Fixed Point Theory, 10(2009), No. 2, 221-228
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ABSTRACT AND CONCRETE GRONWALL LEMMAS

CECILIA CRĂCIUN* AND NICOLAE LUNGU**

*Colfe's School
Horn Park Lane
London, SE12 8AW, United Kingdom
and
Babeş-Bolyai University Cluj-Napoca
Department of Applied Mathematics
Kogălniceanu Street, No. 1, 400084 Cluj-Napoca, Romania.
E-mail: ceciliacraciun@yahoo.com

**Technical University of Cluj-Napoca
Department of Mathematics
C. Daicoviciu 15, Cluj-Napoca, Romania.
E-mail: nlungu@math.utcluj.ro

Abstract. Let $(X, d, \leq)$ be a n ordered metric space and $A : X \to X$ be an increasing Picard operator. It is well known that if we have $x \leq Ax$ for all $x \in X$, then $x \leq x^*$, where x^* is the unique fixed point of A. In this paper we investigate which concrete Gronwall lemmas can be derived from this abstract result.

Key Words and Phrases: Inequalities, Gronwall inequalities, Bihari inequalities, Wendorff-type inequalities, abstract Gronwall lemma, Picard operators.

2000 Mathematics Subject Classification: 47H10, 54H25, 45G10, 45M10.

1. Introduction

In this paper we present some results relative to abstract and concrete Gronwall lemmas.

We begin our considerations with some notions from Picard operators theory (see I.A. Rus [11]-[14]).

Let $(X, \to)$ be an L-space ([13]), $A : X \to X$ an operator. We denote by F_A the fixed points set of A.

Definition 1.1. (I.A. Rus [12]-[14]). A is a Picard operator (PO) if there exists $x_A^* \in X$ such that:

(i) $F_A = \{x_A^*\}$;

(ii) $A^n(x) \to x_A^*$ as $n \to \infty$, $\forall\, x \in X$.

We have the following abstract lemma ([11]-[13], see also [3], [4], [7]).

Lemma 1.1. (Abstract Gronwall lemma). *Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ an operator. We assume that:*

(i) A is PO;

(ii) A is increasing.

If we denote by x_A^* the unique fixed point of A, then

(a) $x \leq A(x) \;\Rightarrow\; x \leq x_A^*$;

(b) $x \geq A(x) \;\Rightarrow\; x \geq x_A^*$.

In this paper we investigate which concrete Gronwall lemmas can be derived from this abstract Gronwall lemma.

In a recent paper [11], I.A. Rus has formulated ten research problems of the theory of Gronwall lemmas. In this paper we concentrate on **Problem 5** and on **Problem 6**. These problems are presented in the following.

Problem 5. *In which Gronwall lemmas the upper bounds are fixed points of the operator A?*

Problem 6. *If the answer to Problem 5 is positive, which of them are a consequence of some abstract Gronwall lemmas?*

We use the notations from I.A. Rus [11], [12], [13], [14].

Let $(X, \leq)$ be an ordered set, and $A : X \to X$ an operator such that the equation

$$x = A(x) \tag{1.1}$$

has an unique solution x_A^*.

The operatorial inequality problem (see I.A. Rus [11], [12], [13]) is the following:

Find the conditions under which

$$\begin{array}{ll} (i) & x \leq A(x) \;\Rightarrow\; x \leq x_A^*; \\ (ii) & x \geq A(x) \;\Rightarrow\; x \geq x_A^*. \end{array} \tag{1.2}$$

To have a concrete result for this problem it is necessary to either determine the solution x_A^* of the equation (1.1), or to find $y, z \in X$ such that $x_A^* \in [y, z]$. In the former case we have a Gronwall type inequality.

2. Some consequences of Abstract Gronwall Lemma

The following concrete Gronwall lemmas are well known (see for instance [1], [8], [9], [16]).

Lemma 2.1. ([8]) *We assume that:*

(i) x, K, a and b are continuous functions on $J = [\alpha, \beta]$;

(ii) b and K are nonnegative on J.

