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VARIATIONAL INEQUALITIES GOVERNED BY BOUNDEDLY LIPSCHITZIAN AND STRONGLY MONOTONE OPERATORS

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Abstract. Consider the variational inequality $VI(C,F)$ of finding a point $x^* \in C$ satisfying the property $\langle Fx^*, x - x^*\rangle \geq 0$ for all $x \in C$, where C is a nonempty closed convex subset of a real Hilbert space H and $F : C \to H$ is a nonlinear mapping. If F is boundedly Lipschitzian and strongly monotone, then we prove that $VI(C,F)$ has a unique solution and iterative algorithms can be devised to approximate this solution. In the case where C is the set of fixed points of a nonexpansie mapping, we also invent a hybrid iterative algorithm to approximate the unique solution of $VI(C,F)$.

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1. Introduction and Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, let C be a nonempty closed convex subset of H, and let $F : C \to H$ be

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a nonlinear operator. We consider the problem of finding a point x^* with the property

$$VI(C,F): \qquad x^* \in C, \quad \langle Fx^*, x - x^*\rangle \geq 0, \quad \forall\, x \in C. \tag{1.1}$$

This is known as the variational inequality problem, initially introduced and studied by Stampacchia [15] in 1964. In recent years, variational inequality problems have been extended to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences, see [1-19] and the references therein. Using the projection technique, it has been shown that the variational inequality problems are equivalent to the fixed point problems.

Lemma 1.1. *Given a point $z \in H$. Then $u \in C$ satisfies the inequality*

$$\langle z - u, v - u\rangle \leq 0, \quad \forall\, v \in C \tag{1.2}$$

if and only if $u = P_C z$, where P_C is the metric projection operator of H onto the closed convex set C; that is, u is the unique point in C such that

$$\|u - z\| = \inf_{v \in C} \|v - z\|.$$

It is well known that the projection P_C is nonexpansive; namely,

$$\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall\, x, y \in H. \tag{1.3}$$

Using Lemma 1.1, one can easily show that $VI(C,F)$ (1.1) is equivalent to the fixed point problem (see, for example, [13]).

Lemma 1.2. *$x^* \in C$ is a solution of variational inequality (1.1) if and only if $x^* \in C$ satisfies the fixed-point relation:*

$$x^* = P_C(I - \lambda F)x^*, \tag{1.4}$$

where $\lambda > 0$ is an arbitrary constant.

Recall that an operator $F : C \to H$ is called monotone, if

$$\langle Fx - Fy, x - y\rangle \geq 0 \quad \text{for all } x, y \in C.$$

Moreover, a monotone operator F is called strictly monotone if the equality '=' holds only when $x = y$ in the last relation. It is easy to see that $VI(C,F)$ (1.1) has at most one solution if F is strictly monotone.

For variational inequality (1.1), F is generally assumed to be κ-Lipschitzian and η-strongly monotone on C, that is, for some constants κ, $\eta > 0$, F satisfies the conditions

$$\|Fx - Fy\| \le \kappa\|x - y\|, \quad \forall\, x, y \in C \tag{1.5}$$

and

$$\langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall\, x, y \in C. \tag{1.6}$$

Under these two conditions, it is not difficult to show that the operator $P_C(I - \lambda F) : C \to C$ is a contraction provided the constant λ is selected such that $0 < \lambda < 2\eta/\kappa^2$. By using the well-known Banach contraction mapping principle, $P_C(I - \lambda F)$ has a unique fixed point. This fact together with Lemma 1.2 leads to the following result which we will use in the sequel repeatedly.

Lemma 1.3. *Assume that F satisfies the conditions (1.5) and (1.6). Then the variational inequality problem (1.1) has a unique solution. Moreover, for any $0 < \lambda < 2\eta/\kappa^2$, the sequence $\{x_n\}$ with initial guess $x_0 \in C$ and defined recursively by*

$$x_{n+1} = P_C(I - \lambda F)x_n, \quad n \ge 0$$

converges strongly to the unique solution of VI (1.1).

