Fixed Point Theory, 10(2009), No. 2, 229-243
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SHADOWING IN AFFINE ITERATED FUNCTION SYSTEMS

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Abstract. We are concerned with the Shadowing in set-valued dynamical systems, including Iterated Function Systems. We prove that a scalar affine IFS has the Shadowing Property iff it is contracting or strictly expanding. The latter, in turn, is equivalent to hyperbolicity of the corresponding linear skew-product flow over the Bernoulli shift.
Key Words and Phrases: shadowing property, iterated function system, linear skew-product flow.
2000 Mathematics Subject Classification: 37C50, 37F05.

Introduction

The Shadowing theory began in the middle of 70th by the works of D. V. Anosov and R. Bowen as a powerful tool for studying the behaviour of diffeomorphisms near hyperbolic sets. Later this concept became itself an object in its own rights with various applications, mainly for detecting chaos in concrete dynamical systems (see [8, 9]).

In [6, 7] some criteria for a linear endomorphism to have the Shadowing Property have been stated; mainly, it was proved that a hyperbolic linear

This work is partially supported by Grants 08.820.08.04 RF of HCSTD ASM and CERIM-1006-06 of CRDF/MRDA.

operator in a Banach space has the Shadowing Property. The converse is also true in finite-dimensional vector spaces.

In [3, 4] the authors proposed a generalization of the notion of Shadowing to set-valued dynamics, including IFS and relations. It was proved that (weakly) contracting relations have the Shadowing Property. It can be easily checked that a linear IFS fails to possess the Shadowing Property if the associated linear semigroup contains an invariant subspace on which it acts equicontinuously. In [5] the Shadowing Property for parameterized IFS was studied, including for affine relations.

The linear and affine cases represent a first step in researching the nonlinear IFS, although not so powerful as the linearization procedure in the case of a single mapping, they give us a portrait of the asymptotical behaviour of IFS.

In this connection the problem to find necessary and sufficient conditions for an affine IFS to have the Shadowing Property arises.

We hope to obtain necessary and sufficient conditions for a linear IFS to have the Shadowing Property, similar to those for a single linear operator (see, e.g. [6, 7]).

The notions of pseudo-chains in this paper and in [10] are different. This conducts to different criteria of Shadowing.

In this paper we consider the case of scalar affine IFS, for which we give a Shadowing criterion. More precisely, we prove that a scalar affine IFS has the Shadowing Property if and only if it is contracting or (strictly) expanding.

We find also a connection with Shadowing in linear skew-products over the Bernoulli shifts and prove that a scalar linear IFS has the Shadowing Property if and only if the corresponding linear skew-product has, which, in turn, is equivalent for the latter to be hyperbolic.

1. Preliminaries

Let (X, d) be a metric space. Consider an *Iterated Function System* (*IFS*) $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$, consisting of pairwise distinct continuous functions $f_j : X \to X$ $(1 \le j \le m)$. Let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ = \mathbb{N} = \{n \in \mathbb{Z} : n \ge 0\}$.

A sequence $(x_n)_{n\in T}$ in X is called a *chain* of the IFS $\mathcal{F}$ if for every $n \in T$ there exists $j_n \in \{1, 2, \ldots, m\}$ such that $x_{n+1} = f_{j_n}(x_n)$. Given $\delta \ge 0$ a sequence $(x_n)_{n\in T}$ in X is called a *δ-chain* (*pseudo-chain*) if for every $n \in T$ there exists $j_n \in \{1, 2, \ldots, m\}$ such that $d(x_{n+1}, f_{j_n}(x_n)) \le \delta$. Denote $J =$

$(j_n)_{n\in T}$ and call it a *control sequence* for the δ-*chain* $(x_n)_{n\in T}$, including the case of chains as 0-chains. Two chains (or pseudo-chains) are called *concordant*, if they admit a common control sequence.

Remark 1.1. The control sequence need not be determined uniquely by the pseudo-chain.

One says that the IFS has the *Shadowing Property* (*on* T) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-chain $(x_n)_{n\in T}$ there is a chain $(y_n)_{n\in T}$, satisfying $d(x_n, y_n) \leq \varepsilon$ for all $n \in T$. In this case we say that $(y_n)_{n\in T}$ ε-*shadows* $(x_n)_{n\in T}$. If the chain $(y_n)_{n\in T}$ can be chosen to be concordant with $(x_n)_{n\in T}$, then we will speak about the *concordant Shadowing Property.*

In what follows we are concerned with the Shadowing Property for a complex affine scalar IFS $\mathcal{F} = \{\mathbb{C}\,; f_1, f_2, \ldots, f_m\}$, $f_j(z) = a_j z + b_j$ $(1 \leq j \leq m)$. Let $\mathcal{F}_0 = \{\mathbb{C}\,; a_1, a_2, \ldots, a_m\}$ denote the corresponding linear IFS.

