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# APPROXIMATE COMMON FIXED POINTS FOR ONE-PARAMETER FAMILY OF NONEXPANSIVE NONSELF-MAPPINGS

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**Abstract.** Let $\mathcal{T}$ be a one-parameter family of nonexpansive nonself-mappings on a nonempty closed convex subset of a smooth and uniformly convex Banach space $X$ such that the set of common fixed points is nonempty. In this paper, we suggest and analyze a modified viscosity approximation method for the family $\mathcal{T}$ of nonexpansive nonself-mappings. We also prove that the approximate solution obtained by the proposed method converges strongly to a solution of a variational inequality.

**Key Words and Phrases**: Viscosity approximation method, fixed point problem, variational inequality, nonexpansive mapping, strong convergence, smooth and uniformly convex Banach space, demiclosedness.

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## 1. Introduction

Let $X$ be a real Banach space with norm $\|\cdot\|$ and let $X^*$ be its dual. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$. The (normalized) duality mapping $J$ from $X$ into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual $X^*$ is defined by

$$J(x) = \{\varphi \in X^* : \langle x, \varphi \rangle = \|x\|^2 = \|\varphi\|^2\}, \quad \forall x \in X.$$

It is known that the norm of $X$ is said to be Gâteaux differentiable (and $X$ is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y$ in $U = \{x \in X : \|x\| = 1\}$ the unit sphere of $X$. It is said to be uniformly Gâteaux differentiable if for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and $X$ is said to be uniformly smooth) if the limit in (1.1) is attained uniformly for $(x, y) \in U \times U$. Since the dual $X^*$ of $X$ is uniformly convex if and only if the norm of $X$ is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The converse implication is false. A discussion of these and related concepts may be found in [21].

Recall also that if $X$ is smooth then $J$ is single-valued and continuous from the norm topology of $X$ to the weak star topology of $X^*$, i.e., norm-to-weak* continuous. It is also well-known that if $X$ has a uniformly Gâteaux differentiable norm, then $J$ is uniformly continuous on bounded subsets of $X$ form the strong topology of $X$ to the weak star topology of $X^*$, i.e., uniformly norm-to-weak* continuous on each bounded subset of $X$. Moreover, if $X$ is uniformly smooth then $J$ is uniformly continuous on bounded subsets of $X$ form the strong topology of $X$ to the strong topology of $X^*$, i.e., uniformly norm-to-norm continuous on each bounded subset of $X$. See [13] for more details.

Let $C$ be a nonempty closed convex subset of a real Banach space $X$. A mapping $f : C \to C$ is said to be a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha \|x - y\|, \quad \forall x, y \in C.$$

We use $\Pi_C$ to denote the collection of all contractions on $C$. That is,

$$\Pi_C = \{f : f : C \to C \text{ a contraction}\}.$$

Let $D$ be a nonempty subset of $C$. A retraction from $C$ to $D$ is a mapping $Q : C \to D$ such that $Qx = x$ for all $x \in D$. A retraction $Q$ from $C$ to $D$ is nonexpansive if $Q$ is nonexpansive (i.e., $\|Qx - Qy\| \le \|x - y\|$ for all $x, y \in C$). A retraction $Q$ from $C$ to $D$ is sunny if $Q$ satisfies the property: $Q(Qx+t(x-Qx)) = Qx$ for each $x \in C$ and $t \ge 0$ whenever $Qx+t(x-Qx) \in C$. A retraction $Q$ from $C$ to $D$ is sunny nonexpansive if $Q$ is both sunny and nonexpansive.

It is well known that in a smooth Banach space $X$, a retraction $Q$ from $C$ to $D$ is a sunny nonexpansive retraction from $C$ to $D$ if and only if the following inequality holds:

$$\langle x - Qx, J(y - Qx)\rangle \le 0, \quad \forall x \in C,\ y \in D.$$

If $C$ is a nonempty closed convex subset of a Hilbert space $H$, then the nearest point projection $P_C$ from $H$ onto $C$ is a sunny nonexpansive retraction. This however is not true for Banach spaces. It is known that if $C$ is a closed convex subset of a uniformly smooth Banach space $X$ and there is a nonexpansive retraction from $X$ to $C$, then there exists a sunny nonexpansive retraction from $X$ to $C$. See [1, 5, 11, 15] for more details.

