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COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS DEFINED ON $\mathscr{L}$-FUZZY METRIC SPACES WITH NONLINEAR CONTRACTIVE TYPE CONDITION

SINIŠA N. JEŠIĆ*, NATAŠA A. BABAČEV*,
DONAL O'REGAN** AND RALE M. NIKOLIĆ***

*Department of Mathematics, Faculty of Electrical Engineering,
Bulevar Kralja Aleksandra 73, P.O. Box 35-54, 11120 Beograd, Serbia
E-mail: jesha@eunet.rs

**Department of Mathematics National University of Ireland Galway, Ireland
E-mail: donal.oregan@nuigalway.ie

***Department of Mathematics, Technical Faculty Čačak, University of Kragujevac,
Svetog Save 65, 32000 Čačak, Serbia

Abstract. This paper presents a common fixed point theorem for four mappings defined on $\mathscr{L}$-fuzzy metric spaces. Some comments and examples are given.
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1. Introduction

There are many papers that provide fixed or common fixed point theorems defined on spaces with non-deterministic distances. Most of them ([3], [15], [16]) include conditions with restrictions, like the condition that the t-norm $\mathscr{T}$ must satisfy $\mathscr{T}(a,a) \geq a$. It is known that only the t-norm $\mathscr{T}(a,b) = \min\{a,b\}$ satisfies this property, and consequently, the results obtained with this property are quite restrictive. Also, fixed and common fixed point theorems proved on spaces satisfying this condition are analogous to results for mappings defined on metric spaces and they are proved with the same technique as results on metric spaces.

In this paper we present some results satisfying nonlinear contractive type condition defined using the function φ which satisfies $\varphi(t) < t$. Nonlinear contractive type conditions for mappings defined on metric spaces were discussed by D.W. Boyd et al. in [2], R.P. Pant in [12], for mappings defined on fuzzy metric spaces by D. Miheţ in [10] and for mappings defined on probabilistic metric spaces by S. Ješić et al. in [8] and by D. O'Regan et al. in [13] and many others. Also, a common fixed point theorem with a nonlinear contractive type condition for mappings defined on intuitionistic and $\mathscr{L}$-fuzzy metric spaces was proved by S. Ješić and N. Babačev in [7], and for mappings defined on transversal spaces by S. Ješić et al. in [9]. On the other hand R. Saadati et al. in [14] and Adibi et al. in [1] have proved common fixed point theorems for mappings satisfying linear contracitve type condition, defined on $\mathscr{L}$-fuzzy metric spaces. The following result will be an extension and improvement for most of the previous results, and also will extend the results from metric to $\mathscr{L}$-fuzzy metric spaces.

2. Preliminaries

The concept of $\mathscr{L}$-fuzzy metric spaces was introduced by R. Saadati et al. in [14].

Definition 2.1. [14] Let $\mathscr{L} = (L, \leq_L)$ be a complete lattice, and U a nonempty set called universe. An $\mathscr{L}$-fuzzy set $\mathscr{A}$ on U is defined as a mapping $\mathscr{A} : U \to L$. For each u in U, $\mathscr{A}(u)$ represents the degree (in L) to which u satisfies $\mathscr{A}$. We define $0_{\mathscr{L}} = \inf L$ and $1_{\mathscr{L}} = \sup L$.

Definition 2.2. [14] A triangular norm (t-norm) on $\mathscr{L}$ is a mapping $\mathscr{T} : L^2 \to L$ satisfying the following conditions for all $x, y, z, x_1, y_1 \in L$

(i) $\mathscr{T}(x, 1_{\mathscr{L}}) = x$,

(ii) $\mathscr{T}(x, y) = \mathscr{T}(y, x)$,

(iii) $\mathscr{T}(x, \mathscr{T}(y, z)) = \mathscr{T}(\mathscr{T}(x, y), z)$,

(iv) $x \leq_L x_1$ and $y \leq_L y_1 \Rightarrow \mathscr{T}(x, y) \leq_L \mathscr{T}(x_1, y_1)$.

A t-norm $\mathscr{T}$ can also be defined recursevely as an $(n+1)$-ary operation, $n \in \mathbb{N}$, by $\mathscr{T}^1 = \mathscr{T}$ and

$$\mathscr{T}^n(x_1, \ldots, x_{n+1}) = \mathscr{T}\left(\mathscr{T}^{n-1}(x_1, \ldots, x_n), x_{n+1}\right), \tag{1}$$

for $n \geq 2$ and $x_1, \ldots, x_{n+1} \in L$.