Every δ-chain $(z_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$ can be written under the following recurrent form

$$z_{n+1} = a_{j_n} z_n + b_{j_n} + \delta_n \quad \text{with} \quad |\delta_n| \leq \delta, \tag{1.1}$$

or, explicitly,

$$z_n = \Big(\prod_{k=0}^{n-1} a_{j_k}\Big) z_0 + \sum_{i=1}^{n-1} \Big[\Big(\prod_{k=i}^{n-1} a_{j_k}\Big)(b_{j_{i-1}} + \delta_{i-1})\Big] + b_{j_{n-1}} + \delta_{n-1} \quad (n \geq 0).$$

Lemma 1.1. *For every scalar affine IFS there exist reals $\gamma > 0, \delta > 0$ and $R > 0$ such that two δ-chains $(z_n)_{n\in T}$ and $(w_n)_{n\in T}$ are concordant, provided they satisfy:*

$$|z_n| > R, \quad |z_n - w_n| \leq \gamma \quad (n \in T). \tag{1.2}$$

Proof. Denote $c = \max\limits_{1\leq j\leq m} |a_j|$, $M_a = \max\limits_{1\leq i,j\leq m} |a_i - a_j|$, $M_b = \max\limits_{1\leq i,j\leq m} |b_i - b_j|$, and

$$\alpha = \begin{cases} 0, & \text{if} \quad M_a = 0, \\ \min\limits_{a_i\neq a_j} |a_i - a_j|, & \text{if} \quad M_a > 0, \end{cases} \quad \beta = \begin{cases} 0, & \text{if} \quad M_b = 0, \\ \min\limits_{b_i\neq b_j} |b_i - b_j|, & \text{if} \quad M_b > 0. \end{cases}$$

Notice that α and β are not both 0. Put

$$\gamma = \begin{cases} \beta/(4+4c), & \text{if} \quad \beta \neq 0, \\ 1, & \text{if} \quad \beta = 0, \end{cases} \quad \delta = \begin{cases} \beta/8, & \text{if} \quad \beta \neq 0, \\ 1, & \text{if} \quad \beta = 0. \end{cases} \tag{1.3}$$

Similarly, put

$$R = \begin{cases} (\gamma + \gamma c + M_b + 2\delta)/\alpha, & \text{if} \quad \alpha \neq 0, \\ 1, & \text{if} \quad \alpha = 0. \end{cases} \tag{1.4}$$

Suppose the lemma is false. Then we could find two nonconcordant δ-chains $(z_n)_{n\in T}$, $z_{n+1} = a_{i_n} z_n + b_{i_n} + \delta'_n$, and $(w_n)_{n\in T}$, $w_{n+1} = a_{j_n} w_n + b_{j_n} + \delta''_n$, satisfying (1.2). Nonconcordance means that there is a natural k such that $i_k \neq j_k$, or, which is the same, $(a_{i_k}, b_{i_k}) \neq (a_{j_k}, b_{j_k})$.

One has

$$z_{n+1} - w_{n+1} = a_{j_n}(z_n - w_n) + (a_{i_n} - a_{j_n}) z_n + b_{i_n} - b_{j_n} + \delta'_n - \delta''_n. \tag{1.5}$$

If $a_{i_k} \neq a_{j_k}$, then (1.2), (1.4) and (1.5) give

$$\alpha R < |a_{i_k} - a_{j_k}| \cdot |z_k| \leq |z_{k+1} - w_{k+1}| + |a_{j_k}| \cdot |z_k - w_k| + \\ |b_{i_k} - b_{j_k}| + |\delta'_k| + |\delta''_k| \leq \gamma + \gamma c + M_b + 2\delta = \alpha R,$$

a contradiction.

If $a_{i_k} = a_{j_k}$, then (1.2), (1.3), (1.5) and the inequality $b_{i_k} \neq b_{j_k}$ imply

$$0 < \beta \leq |b_{i_k} - b_{j_k}| \leq |z_{k+1} - w_{k+1}| + |a_{j_k}| \cdot |z_k - w_k| + |\delta'_k| + |\delta''_k| \leq \\ \gamma + \gamma c + 2\delta = \beta/2,$$

also a contradiction.

Hence, $i_n = j_n$ for all $n \in T$, which means that $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ are concordant. □

Remark 1.2. One can state the analog of Lemma 1.1 for a segment of values $n \in [n_1, n_2] \subset T$.

Lemma 1.2. *If the sequences $(z_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ satisfy the relations*

$$z_0 = u_0, \quad z_{n+1} = a_{j_n} z_n + b_{j_n} + \delta_n, \quad u_{n+1} = a_{j_n} u_n + \delta_n \quad (n \geq 0), \tag{1.6}$$

$$w_0 = v_0, \quad w_{n+1} = a_{j_n} w_n + b_{j_n}, \quad v_{n+1} = a_{j_n} v_n \quad (n \geq 0), \tag{1.7}$$

then $|z_n - w_n| \leq \varepsilon$ $(n \geq 0)$ if and only if $|u_n - v_n| \leq \varepsilon$ $(n \geq 0)$.

Proof. One can prove by induction that $z_n - w_n = u_n - v_n$ $(n \geq 0)$. □

Lemma 1.3. *An affine IFS has the concordant Shadowing Property on $\mathbb{Z}_+$ if and only if the corresponding linear IFS has.*

Proof. Suppose the linear IFS $\mathcal{F}_0$ has the concordant Shadowing Property. Fix an arbitrary $\varepsilon > 0$ and take the respective $\delta > 0$ given by the definition of Shadowing.

For every δ-chain $(z_n)_{n\in\mathbb{N}}$ of the affine IFS $\mathcal{F}$ one can construct a concordant (with $(z_n)_{n\in\mathbb{N}}$) δ-chain $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{F}_0$, satisfying both relations (1.6).

The δ-chain $(u_n)_{n\in\mathbb{N}}$ is ε-shadowed by a concordant (with $(u_n)_{n\in\mathbb{N}}$) chain $(v_n)_{n\in\mathbb{N}}$ of $\mathcal{F}_0$. Construct a concordant chain $(w_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$, satisfying together with $(v_n)_{n\in\mathbb{N}}$ the relations (1.7). By Lemma 1.2 $(w_n)_{n\in\mathbb{N}}$ ε-shadows $(z_n)_{n\in\mathbb{N}}$ and these sequences are concordant. Therefore the affine IFS $\mathcal{F}$ has the concordant Shadowing Property as well.