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Denote by $\mathrm{Fix}(T)$ the set of fixed points of $T$, that is, $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. See, e.g., [2, 7]. Let $G$ be an unbounded subset of $[0, \infty)$ such that $t + h \in G$ for all $t, h \in G$ and $t - h \in G$ for all $t, h \in G$ with $t > h$ (for instance, $G = [0, \infty)$ or $G = N$, the set of nonnegative integers). Recall that a one-parameter family $\mathcal{T} = \{T_t : t \in G\}$ of self-mappings of $C$ is said to be a nonexpansive semigroup on $C$ if the following conditions are satisfied:

(H1) $T_0 x = x,\ \forall x \in C$;

(H2) $T_{t+s} x = T_t T_s x,\ \forall t, s \in G,\ x \in C$;

(H3) for each $x \in C$, $T_t x$ is continuous in $t \in G$ when $G$ has the relative topology of $[0, \infty)$;

(H4) for each $t \in G$, there holds

$$\|T_t x - T_t y\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Denote by $F$ the set of common fixed points of $\mathcal{T}$, i.e.,

$$F = \{x \in C : T_s x = x, \ \forall s \in G\}.$$

Let $C$ be a nonexpansive retract of a smooth Banach space $X$ and $f \in \Pi_C$. The purpose of this paper is to propose and analyze the following iterative scheme for a family $\mathcal{T}= \{T_t : t \in G\}$ of nonexpansive nonself-mappings from $C$ to $X$, that is,

**Algorithm 1.1.** Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and let $\{r_n\}$ be a sequence in $G$ with $r_n \to \infty$. For an arbitrarily initial $x_0 \in C$ define a sequence $\{x_n\}$ recursively by the following explicit iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Q T_{r_n} x_n, \quad n \geq 0,$$

where $Q$ is the nonexpansive retraction of $X$ onto $C$.

If the family $\mathcal{T}= \{T^n : n \in N\}$, where $T$ is a nonexpansive self-mapping on $C$, then Algorithm 1.1 reduces to the following algorithm of Benavides, Acedo and Xu [11]:

**Algorithm 1.2.** Take a sequence $\{r_n\}$ in $N_+$ with $r_n \to \infty$ and a sequence $\{\alpha_n\}$ in $[0, 1]$. Starting with an arbitrarily initial $x_0 \in C$, define a sequence $\{x_n\}$ recursively by the following explicit iterative scheme

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T^{r_n} x_n, \quad n \geq 0,$$

where $u$ is an arbitrary but fixed point in $C$. For the convergence analysis of Algorithm 1.2; see [11].

First, under the lack of the assumptions that every weakly compact convex subset of $X$ has the fixed point property for nonexpansive mappings and that the family $\mathcal{T}= \{T_t : t \in G\}$ of nonexpansive mappings is a semigroup which were imposed in [20], we prove that as $s \to \infty$, $z_s$ converges strongly to a common fixed point of $\mathcal{T}$ in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, where

$$z_s = \alpha_s f(z_s) + (1 - \alpha_s)QT_s z_s.$$

Second, we establish the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 1.1 under some control conditions in a uniformly convex Banach space with both a uniformly Gâteaux differentiable norm and a weakly sequentially continuous duality mapping. Moreover, we deduce that these strong limits are solutions of certain variational inequality. Results in the recent literature related to the results of this paper can be found, e.g., in [3-5, 20, 23-27].

**Notation:** $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.

## 2. Preliminaries

Let $X$ be a real Banach space with the dual $X^*$. We denote by $J$ the normalized duality mapping from $X$ to $2^{X^*}$ defined by

$$J(x) = \{\varphi \in X^* : \langle x, \varphi \rangle = \|x\|^2 = \|\varphi\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between $X$ and $X^*$. Let $C$ be a nonempty closed convex subset of $X$ and let $T : C \to C$ be a mapping. Then we denote by $\mathrm{Fix}(T)$ the set of fixed points of $T$, i.e., $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. A mapping $f : C \to C$ is said to be a contraction on $C$ with a contractive constant $\alpha \in (0, 1)$, if

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

In the sequel, we always use $\Pi_C$ to denote the collection of all contractions on $C$ with a suitable contractive constant $\alpha \in (0, 1)$. That is,

$$\Pi_C := \{f : C \to C, \text{ a contraction with a suitable contractive constant}\}.$$

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Let $U = \{x \in X : \|x\| = 1\}$ be a unit sphere of $X$. Then the Banach space $X$ is called smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. Recall also that if $X$ is smooth then $J$ is single-valued and if $X$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on bounded subsets of $X$. We shall still denote the single-valued duality mapping by $J$.