(a) If $x \in C([\alpha, \beta])$ satisfies the inequality

$$x(t) \leq a(t) + b(t) \int_{\alpha}^{t} K(s)x(s)ds, \quad \forall\ t \in J \tag{2.1}$$

then

$$x(t) \leq a(t) + b(t) \int_{\alpha}^{t} a(s)K(s) \exp\left(\int_{s}^{t} b(r)K(r)dr\right) ds. \tag{2.2}$$

Moreover, equality holds in (2.2) for a subinterval $J_1 = [\alpha, \beta]$ of J if equality holds in (2.1) for $t \in J_1$.

(b) The result remains valid if $\leq$ is replaced with $\geq$ in both (2.1) and (2.2).

Theorem 2.1. *The concrete Lemma 2.1 is a consequence of the abstract Gronwall Lemma 1.1.*

Proof. Let $(X, \to, \leq) := (C[\alpha, \beta], \overset{\|\cdot\|_\tau}{\to}, \leq)$ be as in Lemma 1.1, where $\|\cdot\|_\tau$ is the Bielecki norm on $C[\alpha, \beta]$:

$$\|x\|_\tau := \max_{t \in [\alpha, \beta]} (|x(t)| \exp(-\tau(t - \alpha))), \quad \tau \in \mathbb{R}_+^*. \tag{2.3}$$

We consider the operator $A : X \to X$ defined by

$$A(x)(t) = a(t) + b(t) \int_{\alpha}^{t} K(s)x(s)ds. \tag{2.4}$$

The fixed point of the operator A is

$$x^*(t) = a(t) + b(t) \int_{\alpha}^{t} a(s)K(s) \exp\left(\int_{s}^{t} b(r)K(r)dr\right) ds.$$

The operator A is increasing and

$$\|A(u) - A(v)\|_\tau \leq \frac{1}{\tau} M_b M_K \|u - v\|_\tau \tag{2.5}$$

where

$$M_b = \max_{t \in [\alpha, \beta]} |b(t)|,$$

$$M_K = \max_{t \in [\alpha, \beta]} |K(t)|.$$

We can choose $\tau > 0$ such that $\frac{M_b M_K}{\tau} < 1$, so the operator A is a contraction with respect to $\|\cdot\|_\tau$. This implies that A is PO and applying Lemma 1.1 gives the conclusion. □

Lemma 2.2. (Bihari-type inequality, see [9], [16]). *We assume that*

(i) $c \in \mathbb{R}$, $p \in C([\alpha, \beta], \mathbb{R}_+)$

(ii) V *is a continuous, positive, increasing, and Lipschitz function.*

If $y \in C[\alpha, \beta]$ *is a solution of the inequality*

$$y(x) \leq c + \int_\alpha^x p(s)V(y(s))ds, \quad x \in [\alpha, \beta] \tag{2.6}$$

then

$$y(x) \leq y^*(x) \tag{2.7}$$

where $y^*(x) = F^{-1}(\phi(x) + F(c))$.

Here we have:

$$F(y) = \int_\alpha^y \frac{dy}{V(y)}, \quad \phi(x) = \int_\alpha^x p(s)ds$$

and F^{-1} *is the inverse of* F.

Theorem 2.2. *The concrete Gronwall Lemma 2.2 is a consequence of the abstract Lemma 1.1.*

Proof. Let $(X, \to, \leq) := (C([\alpha, \beta], \overset{\|\cdot\|_\tau}{\to}, \leq)$, where $\|\cdot\|_\tau$ is the Bielecki norm defined by (2.3).

We consider the operator $A : X \to X$

$$A(y)(x) = c + \int_\alpha^x p(s)V(y(s))ds, \quad x \in [\alpha, \beta]. \tag{2.8}$$

The fixed point of the operator A is

$$y_A^*(x) = F^{-1}(\phi(x) + F(c)).$$

Since A is contraction with respect to $\|\cdot\|_\tau$, with $\tau > 0$ suitably chosen, and A is an increasing Picard operator, the result of Lemma 2.2 follows.

Lemmas 2.1 and 2.2 are of Gronwall-type, as the right-hand side terms of (2.2) and (2.7) are fixed points of the corresponding operators.

