***Fixed Point Theory***, 10(2009), No. 2, 275-288
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

# APPROXIMATE FIXED POINTS OF NONEXPANSIVE MAPS

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**Abstract.** A subset $K$ of a Banach space is said to have the approximate fixed point property if $\inf\{\|x - T(x)\| : x \in K\} = 0$ for any nonexpansive mapping $T : K \to K$ . This is a brief overview of what is known about the approximate fixed point property. Many questions remain open.

**Key Words and Phrases**: Approximate fixed points, fixed points, nonexpansive mappings
**2000 Mathematics Subject Classification**: 54H25, 47H09.

## 1. Introduction

Let $K$ be a subset of a Banach space and $T : K \to K$ a nonexpansive mapping ($\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in K$). An *approximate fixed point set* for $T$ is a set of the type

$$F_\varepsilon(T) := \{x \in K : \|x - Tx\| \leq \varepsilon\}$$

for some $\varepsilon > 0$. The set $K$ is said to have the *approximate fixed point property* if $F_\varepsilon(T) \neq \emptyset$ for each $\varepsilon > 0$ and each nonexpansive map $T : K \to K$. It is easy to see that if $K$ is bounded and convex, then $K$ has the approximate fixed point property Indeed, let $T : K \to K$ be nonexpansive and consider the mapping $T_\lambda := \lambda I + (1 - \lambda) T$ for $\lambda \in (0, 1)$. Then $T_\lambda$ is a contraction mapping for each $\lambda$ it has a fixed point $x_\lambda \in K$. Thus

$$\|x_\lambda - Tx_\lambda\| = (1 - \lambda) \|x_\lambda - Tx_\lambda\| \to 0 \text{ as } \lambda \to 1.$$

It is less obvious that some *unbounded* convex sets have the approximate fixed point property. However it has been known for many years that a closed convex subset of a reflexive Banach space has the approximate fixed point property if and only if it is linearly bounded (i.e., its intersection with any line is bounded) (Reich [27]). Subsequently Shafrir introduced the notion of a directionally bounded set in [30] and proved the following facts: (1) A convex subset $K$ of an arbitrary Banach space has the approximate fixed point property if an only if it is directionally bounded. (2) For a Banach space $X$ the following are equivalent: (i) $X$ is reflexive; (ii) every closed convex linearly bounded set is directionally bounded. Combined, these facts yield the following well-known characterization.

**Theorem 1.** *A Banach space $X$ is reflexive $\Leftrightarrow$ every linearly bounded convex subset of $X$ has the approximate fixed point property.*

It is also noteworthy that Reich and Matoušková have recently shown in [29] that every infinite dimensional Banach space (not necessarily reflexive) contains an unbounded closed convex set with the approximate fixed point property. As an aside, it remains unknown whether there exists an unbounded closed convex set in a Banach space with the fixed point property for nonexpansive mappings, although W. O. Ray has shown in [26] that such a set cannot exist in a Hilbert space.

## 2. Preliminaries

A *path* in a metric space $(X, d)$ is a continuous image of the unit interval $I = [0,1] \subset \mathbb{R}$. If $S \equiv f(I)$ is a path then its *length* is defined as

$$\ell(S) = \sup_{(x_i)} \sum_{i=0}^{N-1} d\left(f(x_i), f(x_{i+1})\right)$$

where $0 = x_0 < x_1 < \cdots < x_N = 1$ is any partition of $[0,1]$. If $\ell(S) < \infty$ then the path is said to be *rectifiable*.

A metric space $(X, d)$ is said to be a *length space* if the distance between each two points $x, y$ of $X$ is the infimum of the lengths of all rectifiable paths joining them. In this case, $d$ is said to be a *length metric* (otherwise known an *inner metric* or *intrinsic metric*).