Definition 2.3. [14] The triple $(X, \mathscr{M}, \mathscr{T})$ is said to be an $\mathscr{L}$-fuzzy metric space if X is an arbitrary (non-empty) set, $\mathscr{T}$ is a continuous t-norm on $\mathscr{L}$ and $\mathscr{M}$ is an $\mathscr{L}$-fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for every x, y, z in X and t, s in $(0, \infty)$:

(LF1) $\mathscr{M}(x, y, t) >_L 0_{\mathscr{L}}$;
(LF2) $\mathscr{M}(x, y, t) = 1_{\mathscr{L}}$ for all $t > 0$ if and only if $x = y$;
(LF3) $\mathscr{M}(x, y, t) = \mathscr{M}(y, x, t)$;
(LF4) $\mathscr{T}(\mathscr{M}(x, y, t), \mathscr{M}(y, z, s)) \leq_L \mathscr{M}(x, z, t + s)$;
(LF5) $\mathscr{M}(x, y, \cdot) : (0, \infty) \to L$ is continuous.

In this case $\mathscr{M}$ is called an $\mathscr{L}$-fuzzy metric.

Remark 2.4.[14] *Every fuzzy metric space is an $\mathscr{L}$-fuzzy metric space.*

Definition 2.5. [1] Let $(X, \mathscr{M}, \mathscr{T})$ be an $\mathscr{L}$-fuzzy metric space. $\mathscr{M}$ is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n \to +\infty} \mathscr{M}(x_n, y_n, t_n) = \mathscr{M}(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to $(x, y, t) \in X^2 \times (0, \infty)$.

Lemma 2.6. [1] *Let $(X, \mathscr{M}, \mathscr{T})$ be an $\mathscr{L}$-fuzzy metric space. Then $\mathscr{M}$ is a continuous function on $X^2 \times (0, \infty)$.*

In this paper we will consider $\mathscr{L}$-fuzzy metric spaces that satisfy

$$\mathscr{M}(x, y, 0) = 0_{\mathscr{L}} \text{ for } x \neq y. \tag{2}$$

Remark 2.7. *Let $(X, \mathscr{M}, \mathscr{T})$ be an $\mathscr{L}$-fuzzy metric space and $A \subseteq X$. Then $\mathscr{M}(x, y, \cdot)$ is nondecreasing function for al $x, y \in X$.*

Definition 2.8. [14] A negation on $\mathscr{L}$ is any (strictly) decreasing mapping $\mathscr{N} : L \to L$ satisfying $\mathscr{N}(0_{\mathscr{L}}) = 1_{\mathscr{L}}$ and $\mathscr{N}(1_{\mathscr{L}}) = 0_{\mathscr{L}}$. If $\mathscr{N}(\mathscr{N}(x)) = x$, for all $x \in L$, then $\mathscr{N}$ is called an involutive negation. The negation N_s on

$([0,1],\leq)$ defined as $N_s(x) = 1 - x$ for all $x \in [0,1]$, is called the standard negation on $([0,1],\leq)$.

We will consider the t-norm $\mathscr{T}$ which satisfies the following condition:

For every $\mu \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and arbitrary $n \in \mathbb{N}, n \geq 2$ there exists $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that

$$\mathscr{T}^{n-1}(\mathscr{N}(\lambda), \dots, \mathscr{N}(\lambda)) >_L \mathscr{N}(\mu). \tag{3}$$

Remark 2.9. ([6], [14]) *The condition* (3) *holds for every continuous t-norm $\mathscr{T}$ and for every involutive negation $\mathscr{N}$ on $L \setminus (\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}, \leq_L)$ (follows from the theorem of mean values). Also, all t-norms of h-type with standard negation satisfy condition* (3).

Definition 2.10. [14] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an $\mathscr{L}$-fuzzy metric space $(X, \mathscr{M}, \mathscr{T})$ is called a Cauchy sequence if for each $\varepsilon \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$,

$$\mathscr{M}(x_m, x_n, t) > \mathscr{N}(\varepsilon).$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$ in the $\mathscr{L}$-fuzzy metric space $(X, \mathscr{M}, \mathscr{T})$ if $\mathscr{M}(x_n, x, t) \to 1_{\mathscr{L}}$ whenever $n \to \infty$ for every $t > 0$. An $\mathscr{L}$-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

It is obvious that definitions of convergence and Cauchy sequences depend on the choice of negation $\mathscr{N}$. From this it follows that it needs to be pointed out that we assume the space $(X, \mathscr{M}, \mathscr{T})$ is complete w.r.t. negation $\mathscr{N}$.

The following definition and lemma on $\mathscr{L}$F-strongly bounded sets are given in [7].

Definition 2.11. [7] Let $(X, \mathscr{M}, \mathscr{T})$ be an $\mathscr{L}$-fuzzy metric space and $A \subseteq X$. The $\mathscr{L}$-fuzzy diameter of the set A is defined by

$$\delta_A = \sup_{t>0} \inf_{x,y\in A} \sup_{\varepsilon<t} M(x, y, \varepsilon).$$

If $\delta_A = 1_{\mathscr{L}}$ then we say that the set A is $\mathscr{L}$F-strongly bounded.