The converse follows the same scheme. □

Remark 1.3. If all functions of an affine IFS are invertible, then Lemma 1.3 is true on $\mathbb{Z}$ as well.

2. Shadowing in affine IFS

Recall that a continuous function $f : X \to X$ is said to be *expanding* if there exists $r > 0$ such that the inequality $d(f^n(x), f^n(y)) \leq r\,(\forall\, n \in \mathbb{Z}_+)$ implies $x = y$.

We call an IFS *contracting* or, respectively, *strictly expanding*, if each its component is contracting or, respectively, expanding.

The next theorem completes similar results, obtained for contracting or strictly expanding IFS (see, e.g. [1, 5]), by putting in evidence the concordant attribute of Shadowing.

Theorem 2.1. *Every contracting or strictly expanding affine scalar IFS has the concordant Shadowing Property on $\mathbb{Z}_+$.*

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a δ-chain written under the form (1.1). Given $\varepsilon > 0$ we have to find δ and w_0 such that the concordant (with $(z_n)_{n\in\mathbb{N}}$) chain $(w_n)_{n\in\mathbb{N}}$ satisfies $|z_n - w_n| \leq \varepsilon$ for $n \geq 0$.

For contracting IFS we put $\delta = (1 - c)\varepsilon/2$, where $c = \max\limits_{1\leq j\leq m} |a_j| < 1$. It is easily seen that $(z_n)_{n\in\mathbb{N}}$ is ε-shadowed by every concordant (with $(z_n)_{n\in\mathbb{N}}$) chain $(w_n)_{n\in\mathbb{N}}$, satisfying $|w_0 - z_0| \leq \varepsilon/2$.

For strictly expanding IFS put $\delta = (a - 1)\varepsilon > 0$, where $a = \min\limits_{1\leq j\leq m} |a_j| > 1$. In this case there exists a unique concordant (with $(z_n)_{n\in\mathbb{N}}$) chain $(w_n)_{n\in\mathbb{N}}$, which ε-shadows $(z_n)_{n\in\mathbb{N}}$. The value w_0 is given by

$$w_0 = z_0 + \sum_{i=1}^{\infty} \big[\big(\prod_{k=0}^{i-1} a_{j_k}^{-1} \big) \delta_{i-1} \big]. \tag{2.1}$$

□

Remark 2.1. In [5] the authors have stated the Shadowing Property on $\mathbb{Z}_+$ for affine relations on $\mathbb{C}$.

Remark 2.2. Isometrical changes of coordinates do not affect the Shadowing Property.

Lemma 2.2. *If an affine IFS contains an isometry, then it does not possess the Shadowing Property on* $\mathbb{Z}_+$.

Proof. Assume that an affine IFS with an isometry possesses the Shadowing Property. After making a translation, if necessary, we may assume that this isometry is linear, say $f(z) = az$ with $|a| = 1$.

Choose γ, δ and R as in Lemma 1.1. Given $\varepsilon \in (0, \gamma)$ and $\tilde{\delta} \in (0, \delta)$ construct a $\tilde{\delta}$-chain $(z_n)_{n\in\mathbb{N}}$,

$$|z_0| > R, \quad z_{n+1} = f(z_n) + \delta_n = az_n + \delta_n,\ \delta_n = \tilde{\delta} a z_n / |z_n|\ (n \geq 0).$$

This $\tilde{\delta}$-chain is unbounded, since

$$|z_n| = |z_{n-1}| + \tilde{\delta} = |z_0| + n\tilde{\delta} \to +\infty \text{ as } n \to +\infty. \tag{2.2}$$

If there existed a chain $(w_n)_{n\in\mathbb{N}}$, ε-shadowing $(z_n)_{n\in\mathbb{N}}$, we would have $w_{n+1} = f(w_n) = aw_n$ by Lemma 1.1 and $|w_n - z_n| \leq \varepsilon$ for all $n \geq 0$. However, in this case $|w_n| = |w_0|$, which contradicts (2.2). □

Lemma 2.3. *If an affine IFS consists of expanding and constant components, then it does not possess the Shadowing Property on* $\mathbb{Z}_+$.

Proof. Assume that the IFS $\mathcal{F}$ consists of functions f_i, $f_i(z) = a_i z + b_i$ $(1 \leq i \leq m)$, where $a_1 = \ldots = a_k = 0$ and $1 < |a_{k+1}| \leq \cdots \leq |a_m|$. Denote $\alpha = |a_{k+1}|$ and $B = \{b_1, \ldots, b_k\}$. We may assume that f_m is linear, say $f_m(z) = a_m z$; otherwise we can obtain this by a translation.

Given $\varepsilon > 0$ and $\delta > 0$, take a natural $p > 2$ such that

$$\delta\alpha^{p-2} > \varepsilon. \tag{2.3}$$

Choose a finite chain $(w_n)_{1\le n\le p}$ of length p with $w_1 \in B$ and with maximal $|w_p|$. Let $w_1 = b_r \in B$ $(1 \le r \le k)$ and

$$w_p = (f_{i_{p-1}} \circ f_{i_{p-2}} \circ \cdots \circ f_{i_1})(w_1). \tag{2.4}$$

We need to consider the following two cases: $w_p = 0$ and $w_p \neq 0$.

