a^\beta - \|f\|_{L_2}\right).$$

From $(\Phi 3)$ it follows that $\|f\|_{L_2} \leq c\,a^\alpha$. Hence, we obtain

$$\sigma \geq \mu a^{\beta+1}\left(h(\mu) - c\,a^{\alpha-\beta}\right) > 0$$

for sufficiently small $\tau > 0$.

If $a = -\frac{\tau}{\|\omega\|_C}$: by virtue of $(g3)$ and $(g4)$ we have

$$- < g(a\omega,\mu),\omega >_{L_2} \geq\ h(\mu)|a|^\beta.$$

Therefore,

$$\begin{aligned}\sigma &= -\mu a\left(\int_\Omega -g(a\omega,\mu)\omega dx + \int_\Omega f\omega dx\right) \\ &\geq -\mu a\left(h(\mu)|a|^\beta - \int_\Omega |f||\omega|dx\right) \\ &\geq -\mu a\left(h(\mu)|a|^\beta - \|f\|_{L_2}\right) \\ &\geq \mu|a|^{\beta+1}\left(h(\mu) - c|a|^{\alpha-\beta}\right) > 0\end{aligned}$$

for sufficiently small $\tau > 0$.

The case $\mu < 0$: analogously we have that $\sigma > 0$ for sufficiently small $\tau > 0$.

So, the fields $i - \Sigma^0_\mu(\cdot,0)$ and μi have no opposite directions, hence they are homotopic on $\partial\mathcal{B}^0_C(0,\tau)$ (see, e.g. [2]). Therefore we have

$$deg(i - \Sigma^0_\mu(\cdot,0), \mathcal{B}^0_C(0,\tau)) = deg(\mu i, \mathcal{B}^0_C(0,\tau)).$$

We obtain

$$
\begin{aligned}
&\mathcal{B}(F) = \mathcal{B}(\Sigma_\mu(\cdot, 1)) \\
&= \lim_{\mu \to 0^+} deg(i - \Sigma_\mu(\cdot, 1), \mathcal{B}_C(0, \pi_\mu)) - \lim_{\mu \to 0^-} deg(i - \Sigma_\mu(\cdot, 1), \mathcal{B}_C(0, \pi_\mu)) \\
&= \lim_{\mu \to 0^+} deg(i - \Sigma_\mu(\cdot, 0), \mathcal{B}_C(0, \pi_\mu)) - \lim_{\mu \to 0^-} deg(i - \Sigma_\mu(\cdot, 0), \mathcal{B}_C(0, \pi_\mu)) \\
&= \lim_{\mu \to 0^+} deg(i - \Sigma^0_\mu(\cdot, 0), \mathcal{B}^0_C(0, \pi_\mu)) - \lim_{\mu \to 0^-} deg(i - \Sigma^0_\mu(\cdot, 0), \mathcal{B}^0_C(0, \pi_\mu)) \\
&= \lim_{\mu \to 0^+} deg(i - \Sigma^0_\mu(\cdot, 0), \mathcal{B}^0_C(0, \tau)) - \lim_{\mu \to 0^-} deg(i - \Sigma^0_\mu(\cdot, 0), \mathcal{B}^0_C(0, \tau)) \\
&= \lim_{\mu \to 0^+} deg(\mu i, \mathcal{B}^0_C(0, \tau)) - \lim_{\mu \to 0^-} deg(\mu i, \mathcal{B}^0_C(0, \tau)) = 2,
\end{aligned}
$$

where $\tau \in (0, \pi_\mu)$ is sufficiently small.