If Banach space $X$ admits sequentially continuous duality mapping $J$ from the weak topology to the weak star topology, then by [22, Lemma 1], we know that duality mapping $J$ is single-valued. In this case, duality mapping $J$ is also said to be weakly sequentially continuous, i.e., for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, then $J(x_n) \overset{*}{\rightharpoonup} J(x)$ [22, 23].

Before starting the main results of this paper, we also include some lemmas.

**Lemma 2.1.** [12] Let $X$ be a real Banach space. Then for all $x, y \in X$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

for all $j(x + y) \in J(x + y)$.

**Lemma 2.2.** (See [6]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{a_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

**Lemma 2.3.** [13] Let $X$ be a reflexive Banach space, $C$ be a nonempty closed convex subset of $X$, and $T : C \to X$ be a nonexpansive mapping. Suppose that $X$ admits a weakly sequentially continuous duality mapping. Then the mapping $I - T$ is demiclosed at zero, i.e.,

$$\left.\begin{array}{r} x_n \rightharpoonup x \\ x_n - Tx_n \to 0 \end{array}\right\} \quad \text{implies} \quad x = Tx.$$

Here $I$ is the identity operator of $X$.

**Lemma 2.4.** [10, 14] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property

$$a_{n+1} \leq (1 - s_n)a_n + s_n t_n, \quad n \geq 0,$$

where $\{s_n\} \subset (0,1)$ and $\{t_n\}$ are such that
(i) $\sum_{n=0}^{\infty} s_n = \infty$,
(ii) either $\limsup_{n\to\infty} t_n \leq 0$ or $\sum_{n=0}^{\infty} |s_n t_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Recall that $\mu$ is said to be a mean on the set $N_+$ of all positive integers if $\mu$ is a continuous linear functional on $l^\infty$ satisfying $\|\mu\| = 1 = \mu(1)$. It is known that $\mu$ is a mean on $N_+$ if and only if

$$\inf\{a_n : n \in N_+\} \leq \mu(a) \leq \sup\{a_n : n \in N_+\}$$

for every $a = (a_1, a_2, ...) \in l^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean $\mu$ on $N_+$ is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, ...) \in l^\infty$. Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. It is also known that if $\mu$ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, ...) \in l^\infty$. Let $\{x_n\}$ be a bounded sequence in $X$. Then we can define the real valued continuous convex function $\phi$ on $X$ by

$$\phi(x) = \mu_n \|x_n - x\|^2$$

for all $x \in X$.

The following proposition is actually a variant of Lemma 1.2 in Reich [1].

**Proposition 2.1.** Let $C$ be a nonempty closed convex subset of a Banach space $X$ with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in $X$. Let $\mu$ be a Banach limit and $p \in C$. Then

$$\mu_n \|x_n - p\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if

$$\mu_n \langle x - p, J(x_n - p) \rangle \leq 0$$

for all $x \in C$, where $J$ is the duality mapping of $X$.

## 3. Main Results

Let $X$ be a smooth Banach space, $C$ be a nonexpansive retract of $X$ and $f \in \Pi_C$. Now we first consider the existence of $Q(f)$ which solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

Let $\{\alpha_s\}_{s\in G}$ be a net in the interval $(0,1)$ such that $\lim_{s\to\infty} \alpha_s = 0$ and let $\mathcal{T}= \{T_t : t \in G\}$ be a family of nonexpansive nonself-mappings from $C$ to $X$. By Banach's contraction principle, for each $s \in G$, we have a unique solution $z_s \in C$ of the equation

$$z_s = \alpha_s f(z_s) + (1-\alpha_s)QT_s z_s. \tag{3.1}$$

**Theorem 3.1.** Let $X$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let $C$ be a nonempty closed convex subset of $X$ and $\mathcal{T}= \{T_s : s \in G\}$ be a family of nonexpansive nonself-mappings on $C$. Suppose that $C$ is a nonexpansive retract of $X$, that for a fixed contraction $f \in \Pi_C$, $\{z_s\}_{s\in G}$ is the net generated by (3.1), and that $\mathcal{T}$ satisfies the uniformly left asymptotically regular condition on bounded subsets of $C$, i.e., for each bounded subset $\widetilde{C}$ of $C$, there holds

$$\lim_{s\in G, s\to\infty} \sup_{x\in\widetilde{C}} \|T_r QT_s x - QT_s x\| = 0, \quad r \in G. \tag{ULARC}$$