Lemma 2.12. [7] *The set $A \subseteq X$ is $\mathscr{L}F$-strongly bounded if and only if for arbitrary negation $\mathscr{N}(\lambda)$ and each $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ there exists $t > 0$ such that $\mathscr{M}(x, y, t) > \mathscr{N}(\lambda)$ for all $x, y \in A$.*

Proof. Let $A \subseteq X$ be an $\mathscr{L}$F-strongly bounded set. The statement follows trivially for arbitrary negation $\mathscr{N}(\lambda) \in [0_{\mathscr{L}}, 1_{\mathscr{L}}]$, from the definitions of inf and sup of a set.

Conversely, since for arbitrary negation $\mathscr{N}(\lambda)$, and each $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ there exists $t > 0$ such that $\mathscr{M}(x, y, t) > \mathscr{N}(\lambda)$ for all $x, y \in A$, then this holds for every arbitrary involutive negation. If $\mathscr{N}(\lambda)$ is involutive negation then for each $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ there exists $\mu = \mathscr{N}(\lambda)$ such that $\mathscr{N}(\mu) = \lambda$. This means that $\sup_{\mu \in L} \mathscr{N}(\mu) = \sup_{\lambda \in L} \lambda = 1_{\mathscr{L}}$ and we are finished.

Following A. George et al. ([4]), we introduce the concept of $\mathscr{L}$-fuzzy diameter zero for a collection of sets in $\mathscr{L}$ fuzzy metric spaces.

Definition 2.13. Let $(X, \mathscr{M}, \mathscr{T})$ be an $\mathscr{L}$-fuzzy metric space. A collection $\{F_n\}_{n\in\mathbb{N}}$ is said to have $\mathscr{L}$-fuzzy diameter zero if for each $r \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and every $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathscr{M}(x, y, t) >_L \mathscr{N}(r)$ for all $x, y \in F_{n_0}$.

Theorem 2.14. *Let $(X, \mathscr{M}, \mathscr{T})$ be a complete $\mathscr{L}$-fuzzy metric space, w.r.t. continuous negation $\mathscr{N}$. Then every collection of nonempty, nested, closed sets $\{F_n\}_{n\in\mathbb{N}}$ with $\mathscr{L}$-fuzzy diameter zero has a nonempty intersection. Also, the element $x \in \bigcap_{n\in\mathbb{N}} F_n$ is unique.*

Proof. Let $(X, \mathscr{M}, \mathscr{T})$ be a complete $\mathscr{L}$-fuzzy metric space and $\{F_n\}_{n\in\mathbb{N}}$ a collection of nonempty, nested, closed sets with $\mathscr{L}$-fuzzy diameter zero. Let $x_n \in F_n$ be arbitrary for every $n \in \mathbb{N}$. We will prove that the sequence $\{x_n\}$ is a Cauchy sequence. Let $r \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$ be arbitrary. Since $\{F_n\}$ has $\mathscr{L}$-fuzzy diameter zero, it follows that there exists $n_0 \in \mathbb{N}$ such that $\mathscr{M}(x, y, t) >_L \mathscr{N}(r)$ for every $x, y \in F_{n_0}$. Since F_n is a nested sequence, it follows that $\mathscr{M}(x_n, x_m, t) >_L \mathscr{N}(r)$ for all $n, m \geq n_0$, i.e. the sequence $\{x_n\}$ is a Cauchy sequence. Since the space is complete, it follows that there exists $x \in X$ such that $x_n \to x$. Since $x \in F_n$ for all n, it follows that $x \in \bigcap_{n\in\mathbb{N}} F_n$.

We now prove that $x \in \bigcap_{n\in\mathbb{N}} F_n$ is the unique element that belongs to this intersection. Assume that there exists $y \in \bigcap_{n\in\mathbb{N}} F_n,\ x \neq y$. Since $\{F_n\}$ has

$\mathscr{L}$-fuzzy diameter zero, for arbitrary $t > 0$ it follows that $\mathscr{M}(x, y, t) >_L \mathscr{N}(\frac{1}{n})$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get $\mathscr{M}(x, y, t) = 1_{\mathscr{L}}$, i.e. $x = y$.

In fixed point theory a very important role is played by generalizations of commutativity. The concept of compatible mappings was introduced by G. Junck ([5]) and S.N. Mishra ([11]). There are many generalizations of compatibility in different senses. Recently, B. Singh et al. introduced the concept of weak compatibility in [16]. We fuzzify these definitions.

Definition 2.15. Let $(X, \mathscr{M}, \mathscr{T})$ be a $\mathscr{L}$-fuzzy metric space and S and T self-mappings on X. We say that the mappings S and T are compatible if

$$\lim_{n\to\infty} \mathscr{M}(STx_n, TSx_n, t) = 1_{\mathscr{L}} \text{ for every } t > 0, \tag{4}$$

holds whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \in X$ holds.

Definition 2.16. Let $(X, \mathscr{M}, \mathscr{T})$ be an $\mathscr{L}$-fuzzy metric space and S and T self-mappings on X. We say that the mappings S and T are weakly compatible if for some $z \in X$ holds that $Sz = Tz$ then $STz = TSz$.