A length space $X$ is called a *geodesic space* if there is a path $S$ joining each two points $x, y \in X$ for which $\ell(S) = d(x, y)$. Such a path is often called a *metric segment* (or *geodesic segment*) with endpoints $x$ and $y$. There is a simple criterion which assures the existence of metric segments. A metric space $(X, d)$ is said to be *metrically convex* if given any two points $p, q \in M$ there exists a point $z \in X$, $p \neq z \neq q$, such that

$$d(p, z) + d(z, q) = d(p, q).$$

**Theorem 2** (Menger [25]). *Any two points of a complete and metrically convex metric space are the endpoints of at least one metric segment.*

Menger based the proof of his classical result on transfinite induction. Since then, other proofs have been given - see, e.g., [19] for a proof and citations.

There is an analog of Menger's criterion for length spaces. Here we use $B(x; r)$ to denote the closed ball centered at $x \in X$ with radius $r \geq 0$.

**Definition 3** ([15]). A metric space $(X, d)$ is said to satisfy *property* $(A)$ if given any two points $x, y \in X$, any two numbers $b, c \geq 0$ such that $b + c = d(x, y)$, and any $\varepsilon > 0$,

$$B(x; b + \varepsilon) \cap B(y; c + \varepsilon) \neq \emptyset. \tag{A}$$

The proof of Theorem 1 of [15] yields the following fact.

**Theorem 4.** *If a complete metric space* $(X, d)$ *satisfies property* $(A)$ *then each two points of* $X$ *can be joined by a rectifiable path.* (*Thus* $X$ *has an intrinsic metric.*)

## 3. Structure of approximate fixed point sets

Little is known about the structure of the sets $F_\varepsilon(T)$ in general, although such sets are known to be pathwise connected for bounded convex domains. One proof of this fact uses the following result of Edelstein and O'Brien [8]. A non-uniform version of this result is due to Ishikawa [14] and, in fact, the result is known to hold uniformly over the class of all nonexpansive self-mappings of $K$ (see [12]). This result shows that there is always a nonexpansive mapping of $K$ *into* $F_\varepsilon(T)$, although there is nothing to assure that this mapping is a retraction, or that such a retraction exists.

**Theorem 5** ([8])**.** *Suppose $K$ is a nonempty bounded convex subset of a Banach space and suppose $T : K \to K$ is nonexpansive. Then $f := \lambda I + (1-\lambda) T$ is uniformly asymptotically regular for each $\lambda \in (0,1)$. That is, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left\| f^n x - f^{n+1} x \right\| \leq \varepsilon$ for all $n \geq N$ and all $x \in K$. In particular, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$, $f^n : K \to F_\varepsilon (T)$.*

If $x \in F_{(1-\lambda)\varepsilon} (f)$ then $x \in F_\varepsilon (T)$. Also if $y$ is on the segment joining $x$ and $f(x)$ then

$$\|y - f(y)\| \leq \|y - f(x)\| + \|f(x) - f(y)\|$$

$$\leq \|y - f(x)\| + \|x - y\| = \|x - f(x)\| .$$

Thus if $x \in F_\varepsilon (T)$ then every point on the segment joining $x$ and $f(x)$ lies in $F_\varepsilon (T)$. To see that the above theorem implies $F_\varepsilon (T)$ is pathwise connected, let $u, v \in F_\varepsilon (T)$ and choose $N$ so large that $f^N (K) \subseteq F_\varepsilon (T)$. Then the image under $f^N$ of the segment joining $u$ and $v$ maps into a path joining $f^N u$ and $f^N v$. Moreover the segments joining $f^i u$ and $f^{i+1} u$, $i = 0, \cdots, N-1$ all lie in $F_\varepsilon (T)$. Similarly the segments joining $f^i v$ and $f^{i+1} v$, $i = 0, \cdots, N-1$. By piecing these together one obtains a path $S$ in $F_\varepsilon (T)$ joining $u$ and $v$. Moreover, $\ell (S) \leq 2\varepsilon N + \|u - v\|$. Thus we have the following fact, first noticed by Bruck [6].