Then $F = \{x \in C : T_s x = x,\ \forall s \in G\}$ is nonempty if and only if $\{z_s\}_{s\in G}$ is bounded and in this case, $z_s$ converges strongly as $s \to \infty$ to a common fixed point of $\mathcal{T}$. If we define $Q : \Pi_C \to F$ by

$$Q(f) = \lim_{s\to\infty} z_s, \quad f \in \Pi_C, \tag{3.2}$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

In particular, if $f = u \in C$ is a constant, then the limit (3.2) defines the sunny nonexpansive retraction $Q$ from $C$ to $F$,

$$\langle Q(u)-u, J(Q(u)-p)\rangle \leq 0, \quad u \in C,\ p \in F.$$

**Proof.** If the common fixed point set $F$ of $\mathcal{T}$ is nonempty, then $\{z_s\}_{s\in G}$ is bounded. Indeed, take $p \in F$ arbitrarily. Then we have for each $s \in G$

$$\begin{aligned}\|z_s - p\| &\leq \alpha_s\|f(z_s) - p\| + (1-\alpha_s)\|QT_s z_s - p\| \\ &\leq \alpha\alpha_s\|z_s - p\| + \alpha_s\|f(p) - p\| + (1-\alpha_s)\|z_s - p\|.\end{aligned}$$

which implies that

$$\|z_s - p\| \leq \frac{1}{1-\alpha}\|f(p) - p\|,$$

and hence $\{z_s\}_{s\in G}$ is bounded.

Suppose conversely that $\{z_s\}_{s\in G}$ is bounded. Let us show that $F$ is nonempty and that $z_s$ converges strongly as $s \to \infty$ to a common fixed point of $\mathcal{T}$. Next we divide the proof into several steps.

**Step 1.** We claim that for each $\{s_n\} \subset G$ with $s_n \to \infty$ $(n \to \infty)$ the following set

$$D = \{u \in C : \mu_n\|z_{s_n} - u\|^2 = \min_{x\in C} \mu_n\|z_{s_n} - x\|^2\}$$

consists of one point, where $\mu$ is a Banach limit.

Indeed, let $\{s_n\}$ be a sequence of $G$ such that $\lim_{n\to\infty} s_n = \infty$. Define a function $\phi : C \to [0,\infty)$ by

$$\phi(x) := \mu_n\|z_{s_n} - x\|^2, \quad x \in C.$$

Since $\phi$ is continuous and convex, $\phi(x) \to \infty$ as $\|x\| \to \infty$, and $X$ is reflexive, $\phi$ attains its infimum over $C$ (cf. [8, p. 79]), that is, there exists $z \in C$ such that

$$\mu_n\|z_{s_n} - z\|^2 = \min_{x\in C} \mu_n\|z_{s_n} - x\|^2.$$

Then

$$D = \{u \in C : \phi(u) = \min_{x\in C} \phi(x)\}.$$

is nonempty because $z \in D$. From Proposition 2.1 we know that $z \in D$ if and only if

$$\mu_n\langle x - z, J(z_{s_n} - z)\rangle \leq 0$$

for all $x \in C$. Now let $w \in D$ such that $w \neq z$. Then, by [9, Theorem 1], there exists a positive number $k > 0$ such that

$$\langle z_{s_n} - z - (z_{s_n} - w), J(z_{s_n} - z) - J(z_{s_n} - w)\rangle \geq k > 0$$

for every $n$. Therefore we get

$$\mu_n\langle w-z, J(z_{s_n}-z)-J(z_{s_n}-w)\rangle \geq k>0.$$

Furthermore, since $z, w \in D$, from Proposition 2.1 we have

$$\mu_n\langle w-z, J(z_{s_n}-z)\rangle \leq 0 \quad \text{and} \quad \mu_n\langle z-w, J(z_{s_n}-w)\rangle \leq 0.$$

Hence we have

$$\mu_n\langle w-z, J(z_{s_n}-z)-J(z_{s_n}-w)\rangle \leq 0.$$

This leads to a contradiction. Therefore $z=w$ and so $D=\{z\}$.