It is easy to see that the class of compatible mappings is broader than the class of commuting mappings. Indeed, every pair of commuting mappings is also compatible, while the converse is not true ([16]). Also, every pair of compatible mappings is weakly compatible, as the following remark shows.

Remark 2.17. *Let S and T be compatible mappings on $\mathscr{L}$-fuzzy metric space $(X, \mathscr{M}, \mathscr{T})$. Then the following holds:*

If for some $z \in X$ we have $Sz = Tz$ then $STz = TSz$.

This follows directly from Definition 2.15 taking $x_n = z$ for every $n \in \mathbb{N}$ for some point $z \in X$.

Examples of compatible and weak compatible mappings can be found in [5], [11] and [16].

3. Main results

Lemma 3.1. *Let $(X, \mathscr{M}, \mathscr{T})$ be a $\mathscr{L}$-fuzzy metric space which satisfies* (2)*. Let $\varphi : (0,\infty) \to (0,\infty)$ be a continuous, non-decreasing mapping such that $\varphi(t) < t$ holds for every $t > 0$. Then the following statement holds:*

If for every $x, y \in X$ we have $\mathscr{M}(x,y,\varphi(t)) \geq_L \mathscr{M}(x,y,t)$ for every $t > 0$ then $x = y$.

Proof. Let us suppose that $\mathscr{M}(x,y,\varphi(t)) \geq_L \mathscr{M}(x,y,t)$ and $x \neq y$. From this condition, by induction it follows that $\mathscr{M}(x,y,\varphi^n(t)) \geq_L \mathscr{M}(x,y,t)$. Taking the limit when $n \to \infty$, we get that $\mathscr{M}(x,y,t) = 0_L$ for every $t > 0$, which is a contradiction with (LF-2) i.e. $x = y$.

Lemma 3.2. *Let $(X, \mathscr{M}, \mathscr{T})$ be a $\mathscr{L}$-fuzzy metric space with the continuous triangular norm $\mathscr{T}$. Let S and T be compatible self-mappings on X and let Sx_n and Tx_n converge to some point $z \in X$ for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X. If S is continuous then*

$$\lim_{n\to\infty} TSx_n = Sz.$$

Proof. Let $\mu \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$ be arbitrary. From the continuity of the triangular norm $\mathscr{T}$ it follows that condition (3) holds, and for $n = 2$ it follows that for the involutive negation $\mathscr{N}$ we have that there exists $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that

$$\mathscr{T}\big(\mathscr{N}(\lambda), \mathscr{N}(\lambda)\big) >_L \mathscr{N}(\mu).$$

Since S and T are compatible, it follows that $\mathscr{M}\left(TSx_n, STx_n, \frac{t}{2}\right) >_L \mathscr{N}(\lambda)$. Also, Sx_n and Tx_n converge to z, so $\mathscr{M}\left(Tx_n, x_n, \frac{t}{2}\right) >_L \mathscr{N}(\lambda)$ and $\mathscr{M}\left(Sx_n, x_n, \frac{t}{2}\right) >_L \mathscr{N}(\lambda)$. From the continuity of S it follows that

$$\begin{aligned}\mathscr{M}(TSx_n, Sz, t) &\geq_L \mathscr{T}\left(\mathscr{M}\left(TSx_n, STx_n, \frac{t}{2}\right), \mathscr{M}\left(STx_n, Sz, \frac{t}{2}\right)\right)\\ &\geq_L \mathscr{T}\big(\mathscr{N}(\lambda), \mathscr{N}(\lambda)\big) >_L \mathscr{N}(\mu)\end{aligned}$$

holds. Taking $\mu \to 0$ we get that

$$\lim_{n\to\infty} \mathscr{M}(TSx_n, Sz, t) = 1_{\mathscr{L}},$$

i.e. $\lim\limits_{n\to\infty} TSx_n = Sz$.

Theorem 3.3. *Let $(X, \mathscr{M}, \mathscr{T})$ be a $\mathscr{L}$-fuzzy metric space which is complete w.r.t. continuous negation $\mathscr{N}$ and satisfies condition (3). Let A, B, S and T be self-mappings on X such that $A(X)$ and $B(X)$ are $\mathscr{L}F$-strongly bounded and let the following conditions be satisfied:*

(a) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,
(b) One of the mappings A and S is continuous,
(c) The pair $\{A, S\}$ is compatible and $\{B, T\}$ is weakly compatible,
(d) There is a continuous, non-decreasing function $\varphi : (0, \infty) \to (0, \infty)$, satisfying $\varphi(t) < t$ for every $t > 0$ and

$$\mathscr{M}(Ax, By, \varphi(t)) \geq_L \mathscr{M}(Sx, Ty, t), \text{ for every } t > 0 \text{ and } x, y \in X. \quad (5)$$

Then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. From *(a)* it follows that there exists $x_1 \in X$ such that $A(x_0) = T(x_1)$ and for such a point x_1 there exists $x_2 \in X$ such that $B(x_1) = S(x_2)$. By induction we can construct the following sequence $\{z_n\}_{n \in \mathbb{N}}$

$$\begin{cases} z_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \\ z_{2n} = Sx_{2n} = Bx_{2n-1} \end{cases}. \quad (6)$$

Let us consider a nested sequence of non-empty, closed sets defined by

$$F_n = \overline{\{z_n, z_{n+1}, \ldots\}},\ n \in \mathbb{N}.$$

We now prove that the family $\{F_n\}_{n \in \mathbb{N}}$ has $\mathscr{L}$-fuzzy diameter zero.