If $w_p = 0$ we claim that $b_i = 0$ for all $1 \le i \le m$. If there exists $b_j \neq 0$, $j \le k$, we may take another chain $(\tilde{w}_n)_{1\le n\le p}$ with

$$\tilde{w}_1 = b_j \in B, \quad \tilde{w}_p = f_m^{p-1}(\tilde{w}_1) = a_m^{p-1} b_j, \tag{2.5}$$

otherwise we have $B = \{b_1\} = \{0\}$, $k = 1$, and for some $b_j \neq 0$, $j \ge 2$, we may take the chain $(\tilde{w}_n)_{1\le n\le p}$ with

$$\tilde{w}_1 = b_1 = 0 \in B, \quad \tilde{w}_p = (f_m^{p-2} \circ f_j)(\tilde{w}_1) = a_m^{p-2} b_j. \tag{2.6}$$

In both cases (2.5) and (2.6) we get a new chain $(\tilde{w}_n)_{1\le n\le p}$ with $|\tilde{w}_p| > |w_p| = 0$, which contradicts the selection of $|w_p|$ to be maximal.

Hence, $b_i = 0$ for all $1 \le i \le m$.

Given $\delta > 0$ and z_0 with $|z_0|$ large enough, define an unbounded δ-chain $(z_n)_{n\in\mathbb{N}}$ as follows:

$$z_1 = f_1(z_0) + \delta = \delta, \quad z_{n+1} = f_m(z_n) = a_m z_n \quad (n \ge 1).$$

If there existed a chain $(u_n)_{n\in\mathbb{N}}$, ε-shadowing $(z_n)_{n\in\mathbb{N}}$, then we would have

$$u_1 = f_1(u_0) = 0, \quad u_{n+1} = f_{i_n}(u_n) = a_{i_n} u_n = 0 \quad (n \ge 1).$$

The chain $(u_n)_{n\in\mathbb{N}}$, being eventually constant, cannot shadow $(z_n)_{n\in\mathbb{N}}$, a contradiction.

If $w_p \neq 0$, we claim that all the functions involved in (2.4) are expanding. If not, some of them are constant, and there exists $i_j = q \le k$ in (2.4), for which $w_{j+1} = f_q(w_j) = b_q$ and $w_p = (f_{i_{p-1}} \circ \cdots \circ f_{i_{j+1}})(w_{j+1})$. In this case we may take another chain $(\tilde{w}_n)_{1\le n\le p}$ with $\tilde{w}_1 = b_q \in B$ and

$$|\tilde{w}_p| = |(f_m^j \circ f_{i_{p-1}} \circ \cdots \circ f_{i_{j+1}})(b_q)| = |f_m^j(w_p)| = |a_m|^j \cdot |w_p| > |w_p|,$$

a contraiction with the selection of $|w_p|$.

Hence, $|a_{i_j}| \ge \alpha > 1$ for all $1 \le j \le p-1$ in (2.4).

Now construct a δ-chain $(z_n)_{n\in\mathbb{N}}$ as follows:

$$z_1 = f_r(z_0) = b_r \in B,$$
$$z_2 = f_{i_1}(z_1) + \delta_1 = a_{i_1} z_1 + \delta_1, \quad \delta_1 = \delta \cdot |\prod_{j=2}^{p-1} a_{i_j}| \cdot w_p / \big(|w_p| \cdot \prod_{j=2}^{p-1} a_{i_j}\big),$$
$$z_{n+1} = f_{i_n}(z_n) \quad (n \geq 2), \tag{2.7}$$

where b_r and the functions f_{i_n} $(1 \leq n \leq p-1)$ are stated above by (2.4).

Easily seen that $z_2 = w_2 + \delta_1$ and

$$|z_p| = |(f_{i_{p-1}} \circ \cdots \circ f_{i_2})(z_2)| =$$
$$\left|(f_{i_{p-1}} \circ \cdots \circ f_{i_2})(w_2) + \delta_1 \cdot \prod_{j=2}^{p-1} a_{i_j}\right| = \left|w_p + \delta \cdot |\prod_{j=2}^{p-1} a_{i_j}| \cdot w_p/|w_p|\right| =$$
$$|w_p| + \delta \cdot |\prod_{j=2}^{p-1} a_{i_j}| \geq |w_p| + \delta\alpha^{p-2} > |w_p| + \varepsilon. \tag{2.8}$$

If there existed a chain $(u_n)_{n\in\mathbb{N}}$, ε-shadowing $(z_n)_{n\in\mathbb{N}}$ with $|z_0|$ large enough, then, following the proof of Lemma 1.1, it would satisfy $u_1 = f_r(u_0) = b_r \in B$, $|u_p| \leq |w_p|$ by the selection of $|w_p|$ and, by (2.8) we get

$$|z_p - u_p| \geq |z_p| - |u_p| \geq |z_p| - |w_p| > \varepsilon,$$

a contradiction with the assumption of shadowing. This completes the proof. □

Corollary 2.4. *The affine IFS $\mathcal{F} = \{\mathbb{C}; f_1, f_2\}$, with $f_1(z) = az + b$ and $f_2(z) = c$, has the Shadowing Property on $\mathbb{Z}_+$ if and only if $|a| < 1$.*

The following two theorems consist the main result of this paper.

Theorem 2.5. *If a linear scalar IFS has the Shadowing Property on $\mathbb{Z}_+$, then it is either contracting or strictly expanding.*

Proof. Suppose, contrary to our claim, that there exists a linear IFS $\mathcal{F}_0 = \{\mathbb{C}; a_1, \ldots, a_m\}$ with the Shadowing Property, and which is neither contracting, nor strictly expanding.