Attempts are worth making to weaken the Lipschitz condition (1.5) or the strong monotonicity condition (1.6) so that existence of solutions of variational inequality (1.1) is still guaranteed. One of the purposes of this paper is to weaken the Lipschitz condition (1.5). We will call that $F : C \to H$ is boundedly Lipschitzian on C if it is Lipschitzian on each bounded subset of C; namely, for each nonempty bounded subset B of C, there exists a positive constant κ_B depending only on the set B such that

$$\|Fx - Fy\| \le \kappa_B\|x - y\|, \quad \forall x, y \in B. \tag{1.7}$$

We will prove that variational inequality (1.1) has a unique solution if F is boundedly Lipschitzian and η-strongly monotone on C. We will also devise three iterative algorithms which generate sequences from an initial point chosen arbitrarily in a certain bounded subset $C^* = S(u, r) \cap C$ of C, where $u \in C$ is an arbitrary fixed point, positive constant $r > \|F(u)\|/\eta$, and $S(u, r)$ is a closed ball of H, i.e., $S(u, r) = \{x : x \in H, \|x - u\| \le r\}$. These sequences are shown to converge strongly to the unique solution x^* of variational inequality

(1.1). In the case where C is the set of fixed points of a nonexpansive mapping, we also establish a hybrid iterative algorithm that converges in norm to the solution of the variational inequality (1.1).

We need the demiclosedness pricinple for nonexpansive mappings.

Lemma 1.4. (Demiclosedness Principle; cf. [6]) *Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$; in particular, $Tx = x$ if $y = 0$.*

We also need the following technical result.

Lemma 1.5. (cf. [17]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ and $\{\beta_n\}$ are sequences in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \gamma_n|\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

We will use the notations:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
- $\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.
- $S(u,r) = \{x : x \in H, \|x-u\| \leq r\}$ denotes a closed ball with center u and radius r.

2. Existence and Uniqueness

It is well-known that if F is (globally) κ-Lipschitzian and η-strongly monotone on C, then $VI(C,F)$ has a unique solution which is also the unique fixed point of the contraction $P_C(I-\lambda F)$ whenever $0 < \lambda < 2\eta/\kappa^2$. In this section we aim to weaken the global Lipschitzian condition of F. Any of such weakenings would make the mapping $P_C(I-\lambda F)$ fail to be (globally) a contraction on the entire set C. We introduce a bounded Lipschitz condition for F which makes the mapping $P_C(I-\lambda F)$ to be a contraction on each of bounded subsets of C which is sufficient to guarantee the existence and uniqueness of solutions of $VI(C,F)$. Below is the main result of this section.

Theorem 2.1. *Assume that $F : C \to H$ is boundedly Lipschitzian on C (i.e., for each bounded subset B of C, F is Lipschizian on B). Assume also that F is η-strongly monotone on C. Then variational inequality (1.1) has a unique solution $x^* \in C$ such that*

$$\|x^* - u\| \le \frac{1}{\eta}\|Fu\|, \tag{2.1}$$

where $u \in C$ is an arbitrary fixed point.

Proof. First observe that VI (1.1) has at most one solution due to the strict monotonicity of F. We next prove the existence of solutions of VI (1.1).

Take a point $u \in C$ and a number $r \in \mathbb{R}$ such that $r \ge \|F(u)\|/\eta$. Let $C_r = S(u,r) \cap C$; then C_r is a bounded closed subset of C. Now since F, restricted to C_r, is Lipschitzian and strongly monotone, the $VI(C_r, F)$

$$x^* \in C_r, \quad \langle Fx^*, x - x^*\rangle \ge 0, \quad x \in C_r \tag{2.2}$$

has a unique solution $x^* \in C_r$ by Lemma 1.3.