Let $\mu \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$ be arbitrary. From $F_k \subseteq \overline{A(X)} \cup \overline{B(X)}$ it follows that F_k is a $\mathscr{L}$F-strongly bounded set for arbitrary $k \in \mathbb{N}$. That means that there exists $t_0 > 0$ such that

$$\mathscr{M}(x, y, t_0) >_L \mathscr{N}(\mu) \text{ for all } x, y \in F_k. \quad (7)$$

From $\lim_{n \to \infty} \varphi^n(t_0) = 0$ it follows that there exists $m \in \mathbb{N}$ such that $\varphi^m(t_0) < t$. Let $n = m + k$ and $x, y \in F_n$ be arbitrary. There are sequences $\{z_{n(i)}\}$, $\{z_{n(j)}\}$ in F_n $\quad (n(i), n(j) \geq n \quad i, j \in \mathbb{N})$ such that $\lim_{i \to \infty} z_{n(i)} = x$ and $\lim_{j \to \infty} z_{n(j)} = y$.

Case I. Let us assume that $n(i) \in 2\mathbb{N} - 1$ and $n(j) \in 2\mathbb{N}$ or vice-versa, for large enough $i, j \in \mathbb{N}$ i.e. $z_{n(i)} = Ax_{n(i)-1}$ and $z_{n(j)} = Bx_{n(j)-1}$.

From (5) it follows that

$$\begin{aligned}\mathscr{M}(z_{n(i)}, z_{n(j)}, \varphi(t)) &= \mathscr{M}(Ax_{n(i)-1}, Bx_{n(j)-1}, \varphi(t))\\ &\geq_L \mathscr{M}(Sx_{n(i)-1}, Tx_{n(j)-1}, t) = \mathscr{M}(Ax_{n(i)-2}, Bx_{n(j)-2}, t)\\ &= \mathscr{M}(z_{n(i)-1}, z_{n(j)-1}, t).\end{aligned}$$

By induction, we get that

$$\mathscr{M}(z_{n(i)}, z_{n(j)}, \varphi^m(t)) \geq_L \mathscr{M}(z_{n(i)-m}, z_{n(j)-m}, t). \tag{8}$$

Since $\varphi^m(t_0) < t$ and $\mathscr{M}(x, y, \cdot)$ is non-decreasing in L, from the last inequalities it follows that

$$\mathscr{M}(z_{n(i)}, z_{n(j)}, t) \geq_L \mathscr{M}(z_{n(i)}, z_{n(j)}, \varphi^m(t_0)) \geq_L \mathscr{M}(z_{n(i)-m}, z_{n(j)-m}, t_0).$$

Since $\{z_{n(i)-m}\}$, $\{z_{n(j)-m}\}$ are sequences in F_k, from (7) it follows that

$$\mathscr{M}(z_{n(i)-m}, z_{n(j)-m}, t_0) >_L \mathscr{N}(\mu) \quad \text{for every} \quad i, j \in \mathbb{N}, \tag{9}$$

i.e. we have

$$\mathscr{M}(z_{n(i)}, z_{n(j)}, t) \geq_L \mathscr{N}(\mu), \text{ for } n(i) \in 2\mathbb{N} - 1, n(j) \in 2\mathbb{N}, \text{ or vice} - \text{versa.} \tag{10}$$

Case II. Let us assume that both $n(i)$ and $n(j)$ are from the set $2\mathbb{N} - 1$ and let $n(l) \geq n$ be an arbitrary positive integer and $n(l) \in 2\mathbb{N}$. Because (3) holds, let $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ be such that

$$\mathscr{T}\big(\mathscr{N}(\lambda), \mathscr{N}(\lambda)\big) >_L \mathscr{N}(\mu). \tag{11}$$

Then, from (10) it follows that

$$\mathscr{M}(Ax_{n(j)-1}, Bx_{n(l)-1}, t) >_L \mathscr{N}(\lambda).$$

There exists $\varepsilon > 0$ such that

$$\mathscr{M}(Ax_{n(j)-1}, Bx_{n(l)-1}, t - \varepsilon) \geq_L \mathscr{N}(\lambda).$$

From $\lim_{n\to\infty} \varphi^n(t_0) = 0$ take $m_0 \in \mathbb{N}$ such that $\varphi^{m_0}(t_0) < \varepsilon$. Let $n_1 = \max\{m, m_0\}$. Then we have that

$$\mathscr{M}(Ax_{n(j)-1}, Bx_{n(l)-1}, t - \varphi^{n_1}(t_0)) \geq_L \mathscr{M}(Ax_{n(j)-1}, Bx_{n(l)-1}, t - \varepsilon).$$