In virtue of Lemmas 2.2 and 2.3 one has $a_i \neq 0$ and $|a_i| \neq 1$ for $1 \leq i \leq m$.

Let, say, $|a_1| \leq |a_2| \leq \ldots \leq |a_m|$ with $0 < |a_1| < 1 < |a_m|$.

Choose γ, δ and R as in Lemma 1.1. For $\varepsilon = \gamma$ take δ' from the Shadowing Property of $\mathcal{F}_0$ and put $\tilde{\delta} = \min\{\delta', \delta\}$. Due to Shadowing Property every

$\tilde{\delta}$-chain $(u_n)_{n\in\mathbb{N}}$ with $|u_n| > R$ $(n \geq 0)$ is ε-shadowed by a chain, which, in turn, must be concordant with $(u_n)_{n\in\mathbb{N}}$ by Lemma 1.1. In what follows we will construct an unbounded $\tilde{\delta}$-chain $(u_n)_{n\in\mathbb{N}}$ such that $|u_n| > R$ for all $n \geq 0$. Since every chain, ε-shadowing $(u_n)_{n\in\mathbb{N}}$ and concordant with it, must be bounded, we will have a contradiction.

We will use only two functions for this construction. So, without loss of generality we may consider a linear IFS, consisting of two functions, say $\mathcal{F}_0 = \{\mathbb{C}\,; a_1, a_2\}$ with $0 < |a_1| < 1 < |a_2|$.

The proof falls naturally into 2 parts.

1) "Resonance": $|a_1|^p \cdot |a_2|^q = 1$ for some integers $p, q \geq 1$. In this case define the sequence $(t_k)_{k\in\mathbb{N}}$ by: $t_0 = 0, t_{2r+1} = t_{2r} + q, t_{2r+2} = t_{2r+1} + p$ $(r \geq 0)$. Take the $\tilde{\delta}$-chain $(u_n)_{n\in\mathbb{N}}$ with the $(p+q)$-periodic control sequence $J = (\underbrace{2,\ldots,2}_{q}, \underbrace{1,\ldots,1}_{p}, \underbrace{2,\ldots,2}_{q}, \underbrace{1,\ldots,1}_{p}, \ldots)$, and defined as follows:

$$|u_0| > R,$$
$$u_{n+1} = \begin{cases} a_2 u_n, & \text{for} \quad t_{2r} \leq n < t_{2r+1}, \\ a_1 u_n, & \text{for} \quad t_{2r+1} \leq n < t_{2r+2} - 1, \\ a_1 u_n + \tilde{\delta} a_1 u_n / |a_1 u_n|, & \text{for} \quad n = t_{2r+2} - 1 \quad (r \geq 0). \end{cases} \tag{2.9}$$

Easily verified by induction that $|u_n| \geq |u_0| > R$ $(n \geq 0)$, and

$$|u_{t_{2r}}| = \big|u_0 + r\tilde{\delta} u_0/|u_0|\big| = |u_0| + r\tilde{\delta} \quad (r \geq 0).$$

Thus, the $\tilde{\delta}$-chain $(u_n)_{n\in\mathbb{N}}$ is unbounded. At the same time every concordant (with $(u_n)_{n\in\mathbb{N}}$) chain $(v_n)_{n\in\mathbb{N}}$ takes the form

$$v_{n+1} = \begin{cases} a_2 v_n, & \text{for} \quad t_{2r} \leq n < t_{2r+1}, \\ a_1 v_n, & \text{for} \quad t_{2r+1} \leq n \leq t_{2r+2} - 1 \quad (r \geq 0), \end{cases} \tag{2.10}$$

and, thus, is bounded (moreover, $(v_n)_{n\in\mathbb{N}}$ is $(p+q)$-periodic). Hence, $(v_n)_{n\in\mathbb{N}}$ cannot shadow any unbounded pseudo-chain, in particular $(u_n)_{n\in\mathbb{N}}$, a contradiction.

2) "Non-resonance": $|a_1|^p \cdot |a_2|^q \neq 1$ for all integers $p, q \geq 1$. This is the same as to say that $\ln|a_1|/\ln|a_2|$ is irrational. According to Kronecker's Theorem the subset $\{|a_1|^m \cdot |a_2|^n \,:\, m, n \in \mathbb{N}; m, n \geq 1\}$ is dense in $\mathbb{R}_+ := [0, +\infty)$.

Hence, for every $r \geq 1$ there exist naturals $p_r, q_r \geq 1$ such that

$$1 < |a_1|^{p_r} \cdot |a_2|^{q_r} < \Big(\sum_{i=0}^{r} 2^{-i}\Big) / \Big(\sum_{i=0}^{r-1} 2^{-i}\Big). \tag{2.11}$$

This, in turn, imply

$$1 < |a_1|^{p_1+p_2+\dots+p_r} \cdot |a_2|^{q_1+q_2+\dots+q_r} < \sum_{i=0}^{r} 2^{-i} = 2 - 2^{-r}. \qquad (2.12)$$

Define the sequence $(t_k)_{k\in\mathbb{N}}$ as follows: $t_0 = 0, t_{2r+1} = t_{2r} + q_{r+1}, t_{2r+2} = t_{2r+1} + p_{r+1}$ $(r \geq 0)$. Take the $\tilde{\delta}$-chain $(u_n)_{n\in\mathbb{N}}$ with the control sequence $J = (\underbrace{2,\dots,2}_{q_1}, \underbrace{1,\dots,1}_{p_1}, \underbrace{2,\dots,2}_{q_2}, \underbrace{1,\dots,1}_{p_2}, \dots)$, and defined as in (2.9).