**Step 2.** We claim that for a given $\{s_n\} \subset G$ with $s_n \to \infty$ $(n \to \infty)$ we have $D=\{z\} \subset F$.

Indeed, since $C$ is a nonexpansive retract of $X$, the point $z$ is also the unique global minimum point (over all of $X$) and hence $T_s z = z$ for all $s \in G$. As a matter of fact, let $Q$ be a nonexpansive retraction from $X$ to $C$. Then for each $x \in X$ we have

$$\begin{aligned}\mu_n\|z_{s_n}-x\|^2 &\geq \mu_n\|Qz_{s_n}-Qx\|^2 \\ &= \mu_n\|z_{s_n}-Qx\|^2 \geq \mu_n\|z_{s_n}-z\|^2\end{aligned}$$

and hence

$$z \in M = \{u \in X : \phi(u) = \min_{x\in X}\phi(x)\}.$$

Repeating the same argument as in Step 1, we can conclude that the above global minimum point $z$ is also unique, that is, $M=\{z\}$. Note that $\{z_s\}_{s\in G}$ is bounded. So it is easy to see that both $\{f(z_s)\}_{s\in G}$ and $\{T_s z_s\}_{s\in G}$ are bounded. Consequently, we have

$$\|z_s - QT_s z_s\| = \alpha_s\|f(z_s)-QT_s z_s\| \to 0 \quad (s\to\infty). \tag{3.3}$$

Utilizing (ULARC) we conclude that for each $r \in G$

$$\begin{aligned}\phi(T_r z) &= \mu_n\|z_{s_n}-T_r z\|^2 \leq \mu_n\{\|z_{s_n}-QT_{s_n}z_{s_n}\| + \|QT_{s_n}z_{s_n}-T_r z\|\}^2 \\ &= \mu_n\|QT_{s_n}z_{s_n}-T_r z\|^2 \leq \mu_n\{\|QT_{s_n}z_{s_n}-T_rQT_{s_n}z_{s_n}\| + \|T_rQT_{s_n}z_{s_n}-T_r z\|\}^2 \\ &= \mu_n\|T_rQT_{s_n}z_{s_n}-T_r z\|^2 \leq \mu_n\|QT_{s_n}z_{s_n}-z\|^2 \\ &\leq \mu_n\{\|QT_{s_n}z_{s_n}-z_{s_n}\| + \|z_{s_n}-z\|\}^2 = \mu_n\|z_{s_n}-z\|^2 = \phi(z).\end{aligned}$$

This implies that $T_r z \in M = \{z\}$ for all $r \in G$. Therefore $z \in F$.

**Step 3.** We claim that $z_s$ converges strongly as $s \to \infty$ to a common fixed point of $\mathcal{T}$.

Indeed, take $w \in F$ arbitrarily. Then we have

$$\begin{aligned}
&\langle z_{s_n} - QT_{s_n} z_{s_n}, J(z_{s_n} - w)\rangle \\
&= \langle z_{s_n} - QT_{s_n} w + QT_{s_n} w - QT_{s_n} z_{s_n}, J(z_{s_n} - w)\rangle \\
&= \|z_{s_n} - QT_{s_n} w\|^2 - \langle QT_{s_n} z_{s_n} - QT_{s_n} w, J(z_{s_n} - w)\rangle \\
&\geq \|z_{s_n} - QT_{s_n} w\|^2 - \|QT_{s_n} z_{s_n} - QT_{s_n} w\|\|z_{s_n} - w\| \\
&\geq \|z_{s_n} - QT_{s_n} w\|^2 - \|z_{s_n} - QT_{s_n} w\|^2 = 0
\end{aligned}$$

for all $n$. Since $z_{s_n} - QT_{s_n} z_{s_n} = \alpha_{s_n}(f(z_{s_n}) - QT_{s_n} z_{s_n})$, we get from the last inequality

$$\begin{aligned}
0 &\leq \langle z_{s_n} - QT_{s_n} z_{s_n}, J(z_{s_n} - w)\rangle \\
&= \alpha_{s_n}\langle f(z_{s_n}) - QT_{s_n} z_{s_n}, J(z_{s_n} - w)\rangle,
\end{aligned}$$

and hence

$$\langle QT_{s_n} z_{s_n} - f(z_{s_n}), J(z_{s_n} - w)\rangle \leq 0$$

for all $n$. This implies that

$$\mu_n\langle QT_{s_n} z_{s_n} - f(z_{s_n}), J(z_{s_n} - w)\rangle \leq 0. \tag{3.4}$$