Also, from (10), $\varphi^{n_1}(t_0) \leq t$ and from the fact that $\mathscr{M}(x, y, \cdot)$ is non-decreasing in $\mathscr{L}$ it follows that

$$\mathscr{M}(Ax_{n(i)-1}, Bx_{n(l)-1}, t) \geq_L \mathscr{M}(Ax_{n(i)-1}, Bx_{n(l)-1}, \varphi^{n_1}(t_0)) \geq_L \mathscr{N}(\lambda)$$

holds. From the previous inequality and from (11) we conclude that

$$\begin{gathered} \mathscr{M}(z_{n(i)}, z_{n(j)}, t) = \mathscr{M}(Ax_{n(i)-1}, Ax_{n(j)-1}, t) \\ \geq_L \mathscr{T}\big(\mathscr{M}(Ax_{n(i)-1}, Bx_{n(l)-1}, \varphi^{n_1}(t_0)), \mathscr{M}(Ax_{n(j)-1}, Bx_{n(l)-1}, t - \varphi^{n_1}(t_0))\big) \\ \geq_L \mathscr{T}\big(\mathscr{N}(\lambda), \mathscr{N}(\lambda)\big) >_L \mathscr{N}(\mu), \end{gathered}$$

holds, i.e.

$$\mathscr{M}(z_{n(i)}, z_{n(j)}, t) \geq_L \mathscr{N}(\mu), \text{ for } n(i), n(j) \in 2\mathbb{N} - 1. \tag{12}$$

Similarly we can prove that (12) holds for $n(i), n(j) \in 2\mathbb{N}$.

Finally, from (10) and (12) we conclude that in both cases we have

$$\mathscr{M}(z_{n(i)}, z_{n(j)}, t) \geq_L \mathscr{N}(\mu)$$

for every $i, j \in \mathbb{N}$. Taking the limit when $i, j \to \infty$, and applying Lemma 2.6 we get that $\mathscr{M}(x, y, t) >_L \mathscr{N}(\mu)$ for every $x, y \in F_n$ i.e. the collection $\{F_n\}_{n\in\mathbb{N}}$ has $\mathscr{L}$-fuzzy diameter zero.

Applying Theorem 2.14 we conclude that this collection has non-empty intersection, that consists of exactly one point z. Since the collection $\{F_n\}_{n\in\mathbb{N}}$ has $\mathscr{L}$-fuzzy diameter zero and $z \in F_n$ for every $n \in \mathbb{N}$ then for every $\mu \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and for all $t > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $\mathscr{M}(z_n, z, t) >_L \mathscr{N}(\mu)$. From this it follows that for every $\mu \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ we have

$$\lim_{n\to\infty} \mathscr{M}(z_n, z, t) >_L \mathscr{N}(\mu).$$

Taking $\mu \to 0$ we get that

$$\lim_{n\to\infty} \mathscr{M}(z_n, z, t) = 1_{\mathscr{L}}$$

i.e. $\lim_{n\to\infty} z_n = z$. From the definition of the sequences $\{Ax_{2n-2}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{T_{2n-1}\}$ it follows that every one of these sequences converges to z.

We shall prove that z is a common fixed point of the mappings A, B, S and T. Let us first assume that S is continuous. Then we have that $\lim\limits_{n\to\infty} SSx_{2n} = Sz$. From the compatibility of the pair $\{A, S\}$ and from Lemma 3.2 it follows that

$\lim_{n\to\infty} ASx_{2n} = Sz$. Using the condition (5) we get that the following inequality holds:

$$\mathscr{M}(ASx_{2n}, Bx_{2n-1}, \varphi(t)) \geq_L \mathscr{M}(SSx_{2n}, Tx_{2n-1}, t).$$

Taking the limit as $n \to \infty$ we get that

$$\mathscr{M}(Sz, z, \varphi(t)) \geq_{L^*} \mathscr{M}(Sz, z, t).$$

From Lemma 3.1 it follows that $Sz = z$. Using condition (5) again, we get that

$$\mathscr{M}(Az, Bx_{2n-1}, \varphi(t)) \geq_L \mathscr{M}(Sz, Tx_{2n-1}, t)$$

and taking the limit as $n \to \infty$ we get that

$$\mathscr{M}(Az, z, \varphi(t)) \geq_L \mathscr{M}(Sz, z, t) = \mathscr{M}(z, z, t) = 1_{\mathscr{L}}.$$

This means that $Az = z$. Since $A(X) \subseteq T(X)$, there exists a point $u \in X$ such that $z = Az = Tu$ and we have that that

$$\mathscr{M}(z, Bu, \varphi(t)) = \mathscr{M}(Az, Bu, \varphi(t)) \geq_L \mathscr{M}(Sz, Tu, t) = \mathscr{M}(z, z, t) = 1_{\mathscr{L}},$$

which means that $Bu = z$. From the weak compatibility of the pair $\{B, T\}$ it follows that $Tz = TBu = BTu = Bz$. Also, from (5) it follows that

$$\mathscr{M}(Ax_{2n}, Bz, \varphi(t)) \geq_L \mathscr{M}(Sx_{2n}, Tz, t).$$

Taking the limit when $n \to \infty$ and from Lemma 3.1, we get that $Bz = z$. Thus, z is a common fixed point of the mappings A, B, S and T.