Using (2.11), we obtain :

$$|u_{t_{2r}}| = \Big| a_1^{t_{2r}-t_{2r-1}} a_2^{t_{2r-1}-t_{2r-2}} u_{t_{2r-2}} +$$

$$\tilde{\delta} a_1^{t_{2r}-t_{2r-1}} a_2^{t_{2r-1}-t_{2r-2}} u_{t_{2r-2}} / |a_1^{t_{2r}-t_{2r-1}} a_2^{t_{2r-1}-t_{2r-2}} u_{t_{2r-2}}| \Big| =$$

$$|a_1|^{p_r} \cdot |a_2|^{q_r} \cdot |u_{t_{2r-2}}| + \tilde{\delta} > |u_{t_{2r-2}}| + \tilde{\delta} \quad (r \geq 1).$$

Easy to see that $|u_n| \geq |u_0| > R$ and $|u_{t_{2r}}| > |u_0| + r\tilde{\delta}$ for all $r \geq 1$. As a result, the subsequence $(u_{t_{2r}})_{r\in\mathbb{N}}$ is unbounded.

At the same time, every chain $(v_n)_{n\in\mathbb{N}}$, concordant with $(u_n)_{n\in\mathbb{N}}$, has the form (2.10) and, due to (2.12), satisfies

$$|v_{t_{2r}}| = |a_1|^{\sum_{i=0}^{r-1}(t_{2i+2}-t_{2i+1})} \cdot |a_2|^{\sum_{i=0}^{r-1}(t_{2i+1}-t_{2i})} \cdot |v_0| =$$

$$|a_1|^{\sum_{i=1}^{r} p_i} \cdot |a_2|^{\sum_{i=1}^{r} q_i} \cdot |v_0| < (2 - 2^{-r})|v_0| < 2|v_0|.$$

The subsequence $(v_{t_{2r}})_{r\in\mathbb{N}}$, being bounded, prevents the chain $(v_n)_{n\in\mathbb{N}}$ to shadow $(u_n)_{n\in\mathbb{N}}$; a contradiction, which completes the proof. □

Theorem 2.6. *If an affine scalar IFS has the Shadowing Property on $\mathbb{Z}_+$, then it is either contracting or strictly expanding.*

Proof. Assume the contrary and let $\mathcal{F} = \{\mathbb{C}; f_1, \dots, f_m\}$ be an affine IFS with the Shadowing Property, and which is neither contracting, nor strictly expanding.

Due to Lemmas 2.2 and 2.3 the IFS $\mathcal{F}$ does not contain isometries and constant functions.

Take $\gamma > 0, \delta > 0$ and $R > 0$ from Lemma 1.1 for the IFS $\mathcal{F}$ and for its associated linear IFS $\mathcal{F}_0$. Following the proof of Theorem 2.5, for $\varepsilon = \gamma$ and for every $0 < \tilde{\delta} < \delta$ one can construct a $\tilde{\delta}$-chain $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{F}_0$ with $|u_n| \geq |u_0| > R$ $(n \geq 0)$, for which there is no chain to ε-shadow it.

Consider a concordant (with $(u_n)_{n\in\mathbb{N}}$) $\tilde{\delta}$-chain $(z_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$, which satisfies (1.6). By choosing $|u_0|$ large enough, one can ensure that $|z_n| > R$ $(n \geq 0)$. In virtue of Shadowing Property and of Lemma 1.1 there exists a chain $(w_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$, which, firstly, ε-shadows $(z_n)_{n\in\mathbb{N}}$, and, secondly, is concordant with $(z_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$.

By Lemma 1.2 the chain $(v_n)_{n\in\mathbb{N}}$ of $\mathcal{F}_0$, satisfying (1.7), ε-shadows $(u_n)_{n\in\mathbb{N}}$, a contradiction. □

The previous results we collect into the following theorem.

Theorem 2.7. *Let $\mathcal{F}$ be an affine scalar IFS and let $\mathcal{F}_0$ be the corresponding linear IFS. The following assertions are equivalent:*

(1) *$\mathcal{F}$ is contracting or strictly expanding.*
(2) *$\mathcal{F}$ has the Shadowing Property on $\mathbb{Z}_+$.*
(3) *$\mathcal{F}$ has the concordant Shadowing Property on $\mathbb{Z}_+$.*
(4) *$\mathcal{F}_0$ has the Shadowing Property on $\mathbb{Z}_+$.*
(5) *$\mathcal{F}_0$ has the concordant Shadowing Property on $\mathbb{Z}_+$.*

Remark 2.3. Theorem 2.7 shows that an IFS need not have the Shadowing Property even in the case, when each its mapping has.

In what follows by a *parameterized IFS* we understand an IFS with an arbitrary index set.

Theorem 2.8. [5] *If a parameterized IFS $\mathcal{F} = \{X; f_\lambda \mid \lambda \in \Lambda\}$ has the Shadowing Property on $\mathbb{Z}$, then it has the Shadowing Property on $\mathbb{Z}_+$, provided it satisfies the equality:* $\bigcup_{\lambda\in\Lambda} \bigcup_{x\in X} f(x) = X$.