**Theorem 6.** *Suppose $K$ is a nonempty bounded convex subset of a Banach space and suppose $T : K \to K$ is nonexpansive. Then for each $\varepsilon > 0$, $F_\varepsilon (T)$ is nonempty and rectifiably pathwise connected.*

A class of mappings more general than the nonexpansive mappings has recently been introduced for which Theorem 5 holds when $\lambda \in (0, 1/2]$. A mapping $T : C \to X$ defined on a subset $C$ of a Banach space $X$ is said to satisfy *condition* $(C)$ (Suzuki [32]) if

$$\frac{1}{2} \|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\| \tag{C}$$

for all $x, y \in C$. Obviously all nonexpansive mappings satisfy condition $(C)$. However the converse is not true. Indeed an example is given in [32] which shows that mappings satisfying condition $(C)$ need not even be continuous.

However they are *locally directionally nonexpansive* in the following sense: Suppose $K$ is convex and let $\lambda \in (0, 1/2]$. Then if $m_\lambda = \lambda x + (1-\lambda)\,Tx$,

$$\frac{1}{2}\,\|x - Tx\| \leq (1-\lambda)\,\|x - Tx\| = \|x - m_\lambda\|\,.$$

Thus

$$\|Tx - Tm_\lambda\| \leq \|x - m_\lambda\|\,.$$

Mappings which satisfy the above condition for all $\lambda \in [0, 1]$ are called *directionally nonexpansive* in [21]. The following is Theorem 1 of [21].

**Theorem 7.** *Let $K$ be a bounded convex subset of a Banach space $X$, let $T : K \to K$ be directionally nonexpansive, fix $\alpha \in (0,1)$, and define $fx = \alpha x + (1-\alpha)\,Tx$ for $x \in K$. Then for each $\varepsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that $\left\|f^n x - f^{n+1}x\right\| \leq \varepsilon$ for all $n \geq N$ and $x \in K$.*

The proof of Theorem 7 carries over without change to prove the following. This observation is due to Suzuki [32].

**Theorem 8.** *Let $K$ be a bounded convex subset of a Banach space $X$, let $T : K \to K$ satisfy condition $(C)$, fix $\alpha \in (0,1/2)$, and define $fx = \alpha x + (1-\alpha)\,Tx$ for $x \in K$. Then for each $\varepsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that $\left\|f^n x - f^{n+1}x\right\| \leq \varepsilon$ for all $n \geq N$ and $x \in K$.*

The proof of Theorem 7 given in [21] is an adaptation of the method of [12]. In in fact shows that the same $N$ works not only for all $x \in K$ but also for all directionally nonexpansive $T \in K^K$.

## 4. Commuting families

It has been known for a long time that the fixed point property for nonexpansive mappings implies the common fixed point property for commuting families of nonexpansive mappings under very general assumptions (R. E. Bruck [5]). A closed convex set $K$ is said to have the *hereditary fixed point* property (HFPP) if every nonexpansive mapping $T : K \to K$ has a fixed point in every nonempty $T$-invariant closed convex subset of $K$. Bruck showed in [4] that if a weakly compact convex set has the HFPP then the fixed point set of every nonexpansive mapping $T : K \to K$ is a nonempty nonexpansive retract of $K$. Using this fact it is easy to see that any two commuting nonexpansive mappings $T, G$ of $K \to K$ have a nonempty common fixed point set. (In [5]

Bruck extended this fact to infinite families.) A long standing and seemingly deep open question (brought to the writer's attention many years ago by J.-B. Baillon) is whether this fact extends to approximate fixed points.

**Question 1.** *If $K$ is a nonempty bounded closed convex subset of a Banach space, and if $T, G : K \to K$ are commuting nonexpansive mappings, then is $F_\varepsilon(T) \cap F_\varepsilon(G) \neq \emptyset$ for each $\varepsilon > 0$?*

The existence of a common fixed point for commuting nonexpansive mappings is trivial if one of the mappings is strictly contractive. (A mapping $T : K \to K$ is strictly contractive if $\|Tx - Ty\| < \|x - y\|$ for all $x, y \in K$, $x \neq y$.) In this case the retraction result is not needed. If a strictly contractive $T : K \to K$ has a fixed point it must be unique and necessarily fixed under any mapping with which it commutes. This suggests that the following special case of Question 1 might be more tractable, although this is by no means obvious.