We next prove that x^* is actually the unique solution of VI (1.1). To see this, take an arbitrary $z \in C$ and consider the closed convex bounded subset of C, $\widetilde{C} := \overline{\mathrm{co}}(\{z\} \cup C_r)$. Since again F is Lipschitzian and strongly monotone on $\widetilde{C}$, there exists a unique solution $\tilde{x} \in \widetilde{C}$ to the variational inequality

$$\tilde{x} \in \widetilde{C}, \quad \langle F\tilde{x}, x - \tilde{x}\rangle \ge 0, \quad x \in \widetilde{C}. \tag{2.3}$$

Since F is η-strongly monotone, we get

$$\begin{aligned} \langle Fu, u - \tilde{x}\rangle &\ge \langle F\tilde{x}, u - \tilde{x}\rangle + \eta\|u - \tilde{x}\|^2 \\ &\ge \eta\|u - \tilde{x}\|^2 \end{aligned} \tag{2.4}$$

since $\langle F\tilde{x}, u - \tilde{x}\rangle \ge 0$ by (2.3). But (2.4) implies that

$$\|u - \tilde{x}\| \le \frac{1}{\eta}\|Fu\|. \tag{2.5}$$

That is, $\tilde{x} \in C_r$. This shows that $\tilde{x}$ also solves VI (2.2). By uniqueness we get $\tilde{x} = x^*$. This combined with (2.3) asserts that (noticing $z \in \widetilde{C}$),

$$\langle Fx^*, z - x^*\rangle = \langle F\tilde{x}, z - \tilde{x}\rangle \ge 0.$$

Therefore (1.1) is satisfied for every $z \in C$. Finally, that (2.1) holds follows from (2.5). $\square$

Remark 2.2. Similarly, we can also introduce bounded strong monotonicity of operator. An operator $F : C \to H$ is called boundedly strong monotone on C, if for arbitrary bounded subset B of C, there exists a positive constant η_B depending only on the set B such that

$$\langle F(x) - F(y), x - y\rangle \geq \eta_B \|x - y\|^2, \quad \forall x, y \in B.$$

So a natural question gives rise to this: is it possible also to replace the strong monotonicity of F by bounded strong monotonicity so that the result of Theorem 2.1 is still guaranteed? The following simple example gives us a negative answer.

Consider the exponential function $f(x) = e^x,\ x \in \mathbb{R}$. It follows from the mean value theorem that, for an arbitrary positive constant $r > 0$,

$$|f(x) - f(y)| \leq e^r |x - y|$$

and

$$(f(x) - f(y))(x - y) \geq e^{-r}|x - y|^2$$

hold for all $x, y \in \mathbb{R}$ with $|x|,\ |y| \leq r$. Obviously, f is boundedly Lipschitzian and boundedly strong monotone on $\mathbb{R}$. But it is easy to see that there is no $x^* \in \mathbb{R}$ so that $f(x^*)(x - x^*) \geq 0$ holds for all $x \in \mathbb{R}$.

3. Iterative algorithms

Throughout this section, we always assume that $F : C \to H$ is boundedly Lipschitzian and η-strongly monotone on C. Denote by u an arbitrary fixed point in C and denote by r a positive fixed constant such that

$$r \geq \frac{1}{\eta}\|Fu\|.$$

Set $C_r = S(u, r) \cap C$ and denote by κ_r the Lipschitz constant of F on the bounded closed convex subset C_r.

Since $VI(C, F)$ (1.1) and $VI(C_r, F)$ (2.2) have the same solution, we can devise iterative methods for $VI(C_r, F)$ (2.2) and get the unique solution of $VI(C, F)$ (1.1). This is our idea. In this regard, we present three results.

Theorem 3.1. *Define a sequence $\{x_n\}$ recursively by the iterative algorithm*

$$\begin{cases} x_0 \in C_r \text{ arbitrarily}, \\ x_{n+1} = P_{C_r}(I - \lambda F)x_n, \end{cases} \tag{3.1}$$

where $0 < \lambda < 2\eta/\kappa_r^2$. *Then* $\{x_n\}$ *converges strongly to the unique solution* x^* *of VI (1.1).*

Proof. By Theorem 2.1, VI (1.1) has a unique solution x^* which is also the unique solution of VI (2.2) on C_r. Now Lemma 1.3 says that the sequence $\{x_n\}$ generated by the algorithm (3.1) converges in norm to the unique solution of VI (2.2) and hence, of VI (1.1). This clearly ends the proof. $\square$

Observe that the stepsize λ in Theorem 3.1 is constant. For varying stepsizes, we have the following result.