Now, let us assume that A is a continuous mapping. Then we have that $\mathscr{M}(AAx_{2n}, Az, t) >_L \mathscr{N}(\mu)$. From the compatibility of the pair $\{A, S\}$ and Lemma 3.2 it follows that $\mathscr{M}(SAx_{2n}, Az, t) >_L \mathscr{N}(\mu)$. Using condition (5) we get that

$$\mathscr{M}(AAx_{2n}, Bx_{2n-1}, \varphi(t)) \geq_L \mathscr{M}(SAx_{2n}, Tx_{2n-1}, t).$$

Taking the limit as $n \to \infty$ we get that

$$\mathscr{M}(Az, z, \varphi(t)) \geq_L \mathscr{M}(Az, z, t).$$

From Lemma 3.1 it follows that $Az = z$. Since $A(X) \subseteq T(X)$, there exists a point $v \in X$ such that $z = Az = Tv$. From $\mathscr{M}(Az, Bv, \varphi(t))$ we have that

$$\mathscr{M}(AAx_{2n}, Bv, \varphi(t)) \geq_L \mathscr{M}(SAx_{2n}, Tv, t).$$

Taking the limit as $n \to \infty$ we get that

$$\mathscr{M}(z, Bv, \varphi(t)) = \mathscr{M}(Az, Bv, \varphi(t)) \geq_L \mathscr{M}(Az, Tv, t) = \mathscr{M}(z, z, t) = 1_{\mathscr{L}},$$

which means that $z = Bv$. Since the pair $\{B, T\}$ is weakly compatible we have that $Tz = TBv = BTv = Bz$. Also, using condition (5) we have

$$\mathscr{M}(Ax_{2n}, Bz, \varphi(t)) \geq_L \mathscr{M}(Sx_{2n}, Tz, t).$$

Taking the limit as $n \to \infty$ we get that

$$\mathscr{M}(z, Bz, \varphi(t)) \geq_L \mathscr{M}(z, Tz, t) = \mathscr{M}(z, Bz, t).$$

This means that $z = Bz = Tz$. Since $B(X) \subseteq S(X)$, there exists a point $w \in X$ such that $z = Bz = Sw$. From (5) it follows that

$$\begin{aligned}\mathscr{M}(Aw, z, \varphi(t)) &= \mathscr{M}(Aw, Bz, \varphi(t)) \\ &\geq_L \mathscr{M}(Sw, Tz, t) = \mathscr{M}(Sw, Bz, t) = \mathscr{M}(z, z, t) = 1_{\mathscr{L}},\end{aligned}$$

i.e. $Aw = z$. Since the pair $\{A, S\}$ is compatible and $z = Aw = Sw$, from Remark 2.17 we have that $Az = ASw = SAw = Sz$. Thus, z is a common fixed point for the mappings A, B, S and T.

Let us now show that z is a unique common fixed point. Let us assume that there exists another common fixed point y. From (5) it follows that

$$\mathscr{M}(z, y, \varphi(t)) = \mathscr{M}(Az, By, \varphi(t)) \geq_L \mathscr{M}(Sz, Ty, t) = \mathscr{M}(z, y, t).$$

Finally, from Lemma 3.1 it follows that $z = y$.

Example 3.4. Let $(X, \mathscr{M}, \mathscr{T})$ be a $\mathscr{L}$-fuzzy metric space induced by metric $d(x, y) = |x - y|$ on $X = [0, +\infty) \subset \mathbb{R}$, i.e. $\mathscr{M}(x, y, t) = \frac{t}{t+|x-y|}$ with standard negation $N_s(x) = 1 - x$. Let

$$Ax = \frac{x}{1+x}, \qquad Sx = 2x,$$

$$Bx = \begin{cases} \frac{x}{1+x}, & x \in [0, 1] \\ 0, & x > 1 \end{cases}, \qquad Tx = \begin{cases} 2x, & x \in [0, 1] \\ 0, & x > 1 \end{cases}$$

and

$$\varphi(t) = \begin{cases} t/(1+t), & t \in (0, 1] \\ t/2, & t \geq 1 \end{cases}.$$