**Question 2.** *If $K$ is a nonempty bounded closed convex subset of a Banach space, and if $T, G : K \to K$ are commuting nonexpansive mappings, at least one of which is strictly contractive, then is $F_\varepsilon(T) \cap F_\varepsilon(G) \neq \emptyset$ for each $\varepsilon > 0$? What if both are strictly contractive?*

Since $F_\varepsilon(T)$ is rectifiably pathwise connected it posseses an intrinsic metric $\rho$ obtained by taking $\rho(x, y)$ to be the infimum of the lengths of all paths joining $x, y \in F_\varepsilon(T)$. If a nonexpansive mapping $G : K \to K$ commutes with $T$, it is always the case that $G(F_\varepsilon(T)) \subseteq F_\varepsilon(T)$. Also $G$ is nonexpansive relative to $\rho$. Question 1 now becomes:

**Question 3.** When does $G$ have approximate fixed points in the space $(F_\varepsilon(T), \rho)$?

**Remark 1**. It might in fact be preferable to study the structure of the possibly smaller set obtained by taking the closure of the set

$$F_\varepsilon^0(T) := \{x \in K : \|x - Tx\| < \varepsilon\}.$$

As the following example illustrates, the set $\overline{F_\varepsilon^0(T)}$ can be much nicer than $F_\varepsilon(T)$, and it is also invariant under $G$.

**Example** (Bruck [6]) Let $C$ be the rectangle $[0,2]\times[-1,1]$ in the Euclidean space $\mathbb{R}^2$, and define

$$T(x,y) = (x - \min(x,1), 0).$$

It is easy to see that $T$ is nonexpansive and that the set $F_1(T)$ consists of the closed unit disk intersected with the right half-plane along with the segment

$$\{(x,0) : 1 \leq x \leq 2\}.$$

However $\overline{F_1^0(T)}$ consists of just the closed unit disk intersected with the right half-plane.

## 5. Almost convex maps

**Definition 9** ([11])**.** A mapping $T : K \to X$ is said to be *$\alpha$-almost convex* for a continuous strictly increasing $\alpha : R^+ \to R^+$ with $\alpha(0) = 0$ if for each $x, y \in K$ and $\lambda \in [0,1]$,

$$J_T(\lambda x + (1-\lambda)y) \leq \alpha(\max\{J_T(x), J_T(y)\})$$

*where* $J_T(u) := \|u - Tu\|$, $u \in K$.

These mappings arise optimization theory (e.g., [7]). A number of examples are given Garcia-Falset, et al. [11]. If $\alpha(t) = rt$ for some $r \in \mathbb{R}^+$ then $T$ is said to be *$r$-almost convex*, and if $r = 1$ $T$ is simply said to be *almost convex.* The following alternative principle is proved in [11].

**Alternative Principle:** *If $K$ is a closed bounded convex set, and $T : K \to K$, then at least one of the following holds:*

(i) *$T$ is $r$-almost convex for some $r > 0$, or*

(ii) $\inf\{J_T(x) : x \in K\} = 0$; *that is, $T$ admits approximate fixed points in $K$.*

**Theorem 10.** *Suppose $K$ is a nonempty bounded convex subset of a Banach space, and suppose $T$ and $G$ are two commuting nonexpansive mappings of $K \to K$ at least one of which is $\alpha$-almost convex. Then $F_\varepsilon(T) \cap F_\varepsilon(G) \neq \emptyset$ for each $\varepsilon > 0$.*