Theorem 2.9. *Let $\mathcal{F} = \{\mathbb{C}\,; f_1, f_2, \ldots, f_m\}$ be an affine IFS with $f_j(z) = a_j z + b_j, a_j \neq 0\,(1 \leq j \leq m)$, and let $\mathcal{F}^{-1} = \{\mathbb{C}\,; f_1^{-1}, f_2^{-1}, \ldots, f_m^{-1}\}$. The following assertions are equivalent:*

(1) *$\mathcal{F}$ is contracting or strictly expanding.*
(2) *$\mathcal{F}$ has the Shadowing Property on $\mathbb{Z}_+$.*
(3) *$\mathcal{F}^{-1}$ has the Shadowing Property on $\mathbb{Z}_+$.*
(4) *$\mathcal{F}$ has the Shadowing Property on $\mathbb{Z}$.*

Proof. The assertions 1) and 2) are equivalent by Theorem 2.7.

Since $a_j \neq 0\,(1 \leq j \leq m)$ the IFS $\mathcal{F}$ is contracting or strictly expanding if and only if $\mathcal{F}^{-1}$ is strictly expanding or respectively contracting. It follows that 1) and 3) are equivalent.

The assertion 4) implies 2) due to Theorem 2.8.

Let us show that 1) implies 4). Assume that $\mathcal{F}$ is contracting (analogously if it is strictly expanding). Then $\mathcal{F}^{-1}$ is strictly expanding. Both have Shadowing property on $\mathbb{Z}_+$.

Fix $\varepsilon > 0$. Let $(z_n)_{n\in\mathbb{Z}}$ be an arbitrary δ-chain of $\mathcal{F}$ with $\delta > 0$ to be chosen later. Easily seen that the sequence $(u_n)_{n\in\mathbb{N}}$, $u_n = z_{-n}$ $(n \geq 0)$ is a $\tilde{\delta}$-chain of $\mathcal{F}^{-1}$ with $\tilde{\delta} = a^{-1}\delta$ and $a = \min\limits_{1\leq j\leq m} |a_j|$. Given $\gamma = \varepsilon/2$, for $\mathcal{F}^{-1}$ there exists $\delta_1 > 0$ such that $(u_n)_{n\in\mathbb{N}}$ is γ-shadowed by a chain $(v_n)_{n\in\mathbb{N}}$, provided $\tilde{\delta} \leq \delta_1$.

For $\mathcal{F}$ there exists $\delta_2 > 0$ such that $(z_n)_{n\in\mathbb{N}}$ is ε-shadowed by a chain $(w_n)_{n\in\mathbb{N}}$ with $w_0 = v_0$, provided $\delta \leq \delta_2$, since $|w_0 - z_0| \leq \varepsilon/2$. Put $\delta = \min\{\delta_1, a\delta_2\}$. The sequence $(w_n)_{n\in\mathbb{Z}}$, defined for $n < 0$ as $w_n = v_{-n}$, is a chain of $\mathcal{F}$, ε-shadowing the δ-chain $(z_n)_{n\in\mathbb{Z}}$. □

3. Affine IFS and affine extensions of Bernoulli shifts

Consider a linear IFS $\mathcal{F}_0 = \{\mathbb{C}; f_1, f_2, \ldots, f_m\}$ with distinct functions $f_j(z) = a_j z\,(1 \leq j \leq m)$. Denote $\Sigma_m = \{1, 2, \ldots, m\}^{\mathbb{Z}}$ and endow it with the product topology.

Given a non-zero chain $(x_n)_{n\in\mathbb{Z}}$, there exists a unique control sequence

$$J = (\ldots, j_{-k}, \ldots, j_{-1}, j_0, j_1, \ldots, j_k, j_{k+1}, \ldots) \in \Sigma_m,$$

such that $x_{n+1} = a_{j_n} x_n\,(n \in \mathbb{Z})$.

Let $\sigma : \Sigma_m \to \Sigma_m$ denote the *Bernoulli shift*: $(\sigma(J))_n := j_{n+1}$ $(n \in \mathbb{Z})$ and consider the *linear extension* $\hat{A} : \Sigma_m \times \mathbb{C} \to \Sigma_m \times \mathbb{C}$, defined by

$$\hat{A}(J, z) := \big(\sigma(J), a_{j_0} z\big).$$

Thus, $\hat{A}^n(J, z) = \big(\sigma^n(J), a_{j_{n-1}} \cdot \cdots \cdot a_{j_1} a_{j_0} z\big)$. Denote $A^n(J) = a_{j_{n-1}} \cdot \cdots \cdot a_{j_1} a_{j_0}$.

Every orbit $\big(\hat{A}^n(J, z)\big)_{n\in\mathbb{N}}$ of the homeomorphism $\hat{A} : \Sigma_m \times \mathbb{C} \to \Sigma_m \times \mathbb{C}$ covers the orbit $\big(\sigma^n(J)\big)_{n\in\mathbb{N}}$ of the shift transformation σ. In other words, the sequence $(z_n)_{n\in\mathbb{Z}}$, with $z_{n+1} = A^n(J) z_0$, represents an orbit of $\hat{A}$, starting at

(J, z_0). We call such orbit a J-*chain* (or a $(J, 0)$-*chain*). In this setting, we will say that the sequence $(y_n)_{n\in\mathbb{Z}}$ is a (J, δ)-*chain* if

$$|y_{n+1} - A\big(\sigma^n(J)\big)y_n| \le \delta \quad (n \in \mathbb{Z}).$$

We say that the linear extension $\hat{A}$ has the *Shadowing Property* if given $\varepsilon > 0$ and $J \in \Sigma_m$ there exists $\delta = \delta(\varepsilon, J) > 0$ such that every (J, δ)-chain is ε-shadowed by a $(J, 0)$-chain. In case δ does not depend on $J \in \Sigma_m$ we say that the extension $\hat{A}$ has the *uniform Shadowing Property.*

Our goal is to prove the following statement.