We shall prove that all the conditions of Theorem 3.3 are satisfied. First notice that $A(X) = [0,1) \subset [0,2] = T(X)$ and $B(X) = [0, \frac{1}{2}) \subset [0, +\infty) = S(X)$. The sets $A(X)$ and $B(X)$ are metrically bounded, i.e. $\mathscr{L}$F-strongly bounded as subsets of the $\mathscr{L}$-fuzzy metric space. Because $A(S(x)) = \frac{2x}{1+2x}$ and $S(A(x)) = \frac{2x}{1+x}$ we conclude that A and S are not commuting. We now prove that they are compatible mappings. Note that

$$\mathscr{M}(A(S(x), S(A(x)), t) = \frac{t}{t + \frac{2x^2}{(1+x)(1+2x)}}$$

$$\text{and} \qquad \mathscr{M}(S(x), A(x), t) = \frac{t}{t + \frac{x+2x^2}{1+x}}.$$

Since $\frac{2x^2}{(1+x)(1+2x)} \leq \frac{x+2x^2}{1+x}$ holds for all $x \geq 0$ we get

$$\mathscr{M}(A(S(x), S(A(x)), t) \geq \mathscr{M}(S(x), A(x), t)$$

for all $x, t \geq 0$. For a sequence $\{x_n\}$ in $[0, +\infty)$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

from the previous inequality it follows that $\lim_{n\to\infty} \mathscr{M}(A(S(x_n), S(A(x_n)), t) = 1$.

Now we prove that the mappings B and T are weakly compatible. If $Bz = Tz$ then $z = 0$ or $z > 1$. In the case when $z = 0$ we get $T(B(0)) = B(T(0)) = 0$. On the other hand, if $z > 1$ then $T(B(z)) = T(0) = 0$ and $B(T(z)) = B(0) = 0$, i.e. the condition $T(B(z)) = B(T(z))$ from Definition 2.16 is satisfied.

We now prove that the condition (5) is satisfied, too. Note that for all $x, y \in X$ we have that $\frac{1}{(1+x)(1+y)} \leq 1$. We will consider two cases.

Case I. Consider $0 < t \leq 1$ and note that $1 + t \leq 2$.

a) For $x, y \in [0,1]$ we get

$$\mathscr{M}(A(x), B(y), t/(1+t)) = \frac{t}{t + (1+t)\frac{|x-y|}{(1+x)(1+y)}} \geq \frac{t}{t + 2|x-y|} = \mathscr{M}(S(x), T(y), t).$$

b) For $x > 1$ and $y > 1$ we get

$$\mathscr{M}(A(x), B(y), t/(1+t) = \frac{t}{t + (1+t)\frac{x}{1+x}} \geq \frac{t}{t + 2x} = \mathscr{M}(S(x), T(y), t).$$

c) If $x \in [0,1]$ and $y > 1$ the proof is reduced to b). If $x > 1$ and $y \in [0,1]$ the proof is reduced to a).

Case II. Consider $t \geq 1$.

d) For $x, y \in [0,1]$ we get

$$\mathscr{M}(A(x), B(y), t/2) = \frac{t}{t + 2\frac{|x-y|}{(1+x)(1+y)}} \geq \frac{t}{t + 2|x-y|} = \mathscr{M}(S(x), T(y), t).$$

e) For $x > 1$ and $y > 1$s we get

$$\mathscr{M}(A(x), B(y), t/2) = \frac{t}{t + 2\frac{x}{1+x}} \geq \frac{t}{t + 2x} = \mathscr{M}(S(x), T(y), t).$$

f) If $x \in [0,1]$ and $y > 1$ the proof is reduced to e). If $x > 1$ and $y \in [0,1]$ the proof is reduced to d).

From the above we conclude that condition (5) is satisfied. Since $\varphi(t)$ satisfies all the conditions of Theorem 3.3 we get that all the mappings have a unique common fixed point. It is easy to see that this point is $x = 0$.

4. Analogue of the main result on probabilistic spaces

Since the structures of $\mathscr{L}$-fuzzy metric spaces and Probabilistic metric spaces are quite similar, in this section we give a version of the main result proved in Theorem 3.3 for mappings defined on Menger probabilistic metric spaces (briefly, Menger PM-spaces). For basic definitions and preliminaries in Menger PM-spaces see [8]. The proof is left out because it is very similar to the proof of Theorem 3.3.

Theorem 4.1. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space. Let A, B, S and T be self-mappings on X such that $A(X)$ and $B(X)$ are probabilistic bounded sets and let the following conditions be satisfied:*

(a) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,

(b) One of the mappings A and S is continuous,

(c) The pair $\{A, S\}$ is compatible and $\{B, T\}$ is weakly compatible,

(d) There is a continuous, non-decreasing function $\varphi : (0, \infty) \to (0, \infty)$, satisfying $\varphi(t) < t$ for every $t > 0$ and

$$F_{Ax,By}(\varphi(t)) \geq F_{Sx,Ty}(t), \text{ for every } t > 0 \text{ and } x, y \in X. \tag{13}$$

Then A, B, S and T have a unique common fixed point.

Finally, many fixed and common fixed point results, for example the Banach PM-contraction theorem, proved for mappings defined on Menger PM-spaces follow from the previous theorem.

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