*Proof.* Suppose $T$ is $\alpha$-almost convex. Let $\varepsilon > 0$, and let $f = (1-\lambda) I + \lambda G$. By Theorem 5 it is possible to choose $N \in \mathbb{N}$ so large that

$$\|f^n x - G \circ f^n x\| \leq \varepsilon$$

for all $x \in K$ and all $n \geq N$. For any $u \in K$,

$$\begin{aligned} J_T(Gu) &= \|Gu - T \circ Gu\| \\ &= \|Gu - G \circ Tu\| \\ &\leq \|u - Tu\| \\ &= J_T(u). \end{aligned}$$

Therefore for any $x \in K$,

$$\begin{aligned} J_T(f^n x) &= J_T\left((1-\lambda) f^{n-1}x + \lambda G \circ f^{n-1}x\right) \\ &\leq \alpha\left(\max\left\{J_T\left(f^{n-1}x\right), J_T\left(G \circ f^{n-1}x\right)\right\}\right) \\ &= \alpha\left(J_T\left(f^{n-1}x\right)\right) \\ &\leq \cdots \\ &\leq \alpha^n\left(J_T(x)\right). \end{aligned}$$

Since $\alpha^n$ is continuous at 0 it is possible to choose $\delta > 0$ so that $J_T(x) \leq \delta \Rightarrow \alpha^n(J_T(x)) \leq \varepsilon$. Further we may assume $\delta \leq \varepsilon$. Therefore if $x \in F_\delta(T)$ and $n \geq N$, then $f^n x \in F_\varepsilon(T) \cap F_\varepsilon(G)$. Since $F_\delta(T) \neq \emptyset$, for each $\delta > 0$ the proof is complete. □

**Remark 2.** The fact that the mapping $T$ in Theorem 10 is $\alpha$-almost convex means that for each $x, y \in K$ and $\lambda \in [0,1]$,

$$J_T(\lambda x + (1-\lambda) y) \leq \alpha\left(\max\left\{J_T(x), J_T(y)\right\}\right).$$

However an inspection of the proof reveals that it suffices to assume only that *there exists* $\lambda \in (0,1)$ *such that for each* $u \in K$

$$J_T((1-\lambda) u + \lambda Tu) \leq \alpha\left(\max\left\{J_T(u), J_T(Gu)\right\}\right). \tag{1}$$

## 6. An ultrapower connection

There is another connection between the existence of common approximate fixed points and the structure of the common fixed point sets. To describe this we need some more notation. Assume that $K$ is a bounded closed convex subset of a Banach space $X$ and let $I$ be a set and let $\mathcal{U}$ be a nontrivial ultrafilter on $I$. Let $\tilde{X}$ denote the Banach space ultrapower of $X$ over $\mathcal{U}$. Thus the elements of $\tilde{X}$ consist of equivalence classes $[(x_i)]_{i \in I}$ for which

$$\|\tilde{x}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\| < \infty,$$

with $(u_i) \in [(x_i)]$ if and only if $\lim_{\mathcal{U}} \|u_i - x_i\| = 0$. (For a more detailed description of this setting see, e.g., [1], [16], [31].)

Let

$$\tilde{K} = \left\{ \tilde{x} = [(x_i)] \in \tilde{X} : x_i \in K \text{ for each } i \right\}.$$

Now assume $T : K \to K$ is nonexpansive. For $\tilde{x} = [(x_i)] \in \tilde{K}$, define $\tilde{T} : \tilde{K} \to \tilde{X}$ by setting

$$\tilde{T}(\tilde{x}) = [(T(x_i))].$$

The mapping $\tilde{T}$ is well-defined and also nonexpansive.