Theorem 3.2. *Define a sequence* $\{x_n\}$ *by the iterative algorithm:*

$$\begin{cases} x_0 \in C_r \textit{ arbitrarily}, \\ x_{n+1} = P_{C_r}(I - \lambda_n F)x_n, \end{cases} \tag{3.4}$$

where the sequence $\{\lambda_n\}$ *satisfies the condition*

$$0 < \varliminf_{n\to\infty} \lambda_n \le \varlimsup_{n\to\infty} \lambda_n < \frac{2\eta}{\kappa_r^2}. \tag{$C1$}$$

Then $\{x_n\}$ *converges strongly to the unique solution* x^* *of VI (1.1).*

Proof. By condition (C1), there exists some natural number N and positive constants a and b such that $0 < a \le b < 2\eta/\kappa_r^2$ and $a \le \lambda_n \le b$ for all $n \ge N$. Set

$$h = \max_{a\le\lambda\le b} \sqrt{1 - \lambda(2\eta - \lambda\kappa_r^2)}.$$

Then $0 \le h < 1$ and it is easy to see that $0 \le \sqrt{1 - \lambda_n(2\eta - \lambda_n\kappa_r^2)} \le h$ for all $n \ge N$.

By using Theorem 2.1, we assert that $x^* \in C_r$. Observing that x^* is also the unique solution of VI (2.2), we have from Lemma 1.2 that $x^* = P_{C_r}(I - \lambda_n F)x^*$ holds for all $n \ge 0$. The algorithm (3.4) implies that $\{x_n\} \subset C_r$. Thus, for all

$n \geq N$, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|P_{C_r}(I - \lambda_n F)x_n - P_{C_r}(I - \lambda_n F)x^*\|^2 \\ &\leq \|(I - \lambda_n F)x_n - (I - \lambda_n F)x^*\|^2 \\ &= \|(x_n - x^*) - \lambda_n(Fx_n - Fx^*)\|^2 \\ &= \|x_n - x^*\|^2 + \lambda_n^2\|Fx_n - Fx^*\|^2 - 2\lambda_n\langle x_n - x^*, Fx_n - Fx^*\rangle \\ &\leq [1 - \lambda_n(2\eta - \lambda_n\kappa_r^2)]\|x_n - x^*\|^2 \\ &\leq h^2\|x_n - x^*\|^2. \end{aligned}$$

Consequently

$$\|x_{n+1} - x^*\| \leq h\|x_n - x^*\| \leq \cdots \leq h^{n-N+1}\|x_N - x^*\|.$$

Clearly $\lim_{n\to\infty} \|x_n - x^*\| = 0$. □

Theorem 3.2 asserts that if the parameter sequence $\{\lambda_n\}$ is bounded away below from zero and above from $2\eta/\kappa_r^2$, then the sequence $\{x_n\}$ generated by the algorithm (3.4) converges in norm to the unique solution of VI (1.1). The result below shows that we can allow $\{\lambda_n\}$ to close either zero or $2\eta/\kappa_r^2$ and still keep the strong convergence of the sequence $\{x_n\}$. (However, the rate of convergence would possibly be slower.)

Theorem 3.3. *Assume that the sequence $\{\lambda_n\}$ satisfies the condition*

$$0 < \lambda_n < \frac{2\eta}{\kappa_r^2} \quad \textit{for all } n \textit{ and } \quad \sum_{n=0}^{\infty} \lambda_n \left(\frac{2\eta}{\kappa_r^2} - \lambda_n\right) = \infty. \tag{$C2$}$$

Then the sequence $\{x_n\}$ generated by the algorithm (3.4) converges strongly to the unique solution x^* of VI (1.1).