3. Improvement of some concrete Gronwall lemmas

In some concrete Gronwall lemmas only the following implication holds:

$$x \le A(x) \ \Rightarrow \ x \le y^* \neq x_A^*.$$

Such a result is presented in the following.

Lemma 3.1. (Wendorff-type inequalities) ([2], see also [1], [4], [8], [9], [10]). *We assume that*

(i) $v \in C([0,a] \times [0,b], \mathbb{R}_+)$, $\alpha \in \mathbb{R}_+$

(ii) v *is increasing.*

If $u \in C([0,a] \times [0,b])$ *is a solution of the inequality*

$$u(x,y) \le \alpha + \int_0^x \int_0^y v(s,t)u(s,t)dsdt, \quad x \in [0,a], \ y \in [0,b] \tag{3.1}$$

then

$$u(x,y) \le \alpha \exp\left(\int_0^x \int_0^y v(s,t)dsdt\right). \tag{3.2}$$

We consider $(X, \to, \le) := C(D, \overset{\|\cdot\|_\tau}{\to}, \le)$ where $D = [0,a] \times [0,b]$ and $\|\cdot\|_\tau$ is the Bielecki norm on $C(D)$:

$$\|u\|_\tau := \max_D(|u(x,y)| \exp(-\tau(x+y))), \quad \tau \in \mathbb{R}_+^*. \tag{3.3}$$

The corresponding operator $A : X \to X$ is defined by

$$A(u)(x,y) := \alpha + \int_0^x \int_0^y v(s,t)u(s,t)dsdt, \quad (x,y) \in D. \tag{3.4}$$

This operator is an increasing Picard operator, but the function:

$$(x,y) \mapsto \alpha \exp\left(\int_0^x \int_0^y v(s,t)dsdt\right) \tag{3.5}$$

is not a fixed point of the operator A. Therefore we have proved the following remark.

Remark 3.1. The right hand side of (3.2) is not a fixed point of the operator A, so the concrete Lemma 3.1 is not a consequence of the abstract Gronwall Lemma 1.1.

On the other hand, Lemma 1.1 allows us to obtain a theoretical upper bound of the type of the right-hand side of (3.2).

Lemma 3.2. *We assume that:*

(i) $v \in C([0,a] \times [0,b], \mathbb{R}_+)$, $\alpha \in \mathbb{R}_+$

(ii) v is increasing.

If $u \in C([0,a] \times [0,b])$ is a solution of the inequality (3.1) then $u(x,y) \leq u_A^*(x,y)$, where $u_A^*(x,y)$ is the unique fixed point of the corresponding operator A, defined by (3.4).

Proof. We consider $(X, \to, \leq) := C([0,a] \times [0,b], \stackrel{\|\cdot\|_\tau}{\to}, \leq)$ where $\|\cdot\|_\tau$ is the Bielecki norm on $C([0,a] \times [0,b])$ defined in (3.3). Consider also the operator $A : X \to X$ defined by (3.4).

Operator A is an increasing Picard operator, so applying Lemma 1.1 gives the conclusion. $\square$

In general, it is difficult to determine the unique fixed point of A. For two particular cases in which the fixed points can be found we obtain upper bounds for the inequalities' solutions which are stronger than those of Lemma 3.1.

Example 3.1. (Wendorff's inequality for $v(x,y) \equiv 1$)

Let $\alpha \in \mathbb{R}_+$ and $c \in \mathbb{R}$ be given. If $u \in C(D, \mathbb{R}_+)$ is a solution of the inequality

$$u(x,y) \leq \alpha + c^2 \int_0^x \int_0^y u(s,t)dsdt \tag{3.6}$$

with conditions:

$$u(x,0) = u(0,y) = \alpha, \quad x \in [0,a],\ y \in [0,b]$$

then:

$$u(x,y) \leq \alpha \exp(c^2 xy),\ \forall\ x \in [0,a],\ y \in [0,b]. \tag{3.7}$$

In this case the corresponding operator $A : X \to X$ is given by:

$$A(u)(x,y) := \alpha + c^2 \int_0^x \int_0^y u(s,t)dsdt, \quad x \in [0,a],\ y \in [0,b]. \tag{3.8}$$

This operator is an increasing Picard operator, but the function

$$(x,y) \mapsto \alpha \exp(c^2 xy)$$

is not a fixed point of the operator A.