Since $F_\varepsilon(T) \neq \emptyset$ for each $\varepsilon > 0$, $T$ has an approximate fixed point sequence, that is, a sequence $(x_n)$ for which $\|x_n - Tx_n\| \to 0$. The point $\tilde{x} := [(x_n)] \in \tilde{K}$ is a fixed point of $\tilde{T}$. Little is known about the structure of the fixed point set of $\tilde{T}$, although it is known to be metrically convex [24] (see also Elton, et al. [9] for a proof). If $T, G : K \to K$ are commuting nonexpansive mappings then the existence of a common approximate fixed point sequence for $T$ and $G$ is equivalent to the existence of a common fixed point of the corresponding mappings $\widetilde{T}, \widetilde{G}$ of $\tilde{K} \to \tilde{K}$. Moreover if the fixed point set of $\tilde{T}$ is a nonexpansive retract of $\tilde{K}$, then such a common fixed point exists. In fact, combining Theorem 3.2 with Corollary 3.3 of [34] yields the following result.

**Theorem 11.** *Suppose $T, G : K \to K$ are commuting nonexpansive mappings, and suppose $Fix\left(\tilde{T}\right)$ is a nonexpansive retract of $\tilde{K}$. Then $Fix\left(\tilde{T}\right) \cap Fix\left(\tilde{G}\right) \neq \emptyset$. Moreover $T$ and $G$ have a common approximate fixed point sequence.*

## 7. Hyperbolic spaces

Approximate fixed points of nonexpansive mappings also exist in a broader context. To describe the setting we adopt the terminology of [17]. A *hyperbolic space* is a triple $(X, \rho, W)$, where $(X, \rho)$ is a metric space and $W : X \times X \times [0,1] \to X$ satisfies

(W1) $\rho(z, W(x, y, \lambda)) \leq (1-\lambda)\rho(z, x) + \lambda\rho(z, y)$,

(W2) $\rho\left(W(x, y, \lambda), W\left(x, y, \bar{\lambda}\right)\right) = \left|\lambda - \bar{\lambda}\right| \rho(x, y)$,

(W3) $W(x, y, \lambda) = W(y, x, 1-\lambda)$,

(W4) $\rho(W(x, z, \lambda), W(y, w, \lambda)) \leq (1-\lambda)\rho(x, y) + \lambda\rho(z, w)$.

If only axiom (W1) is assumed this structure is a convex metric space in the sense of Takahashi [33]. If (W1)-(W3) are assumed the notion is equivalent to spaces called of *hyperbolic type* in [12]. Axiom (W4) is used for example in [28]. However Kohlenbach's definition is less restrictive than that given in [28] in that it does not require the existence of metric lines. Hence it includes all CAT(0) spaces, whereas the definition in [28] includes only those CAT(0) space which have the unique geodesic extension property.

The following fact about approximate fixed points is an immediate consequence of results of [20].

**Theorem 12.** *Let $(M, d)$ be a metric space of hyperbolic type and let $K$ be a bounded convex subset of $M$. Suppose $T : K \to K$ is nonexpansive. Then* $\inf\{d(x, T(x)) : x \in K\} = 0$.

It is natural to ask whether an analog of Theorem 12 holds if $d$ is a length metric. However in this setting the absence of metric segments makes it difficult to formulate an analog of the hyperbolic definition. This difficulty can be circumvented by passing to a metric space ultrapower $\tilde{X}$ of the underlying space $X$. One can do this by isometrically embedding $X$ in a Banach space$E$. If $\tilde{E}$ denotes the Banach space ultrapower of $E$ relative to some nontrivial ultrafilter $\mathcal{U}$, take

$$\tilde{X} := \left\{\tilde{x} = [(x_n)] \in \tilde{E} : x_n \in X \text{ for each } n\right\}.$$

Then for $\tilde{x}, \tilde{y} \in \tilde{X}$, set $\tilde{\rho}(\tilde{x}, \tilde{y}) = \lim_{\mathcal{U}} \|x_n - y_n\| = \lim_{\mathcal{U}} d(x_n, y_n)$.