Proof. The first part of condition $(C2)$ assures that that the mapping $P_{C_r}(I - \lambda_n F) : C_r \to C_r$ is a contraction with the coefficient $\sqrt{1 - \lambda_n(2\eta - \lambda_n\kappa_r^2)}$ for all $n \geq 0$. Observing $\{x_n\} \subset C_r$, $x^* \in C_r$ and $x^* = P_{C_r}(I - \lambda_n F)x^*$ $(n \geq 0)$, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|P_{C_r}(I - \lambda_n F)x_n - P_{C_r}(I - \lambda_n F)x^*\| \\ &\leq \sqrt{1 - \lambda_n(2\eta - \lambda_n\kappa_r^2)}\|x_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.5}$$

Thus $r = \lim_{n\to\infty} \|x_n - x^*\|$ exists. Assume $r > 0$. We then derive from (3.5) that

$$\|x_{n+1} - x^*\| \leq \left(1 - \frac{1}{2}\lambda_n(2\eta - \lambda_n\kappa_r^2)\right)\|x_n - x^*\|.$$

Hence

$$\frac{\kappa_r^2}{2}\lambda_n\left(\frac{2\eta}{\kappa_r^2} - \lambda_n\right)\|x_n - x^*\| \leq \|x_n - x^*\| - \|x_{n+1} - x^*\|. \tag{3.6}$$

Noting $\|x_n - x^*\| \geq r$, we have from (3.6) that

$$\frac{1}{2}r\kappa_r^2\lambda_n\left(\frac{2\eta}{\kappa_r^2} - \lambda_n\right) \leq \|x_n - x^*\| - \|x_{n+1} - x^*\|.$$

It turns out that

$$\sum_{n=1}^{\infty}\lambda_n\left(\frac{2\eta}{\kappa_r^2} - \lambda_n\right) < \infty.$$

This contradicts the second part of condition $(C2)$. Thus we must have $r = 0$; that is, $\{x_n\}$ converges strongly to x^*. □

4. Extension of Yamada's hybrid method

In this section, we extend Yamada's hybrid method using the result of Theorem 2.1. Consider VI (1.1) where the feasible set C is composed of fixed points of a nonexpansive mapping; that is, C if of the form

$$C \equiv Fix(T) := \{x \in H : Tx = x\},$$

where $T : H \to H$ is a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$). We always assume that $Fix(T) \neq \emptyset$ so that $C = Fix(T)$ is closed, convex and nonempty. Thus, our variational inequality is reformulated as

$$x^* \in Fix(T), \quad \langle Fx^*, x - x^*\rangle \geq 0, \quad x \in Fix(T). \tag{4.1}$$

In this regard, Yamada [18] obtained the following hybrid iterative method for solving VI (4.1).

Theorem 4.1. ([18]) Assume that $F : H \to H$ is $\eta-$strongly monotone and κ-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/\kappa^2$. Assume also that a sequence $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions

(i) $\lim_{n\to\infty} \lambda_n = 0$,
(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(iii) either $\sum_{n=1}^{+\infty} |\lambda_{n+1} - \lambda_n| < +\infty$ or $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$.

Define a sequence $\{x_n\}$ recursively by the hybrid iterative algorithm

$$\begin{cases} x_0 \in C \ \textit{arbitrarily}, \\ x_{n+1} = Tx_n - \lambda_n \mu F(Tx_n), \quad n \geq 0. \end{cases} \tag{4.2}$$

Then $\{x_n\}$ converges strongly to the unique solution x^* of VI (4.1).

Using Theorem 2.1, we are able to relax the global Lipschitz condition on F in Theorem 4.1 to the weaker bounded Lipschitz condition.