Thus from (3.3) and (3.4), we obtain

$$\begin{aligned}
&\mu_n\langle z_{s_n} - f(z_{s_n}), J(z_{s_n} - w)\rangle \\
&= \mu_n\langle z_{s_n} - QT_{s_n} z_{s_n} + QT_{s_n} z_{s_n} - f(z_{s_n}), J(z_{s_n} - w)\rangle \\
&= \mu_n\langle z_{s_n} - QT_{s_n} z_{s_n}, J(z_{s_n} - w)\rangle + \mu_n\langle QT_{s_n} z_{s_n} - f(z_{s_n}), J(z_{s_n} - w)\rangle \\
&= \mu_n\langle QT_{s_n} z_{s_n} - f(z_{s_n}), J(z_{s_n} - w)\rangle \leq 0,
\end{aligned}$$

and hence

$$\mu_n\langle z_{s_n} - f(z_{s_n}), J(z_{s_n} - w)\rangle \leq 0 \tag{3.5}$$

for each $w \in F$. From Proposition 2.1 it follows that

$$\mu_n\langle x - z, J(z_{s_n} - z)\rangle \leq 0$$

for all $x \in C$. In particular, we have

$$\mu_n\langle f(z) - z, J(z_{s_n} - z)\rangle \leq 0. \tag{3.6}$$

Since $f \in \Pi_C$, by combining (3.5) with (3.6) we obtain

$$\begin{aligned}
\alpha\mu_n\|z_{s_n} - z\|^2 &\geq \mu_n\langle f(z_{s_n}) - f(z), J(z_{s_n} - z)\rangle \\
&\geq \mu_n\{\langle f(z_{s_n}) - f(z), J(z_{s_n} - z)\rangle + \langle f(z) - z, J(z_{s_n} - z)\rangle\} \\
&= \mu_n\langle f(z_{s_n}) - z, J(z_{s_n} - z)\rangle \\
&= \mu_n\{\langle f(z_{s_n}) - z_{s_n}, J(z_{s_n} - z)\rangle + \langle z_{s_n} - z, J(z_{s_n} - z)\rangle\} \\
&\geq \mu_n\langle z_{s_n} - z, J(z_{s_n} - z)\rangle \\
&= \mu_n\|z_{s_n} - z\|^2,
\end{aligned}$$

and hence

$$\mu_n\|z_{s_n} - z\|^2 \leq 0.$$

Therefore, there exists a subsequence $\{z_{s_{n_j}}\}$ of $\{z_{s_n}\}$ which converges strongly to $z$. In order to show $z_{s_n} \to z$ $(n \to \infty)$, suppose that there is another subsequence $\{z_{s_{m_i}}\}$ of $\{z_{s_n}\}$ which converges strongly to (say) $y \in C$. Then $y$ must be a common fixed point of $\mathcal{T}$. In fact, observe that

$$\begin{aligned}
\|y - T_r y\| &\leq \|y - z_{s_{m_i}}\| + \|z_{s_{m_i}} - QT_{s_{m_i}} z_{s_{m_i}}\| \\
&\quad + \|QT_{s_{m_i}} z_{s_{m_i}} - T_r QT_{s_{m_i}} z_{s_{m_i}}\| + \|T_r QT_{s_{m_i}} z_{s_{m_i}} - T_r z_{s_{m_i}}\| \\
&\quad + \|T_r z_{s_{m_i}} - T_r y\| \\
&\leq 2\|y - z_{s_{m_i}}\| + 2\|z_{s_{m_i}} - QT_{s_{m_i}} z_{s_{m_i}}\| \\
&\quad + \|QT_{s_{m_i}} z_{s_{m_i}} - T_r QT_{s_{m_i}} z_{s_{m_i}}\|.
\end{aligned}$$

Thus, (ULARC) together with (3.3) implies that $y = T_r y$ for all $r \in G$. That is, $y \in F$. Consequently, from (3.5) it follows that

$$\langle z - f(z), J(z - y)\rangle \leq 0$$

and

$$\langle y - f(y), J(y - z)\rangle \leq 0.$$

Adding these two inequalities yields

$$\begin{aligned}
(1 - \alpha)\|z - y\|^2 &\leq \langle z - y, J(z - y)\rangle - \langle f(z) - f(y), J(z - y)\rangle \\
&= \langle z - f(z), J(z - y)\rangle + \langle y - f(y), J(y - z)\rangle \leq 0,
\end{aligned}$$

and thus $z = y$. Therefore, $z_s$ converges strongly as $s \to \infty$ to a point in $F$.