To complete the proof we need only to apply Theorem 3 with the remark that the set $C(\Omega) \times \mathbb{R}$ is unbounded, and so the case (b) of Theorem 3 is imposible. □

Theorem 8. *Let conditions $(A1) - (A3)$, $(g1) - (g2)$ and $(\Phi1) - (\Phi2)$ hold. In addition, assume that*

$(g'3)$ *there are numbers $c_1 > 0$ and $\beta_1 > 0$ such that*

$$\|g(u, \mu)\|_{L_2} \leq c_1 \|u\|^{\beta_1}_{L_2},$$

for $(u, \mu) \in \mathcal{B}_C(0, \varepsilon_0) \times [-\varepsilon_0, \varepsilon_0]$, $\mu \neq 0$;

$(\Phi'3)$ *There exist a function $\ell\colon [-\varepsilon_0, 0) \cup (0, \varepsilon_0] \to (0, +\infty)$ and a number $0 < \alpha_1 < \beta_1$ such that*

$$| < \Phi(u, \mu), \omega >_{L_2} | \geq \ell(\mu) \ \|u\|^{\alpha_1}_{L_2},$$

for every $(u, \mu) \in \mathcal{B}_C(0, \varepsilon_0) \times [-\varepsilon_0, \varepsilon_0]$, $\mu \neq 0$, where

$$< \Phi(u, \mu), \omega >_{L_2} = \{< f, \omega >_{L_2} : f \in \Phi(u, \mu)\};$$

$(\Phi4)$ *for $a \neq 0$ and $\mu \neq 0$ are such that $|a\mu|$ is sufficiently small we have*

$$a\mu < \Phi(a\omega, \mu), \omega >_{L_2} \subset (0, +\infty).$$

Then there exists a connected subset $\mathcal{C} \subset \mathcal{S}$ with $(0, 0) \in \overline{\mathcal{C}}$ and either $\mathcal{C}$ is unbounded or $(0, \mu_*) \in \overline{\mathcal{C}}$ for some $\mu_* \neq 0$.

Proof. The proof of this theorem is analogous to that given in Theorem 7 with one change when we evaluate the bifurcation index $\mathcal{B}(F)$. In this case, to evaluate the bifurcation index we will consider the fields $i - \Sigma^0_\mu(\cdot, 0)$ and $-\mu i$ on the sphere $\partial\mathcal{B}^0_C(0,\tau)$. These fields are homotopic and hence we obtain that $\mathcal{B}(F) = -2$. Finally, we apply Theorem 3 to get the global structure of solutions of family (2). □

References

[1] J.C. Alexander and P.M. Fitzpatrick, *Global bifurcation for solutions of equations involving several parameter multivalued condensing mappings,* Fixed point theory, Proc. Sherbrooke Que. 1980, ed. E. Fadell, G. Fournier, Springer Lect. Notes, 886, 1-19.

[2] Yu.G. Borisovich, B.D. Gelman, A.D. Myshkis and V.V. Obukhovskii, *Introduction to the Theory of Multimap and Differential Inclusions,* Moscow: KomKnhiga, 2005, 216 pp. (Russian)

[3] K.C. Chang, *The obstacle problem and partial differential equations with discontinuous nonlinearities,* Comm. Pure Appl. Math., **33**(1980), no. 2, 117-146.

[4] Z. Denkowski, S. Migórski, N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic Publishers, Boston, MA, 2003.

[5] R.E. Gaines, J.L. Mawhin, *Coincidence degree and nonlinear differential equations,* Lecture Notes in Mathematics, no. 568, Springer-Verlag, Berlin-New York, 1977.

[6] Górniewicz, L., *Topological Fixed Point Theory of Multivalued Mappings*, 2nd edition, Topological Fixed Point Yheory and Its Applications 4, Springer-Verlag, Dordrecht, 2006.

[7] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces,* Walter de Gruyter, Berlin-New York, 2001.

[8] M.A. Krasnosel'skii, A.V. Pokrovskii, *Systems with Hysteresis,* Nauka, Moscow, 1983 (in Russian), English translation, Springer-Verlag, Berlin, 1989.

[9] M.A. Krasnosel'skii, A.V. Pokrovskii, *On elliptic equations with discontinuous nonlinearities* (in Russian), Dokl. Akad. Nauk, **342**(1995), no. 6, 731-734.

[10] V. Obukhovskii, P. Zecca and V. Zvyagin, *On some generalizations of the Landesman-Lazer theorem,* Fixed Point Theory, **8**(2007), no. 1, 69-85.

Received: 27. 04. 2009; Accepted: 09. 07. 2009.