Theorem 4.2. *Assume that $F : H \to H$ is $\eta-$strongly monotone and boundedly Lipschitzian. Fix an $x_0 \in C = Fix(T)$ arbitrarily and let $\hat{C}$ be the closed ball centered at x_0 and with radius $2\|Fx_0\|/\eta$ (i.e., $\hat{C} = S(x_0, 2\|Fx_0\|/\eta)$). Denote by $\hat{\kappa}$ the Lipschitz constant of F on $\hat{C}$, and take a constant μ satisfying $0 < \mu < \eta/\hat{\kappa}^2$. Assume a sequence $\{\lambda_n\}$ in the unit interval $(0,1)$ satisfies the same conditions (i)-(iii) as in Theorem 4.1. Suppose that the sequence $\{x_n\}$ is generated by the following hybrid iterative algorithm*

$$x_{n+1} = Tx_n - \lambda_n \mu F(Tx_n), \quad n \geq 0. \tag{4.3}$$

Then $\{x_n\}$ converges strongly to the unique solution x^* of VI (4.1).

Proof. The core of the proof lies in proving that the sequence $\{x_n\}$ generated by the algorithm (4.3) remains in the set $\hat{C}$ so that the $\hat{\kappa}$-Lipschiz continuity and η-strong monotonicity of F on $\hat{C}$ can be utilized.

First we observe that the $\hat{\kappa}$-Lipschitz continuity and $\eta-$strong monotonicity of F on $\hat{C}$ immediately yields $\eta \leq \hat{\kappa}$.

For each $0 < \lambda < 1$, define a mapping T^λ by

$$T^\lambda = (I - \lambda \mu F)T.$$

Then it is not hard to find that T^λ, restricted to $\hat{C}$, is a contraction with coefficient $1 - \lambda\tau$; that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad x, y \in \hat{C}. \tag{4.4}$$

where

$$\tau = \frac{1}{2}\mu(2\eta - \mu\hat{\kappa}^2) \in (0,1).$$

As a matter of fact, we compute

$$\begin{aligned}\|T^\lambda x - T^\lambda y\|^2 &= \|Tx - Ty - \lambda\mu(F(Tx) - F(Ty))\|^2 \\ &= \|Tx - Ty\|^2 + \lambda^2\mu^2\|F(Tx) - F(Ty)\|^2 \\ &\quad -2\lambda\mu\langle Tx - Ty, F(Tx) - F(Ty)\rangle \\ &\le [1 + \lambda^2\mu^2\hat{\kappa}^2 - 2\lambda\mu\eta]\|T(x) - T(y)\|^2 \\ &\le [1 - \lambda\mu(2\eta - \lambda\mu\hat{\kappa}^2)]\|x - y\|^2.\end{aligned}$$

It turns out that

$$\begin{aligned}\|T^\lambda x - T^\lambda y\| &\le \sqrt{1 - \lambda\mu(2\eta - \lambda\mu\hat{\kappa}^2)}\|x - y\| \\ &\le [1 - \frac{1}{2}\lambda\mu(2\eta - \lambda\mu\hat{\kappa}^2)]\|x - y\| \\ &\le (1 - \lambda\tau)\|x - y\|.\end{aligned}$$

This is (4.4). We can rewrite the algorithm (4.3) as

$$x_{n+1} = T^{\lambda_n} x_n, \quad n \ge 0. \tag{4.5}$$

We claim that $x_n \in \hat{C}$ for all $n \ge 0$. We prove this by induction. It is trivial that $x_0 \in \hat{C}$.

Suppose we have proved $x_n \in \hat{C}$, i,e.,

$$\|x_n - x_0\| \le \frac{2}{\eta}\|Fx_0\|. \tag{4.6}$$

We then derive from (4.5) and (4.6) that

$$\begin{aligned}\|x_{n+1} - x_0\| &= \|T^{\lambda_n} x_n - x_0\| \\ &\le \|T^{\lambda_n} x_n - T^{\lambda_n} x_0\| + \|T^{\lambda_n} x_0 - x_0\| \\ &\le (1 - \lambda_n\tau)\|x_n - x_0\| + \lambda_n\mu\|Fx_0\| \\ &\le (1 - \lambda_n\tau)\|x_n - x_0\| + \lambda_n\tau\frac{\mu}{\tau}\|Fx_0\| \\ &\le \max\left\{\|x_n - x_0\|,\ \frac{\mu}{\tau}\|Fx_0\|\right\} \\ &\le \max\left\{\frac{2}{\eta}, \frac{\mu}{\tau}\right\}\|Fx_0\|. \end{aligned}\tag{4.7}$$