**Step 4.** We claim that if we define $Q : \Pi_C \to F$ by

$$Q(f) = \lim_{s \to \infty} z_s, \quad f \in \Pi_C,$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

Indeed, define $Q : \Pi_C \to F$

$$Q(f) = \lim_{s\to\infty} z_s, \quad f \in \Pi_C.$$

Since $z_s = \alpha_s f(z_s) + (1-\alpha_s)QT_s z_s$, we have

$$(I-f)z_s = -\frac{1-\alpha_s}{\alpha_s}(I-QT_s)z_s.$$

Hence for each $p \in F$,

$$\langle (I-f)z_s, J(z_s-p)\rangle = -\frac{1-\alpha_s}{\alpha_s}\langle (I-T_s)z_s-(I-T_s)p, J(z_s-p)\rangle \leq 0.$$

Letting $s \to \infty$ yields

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0.$$

In particular, if $f = u \in C$ is a constant, then

$$\langle Qu-u, J(Qu-p)\rangle \leq 0, \quad u \in C,\ p \in F.$$

Therefore $Q$ is a sunny nonexpansive retraction from $C$ to $F$. This completes the proof. □

We remark that the important condition (ii) in Theorem 3.2 was introduced by Suzuki [6].

**Theorem 3.2.** Let $X$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let $C$ be a nonempty closed convex subset of $X$, $f \in \Pi_C$ and $\mathcal{T} = \{T_s : s \in G\}$ be a family of nonexpansive nonself-mappings on $C$ such that $F \neq \emptyset$. Suppose that $C$ is a nonexpansive retract of $X$ and that $X$ admits a weakly sequentially continuous duality mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{r_n\}$ be a sequence in $G$. Let $\{\alpha_n\}$ satisfy the control conditions (C1) $\lim_{n\to\infty}\alpha_n = 0$ and (C2)$\Sigma_{n=0}^{\infty}\alpha_n = \infty$. Assume that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $r_n \to \infty$ such that $T_{r_{n+1}}x_n - T_{r_n}x_n \to 0$, $\forall\{x_n\}$ bounded in $C$;

(iv) $\mathcal{T}$ satisfies (ULARC) on bounded subsets of $C$, i.e., for each bounded subset $\widetilde{C}$ of $C$, there holds

$$\lim_{s\in G, s\to\infty} \sup_{x\in\widetilde{C}} \|T_r QT_s x - QT_s x\| = 0, \quad r \in G. \tag{ULARC}$$

Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n QT_{r_n} x_n, \end{cases} \tag{3.7}$$

converges strongly to $Q(f) \in F$, where $Q(f)$ is a solution of the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \le 0, \quad f \in \Pi_C, \ p \in F.$$

**Proof.** We divide the proof into several steps.

**Step 1.** We claim that $\{x_n\}$ is bounded.
Indeed, take $p \in F$ arbitrarily. Then we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n QT_{r_n} x_n - p\| \\ &\le \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|QT_{r_n} x_n - p\| \\ &\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| \\ &\quad + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\ &\le \alpha\alpha_n \|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n \|f(p) - p\| \\ &= [1 - (1 - \alpha)\alpha_n]\|x_n - p\| + \alpha_n \|f(p) - p\| \\ &\le \max\{\|x_n - p\|, \tfrac{\|f(p)-p\|}{1-\alpha}\}. \end{aligned}$$

By induction we have

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \alpha}\}, \quad n \ge 0.$$

Thus, it follows that $\{x_n\}$ is bounded, and so are $\{T_{r_n} x_n\}$ and $\{f(x_n)\}$.