**Theorem 13.** *A complete metric space $(X, \rho)$ is a length space if and only if every nontrivial ultrapower $\tilde{X}$ of $X$ is a geodesic space.*

*Proof.* Let $p, q \in X$ and $\alpha = (1/2)\,\rho\,(p, q)$. Let $(\varepsilon_n) \subset (0, \infty)$ with $(\varepsilon_n) \to 0$. The fact that $X$ is a length space assures the existence of a sequence $\{m_n\} \subset B\,(p; \alpha + \varepsilon_n) \cap B\,(q; \alpha + \varepsilon_n)$. If $\tilde{m} = [(m_n)]$ then $\tilde{\rho}\,(\tilde{p}, \tilde{m}) = \tilde{\rho}\,(\tilde{q}, \tilde{m}) = (1/2)\,\tilde{\rho}\,(\tilde{p}, \tilde{q})$. Since $\tilde{X}$ is complete, $X$ is a geodesic space by the criterion of Menger. On the other hand, if $\tilde{X}$ is a geodesic space then it is easy to verify that $X$ satisfies Property (A) of Section 2; hence $X$ is a length space by Theorem 4. □

One can now say that a length space $(X, \rho)$ is of *hyperbolic type* if some nontrivial ultrapower $\tilde{X}$ of $X$ is of hyperbolic type in the usual sense. It is now possible to extend Theorem 12 as follows.

**Theorem 14.** *Let $(X, \rho)$ be a length space of hyperbolic type, and suppose $T : M \to M$ is nonexpansive. Then* $\inf\{d\,(x, T\,(x)) : x \in X\} = 0$.

.

*Proof.* It follows that if $(X, \rho)$ is a length space of almost hyperbolic type then $\left(\tilde{X}, \tilde{\rho}\right)$ is a metric space of hyperbolic type. Thus

$$\inf\left\{\tilde{\rho}\left(\tilde{x}, \tilde{T}\,(\tilde{x})\right) : \tilde{x} \in \tilde{M}\right\} = 0$$

by Theorem 12 From this one can extract a sequence $(x_n)$ in $X$ such that $\rho\,(x_n, T\,(x_n)) \to 0$. □

**Remark 3.** Frim the above we conclude that if approximate fixed point sets (or sets of the type $F_\varepsilon^0\,(T)$) are of either hyperbolic type, or of almost hyperbolic type relative to their intrinsic metric, then the answer to Question 1 is affirmative.

## 8. Product spaces

The approximate fixed point property has also been considered in product spaces. In [10] it is proved that if $M$ is a metric space which has the approximate fixed point property for nonexpansive mappings and if $K$ is a bounded convex subset of a Banach space, then $(K \times M)_\infty$ also has the approximate fixed point property for nonexpansive mappings. In [23] it is noted that this fact extends to hyperbolic spaces.

**Theorem 15.** *Suppose $C$ is a nonempty bounded convex subset of a hyperbolic space $X$ and suppose $(M, d)$ is a metric space which has the approximate fixed point property. Then the product space*

$$H := (C \times M)_\infty$$

*has the approximate fixed point property.*

This can be proved by following the argument of Theorem 25 of [23] where $X$ is assumed to be a CAT(0) space. It was erroneously claimed in [23] (Remark 26) that the above theorem holds if it is merely assumed that $X$ is of hyperbolic type. In fact the Axiom (W4) seems to be essential. As noted in [18] (where Theorem 15 is extended to unbounded sets), the axiom (W4) is also used in is used in an essential way to establish a key ingredient in the proof due to Borwein, Reich, and Shafrir [2].

## 9. CAT(0) spaces

A CAT(0) space is a globally non-positively curved geodesic space. Such spaces share many properties of Hilbert space in a uniformly convex metric setting. For a detailed definition and properties of such spaces we refer to [3], Chapter II. We conclude this survey by stating a metric analog of Theorem 1. This is Theorem 25 of [22].

**Theorem 16.** *Let $X$ be a complete CAT(0) space with the geodesic extension property. Then a closed convex subset of $X$ has the approximate fixed point property if and only if it is geodesically bounded.*

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*Received: 10. 04. 2009; Accepted: 15. 06. 2009*