However, since $0 < \mu < \frac{\eta}{\hat{k}^2}$, we get

$$\frac{\mu}{\tau} = \frac{\mu}{\frac{1}{2}\mu(2\eta - \mu\hat{\kappa}^2)} = \frac{2}{\eta + (\eta - \mu\hat{\kappa}^2)} \le \frac{2}{\eta}.$$

This together with (4.7) implies that

$$\|x_{n+1} - x_0\| \leq \frac{2}{\eta}\|Fx_0\|. \tag{4.8}$$

This proves that $x_{n+1} \in \hat{C}$. Therefore, $x_n \in \hat{C}$ for all $n \geq 0$. In particular, all sequences $\{x_n\}$, $\{Tx_n\}$ and $\{F(Tx_n)\}$ are bounded.

Since $\lambda_n \to 0$, it is straitforward from (4.3) that

$$\|x_{n+1} - Tx_n\| \to 0 \ (n \to \infty). \tag{4.9}$$

By (4.4) and (4.5), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|T^{\lambda_n}x_n - T^{\lambda_{n-1}}x_{n-1}\| \\
&\leq \|T^{\lambda_n}x_n - T^{\lambda_n}x_{n-1}\| \\
&\quad + \|T^{\lambda_n}x_{n-1} - T^{\lambda_{n-1}}x_{n-1}\| \\
&\leq (1-\tau\lambda_n)\|x_n - x_{n-1}\| + \mu|\lambda_n - \lambda_{n-1}|\|F(Tx_{n-1})\| \\
&\leq (1-\tau\lambda_n)\|x_n - x_{n-1}\| + \mu M|\lambda_n - \lambda_{n-1}|,
\end{aligned}$$

where $M = \sup\|F(Tx_n)\| < \infty$. By Lemma 1.4 together with the conditions (i)-(iii) assumed in the statement of the theorem, we conclude that

$$\|x_{n+1} - x_n\| \to 0 \ (n \to \infty). \tag{4.10}$$

Thus combining (4.9) and (4.10) yields that

$$\|x_n - Tx_n\| \to 0 \ (n \to \infty). \tag{4.11}$$

By Lemma 1.4 and (4.11), we obtain

$$\omega_w(x_n) \subset Fix(T). \tag{4.12}$$

Notice that Theorem 2.1 asserts that VI (4.1) has a unique solution $x^* \in \hat{C}$. Now we turn to prove that $\|x_n - x^*\| \to 0$ $(n \to \infty)$. It is well-known that the following inequality holds: for each $x,\ y \in H$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Using this inequality, (4.3) and (4.4), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|T^{\lambda_n}x_n - x^*\|^2 \\
&= \|(T^{\lambda_n}x_n - T^{\lambda_n}x^*) + (T^{\lambda_n}x^* - x^*)\|^2 \\
&\leq (1-\tau\lambda_n)\|x_n - x^*\|^2 + 2\mu\lambda_n\langle -Fx^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{4.13}$$

Let us next show that

$$\limsup_{n\to\infty}\langle -F(x^*), x_n - x^*\rangle \le 0. \tag{4.14}$$

In fact, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle -F(x^*), x_n - x^*\rangle = \lim_{j\to\infty}\langle -F(x^*), x_{n_j} - x^*\rangle.$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \tilde{x}$. Notice that $\tilde{x} \in Fix(T)$ by virtue of (4.12).

Since x^* is the solution of VI (4.1), we get

$$\begin{aligned}\limsup_{n\to\infty}\langle -F(x^*), x_n - x^*\rangle &= \lim_{j\to\infty}\langle -F(x^*), x_{n_j} - x^*\rangle \\ &= -\langle F(x^*), \tilde{x} - x^*\rangle \le 0.\end{aligned}$$

Finally the conditions (i)-(iii) and (4.14) allow us to apply Lemma 1.5 to the relation (4.13) to conclude that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. □

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