Theorem 3.1. *A linear IFS $\mathcal{F}_0 = \{\mathbb{C}; f_1, f_2, \dots, f_m\}$ with pairwise distinct functions $f_j(z) = a_j z\,(1 \le j \le m)$ has the concordant Shadowing Property if and only if the associated linear extension has the uniform Shadowing Property.*

Proof. Let $\mathcal{F}_0$ have the concordant Shadowing Property. So, given $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-chain $(z_n)_{n\in\mathbb{Z}}$ there is a chain $(w_n)_{n\in\mathbb{Z}}$, verifying $|z_n - w_n| \le \varepsilon$ $(n \in \mathbb{Z})$. The fact that $(z_n)_{n\in\mathbb{Z}}$ is a δ-chain means that for every $n \in \mathbb{Z}$ there exists $j_n \in \{1, 2, \dots, m\}$ such that

$$|z_{n+1} - a_{j_n} z_n| \le \delta \qquad (n \in \mathbb{Z}). \tag{3.1}$$

For $(w_n)_{n\in\mathbb{Z}}$ to be a chain concordant with $(z_n)_{n\in\mathbb{Z}}$ means that $w_{n+1} = a_{j_n} w_n$ for all $n \in \mathbb{Z}$.

Denote $J = (j_n)_{n\in\mathbb{Z}}$. Fix $\varepsilon > 0$ and take δ from (3.1). Hence, $(z_n)_{n\in\mathbb{Z}}$ is a (J, δ)-chain for the linear extension $\hat{A}$ and $(w_n)_{n\in\mathbb{Z}}$ is a $(J, 0)$-chain for $\hat{A}$. Since δ does not depend on $J \in \Sigma_m$, one has the uniform Shadowing Property for the linear extension $\hat{A}$.

The converse is obtained similarly. Assume that the linear extension $\hat{A}$ has the uniform Shadowing Property. So, given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $J \in \Sigma_m$ and every (J, δ)-chain $(z_n)_{n\in\mathbb{N}}$ there is a $(J, 0)$-chain $(w_n)_{n\in\mathbb{N}}$, which ε-shadows $(z_n)_{n\in\mathbb{N}}$.

By construction, $J \in \Sigma_m$ represents a control sequence, the same for the δ-chain $(z_n)_{n\in\mathbb{N}}$ and the chain $(w_n)_{n\in\mathbb{N}}$. □

Recall (see [2]) that the linear extension $\mathrm{Pr}_1 : (\Sigma_m \times \mathbb{C}; \hat{A}) \to (\Sigma_m, \sigma)$ is *hyperbolic* if and only if there exist $c > 0$ and $0 < \lambda < 1$ such that one of the following two inequalities holds

$$|A^n(J)z| \le c\lambda^n|z| \qquad \text{or} \qquad |A^n(J)z| \ge c\lambda^{-n}|z|$$

for every $J \in \Sigma_m$, $z \in \mathbb{C}$, $n \ge 0$.

Theorem 3.2. *The linear extension* $\mathrm{Pr}_1 : (\Sigma_m \times \mathbb{C}; \hat{A}) \to (\Sigma_m, \sigma)$ *is hyperbolic if and only if*

$$|a_j| > 1 \quad (1 \le j \le m), \qquad \textit{or} \qquad |a_j| < 1 \quad (1 \le j \le m). \tag{3.2}$$

Proof. Obviously, (3.2) implies hyperbolicity.

Hyperbolicity (over Σ_m) implies hyperbolicity over fixed points $J = \bar{j} = (\dots, j, \dots, j, \dots)$, $j \in \{1, 2, \dots, m\}$, which, in turn, yields $|a_j| \ne 1\,(j = \overline{1,m})$.

We have to prove (3.2). Assuming the contrary, let $|a_1| < 1$ and $|a_2| > 1$.

Consider the point $J = (j_n)_{n\in\mathbb{Z}} \in \Sigma_m$ with

$$j_n = \begin{cases} 1, & \text{if} \quad n \ge 0, \\ 2, & \text{if} \quad n < 0. \end{cases}$$

The closure of its orbit is an invariant subset of Σ_m, containing the fixed points $\bar{1}$ and $\bar{2}$. The non-zero $(J, 0)$-chain $(w_n)_{n\in\mathbb{Z}}$, defined by

$$w_n = \begin{cases} a_1^n z, & \text{if} \quad n \ge 0, \\ a_2^n z, & \text{if} \quad n < 0, \end{cases}$$

represents a nontrivial bounded trajectory of the linear extension; existence of such orbits prevents the hyperbolicity. □

Associated to any affine IFS there is an affine extension of the Bernoulli shift defined similarly as for the linear case. The notion of (uniform) Shadowing Property is defined similarly.

Theorem 3.3. *An affine IFS* $\mathcal{F} = \{\mathbb{C}; f_1, f_2, \dots, f_m\}$ *with pairwise distinct functions* $f_j(z) = a_j z + b_j\,(1 \le j \le m)$ *has the concordant Shadowing Property if and only if the associated affine extension has the uniform Shadowing Property.*

Proof. The proof is similar to that of Theorem 3.1. □

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Received: 20.03. 2009; Accepted: 10.06. 2009.