Using Lemma 3.2 we can obtain:

$$u(x,y) \leq u_A^*(x,y), \tag{3.9}$$

where the fixed point $u_A^*(x,y)$ is (see [4], [7]):

$$u_A^*(x,y) = \alpha J_0(2c\sqrt{xy}). \tag{3.10}$$

Here $J_0(2c\sqrt{xy})$ is the Bessel function.

Example 3.2. (Wendorff's inequality for $v(x,y)=xy$)

Let $\alpha \in \mathbb{R}_+$ be given. If $u \in C(D,\mathbb{R}_+)$ is a solution of the inequality:

$$u(x,y) \le \alpha + \int_0^x \int_0^y st\, u(s,t)dsdt, \tag{3.11}$$

with conditions:

$$u(x,0)=u(0,y)=\alpha, \quad x\in[0,a],\ y\in[0,b]$$

then:

$$u(x,y) \le \alpha \exp(x^2y^2/4),\ \forall\, x\in[0,a],\ y\in[0,b]. \tag{3.12}$$

In this case the corresponding operator $A : X \to X$ given by:

$$A(u)(x,y) := \alpha + \int_0^x \int_0^y st\, u(s,t)dsdt, \quad x\in[0,a],\ y\in[0,b] \tag{3.13}$$

is an increasing Picard operator. However, the function

$$(x,y) \mapsto \alpha \exp(x^2y^2/4)$$

is not a fixed point of the operator A. From here on the argument is similar to the one in the previous example, but the fixed point is now:

$$u_A^*(x,y) = \alpha J_0(xy). \tag{3.14}$$

Here J_0 is the Bessel function.

In conclusion, we have seen that concrete Lemmas 2.1 and 2.2 can be derived from the abstract Lemma 1.1, while Lemma 3.1 cannot. However, we improved on Lemma 3.1 in the spirit of Lemma 1.1.

References

[1] D. Bainov and P. Simeonov, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Boston, 1992.

[2] V. Lakshmikantham, S. Leela and A.A. Martynyuk, *Stability Analysis of Nonlinear Systems*, Marcel Dekker, New York, 1989.

[3] N. Lungu, *On some Volterra integral inequalities*, Fixed Point Theory, **8**(2007), No. 1, 39-45.

[4] N. Lungu, *Qualitative Problems in the Theory of Hyperbolic Differential Equations*, Digital Data, Cluj-Napoca, 2006.

[5] N. Lungu, D. Popa, *On some differential inequalities*, Seminar of Fixed Point Theory, **3**(2002), 323-327.

[6] N. Lungu, I.A. Rus, *Gronwall inequalities via Picard operators*, (to appear).

[7] N. Lungu, I.A. Rus, *Hyperbolic differential inequalities*, Libertas Mathematica, **21** (2001), 35-40.

[8] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Inequalities Involving Functions and their Integral and Derivatives*, Kluwer Dordrecht, 1991.

[9] B.G. Pachpatte, *Inequalities for Differential and Integral Equations*, Academic Press, New York, London, 1998.

[10] D. Popa, N. Lungu, *On an operatorial inequality*, Demonstratio Mathematica, **38**(2005), 667-674.

[11] I.A. Rus, *Gronwall Lemmas: Ten Open Problems*, Scientaiae Math. Japonicae, (to appear).

[12] I.A. Rus, *Fixed points, upper and lower fixed points: abstract Gronwall lemmas*, Carpathian J. Math., **20**(2004), No. 1, 125-134.

[13] I.A. Rus, *Picard operators and applications*, Scientiae Math. Japonicae, **58**(2003), No. 1, 191-219.

[14] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[15] I.A. Rus, *Weakly Picard mappings*, Comment. Math. Caroline, **34**(1993), 769-773.

[16] V. Ya. Stetsenko, M. Shaban, *On operatorial inequalities analogous to Gronwall-Bihari ones*, D.A.N. Tadj., **29**(1986), 393-398 (in Russian).

Received: 04.03. 2009; Accepted: 12. 07. 2009.