**Step 2.** We claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
Indeed, define a sequence $\{x_n\}$ by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n. \tag{3.8}$$

Then we observe that

$$\begin{aligned} y_{n+1} - y_n &= \tfrac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \tfrac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \tfrac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}QT_{r_{n+1}}x_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n f(x_n)+\gamma_n QT_{r_n}x_n}{1-\beta_n} \\ &= \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1}) - \tfrac{\alpha_n}{1-\beta_n}f(x_n) \\ &+ \tfrac{\gamma_{n+1}}{1-\beta_{n+1}}(QT_{r_{n+1}}x_{n+1} - QT_{r_{n+1}}x_n) \\ &+QT_{r_{n+1}}x_n - QT_{r_n}x_n + \tfrac{\alpha_n}{1-\beta_n}QT_{r_n}x_n - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}QT_{r_{n+1}}x_n. \end{aligned}$$

It follows that

$$\begin{aligned} &\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|QT_{r_{n+1}}x_n\|) + \tfrac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|QT_{r_n}x_n\|) \\ &+\tfrac{\gamma_{n+1}}{1-\beta_{n+1}}\|QT_{r_{n+1}}x_{n+1} - QT_{r_{n+1}}x_n\| \\ &+\|QT_{r_{n+1}}x_n - QT_{r_n}x_n\| - \|x_{n+1} - x_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|QT_{r_{n+1}}x_n\|) + \tfrac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|QT_{r_n}x_n\|) \\ &+\|T_{r_{n+1}}x_n - T_{r_n}x_n\|. \end{aligned} \tag{3.9}$$

According to condition (iii) and $\alpha_n \to 0$, we conclude from (3.9) that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Thus, by Lemma 2.2, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Consequently, it follows from (3.8) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|y_n - x_n\| = 0. \tag{3.10}$$

**Step 3.** We claim that for each fixed $s \in G$, $T_s x_n - x_n \to 0$ $(n \to \infty)$. Indeed, observe that

$$\begin{aligned} \|x_{n+1} - QT_{r_n}x_n\| &\le \alpha_n\|f(x_n) - QT_{r_n}x_n\| + \beta_n\|x_n - QT_{r_n}x_n\| \\ &\le \alpha_n\|f(x_n) - QT_{r_n}x_n\| + \beta_n\|x_n - x_{n+1}\| \\ &+\beta_n\|x_{n+1} - QT_{r_n}x_n\|, \end{aligned}$$

and hence

$$\begin{aligned} \|x_n - QT_{r_n}x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - QT_{r_n}x_n\| \\ &\le \|x_n - x_{n+1}\| \\ &+\tfrac{1}{1-\beta_n}(\alpha_n\|f(x_n) - QT_{r_n}x_n\| + \beta_n\|x_n - x_{n+1}\|). \end{aligned} \tag{3.11}$$

So, from (3.10), (3.11) and conditions (C1), (ii), we derive

$$\lim_{n\to\infty} \|x_n - QT_{r_n}x_n\| = 0. \tag{3.12}$$

Let $\widetilde{C}$ be any bounded subset of $C$ which contains the sequence $\{x_n\}$. It follows that for each fixed $s \in G$

$$\begin{aligned} \|T_s x_n - x_n\| &\le \|T_s x_n - T_s QT_{r_n}x_n\| + \|T_s QT_{r_n}x_n - QT_{r_n}x_n\| \\ &\quad + \|QT_{r_n}x_n - x_n\| \\ &\le 2\|QT_{r_n}x_n - x_n\| + \|T_s QT_{r_n}x_n - QT_{r_n}x_n\| \\ &\le 2\|QT_{r_n}x_n - x_n\| + \sup_{x\in\widetilde{C}} \|T_s QT_{r_n}x - QT_{r_n}x\|. \end{aligned}$$

So (ULARC) together with (3.12) yields

$$\lim_{n\to\infty} \|T_s x_n - x_n\| = 0. \tag{3.13}$$

**Step 4.** We claim that $\limsup_{n\to\infty}\langle (I-f)Q(f), J(Q(f)-x_{n+1})\rangle \le 0$, where $Q : \Pi_C \to F$ is defined in Theorem 3.1.

Indeed, the mapping $Q : \Pi_C \to F$ is defined well by virtue of Theorem 3.1. Since $X$ is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_n\}$ such that $x_{n_j+1} \rightharpoonup q$ and

$$\limsup_{n\to\infty}\langle (I-f)Q(f), J(Q(f)-x_{n+1})\rangle = \lim_{j\to\infty}\langle (I-f)Q(f), J(Q(f)-x_{n_j+1})\rangle.$$

It follows from (3.13) that $\lim_{j\to\infty}\|x_{n_j+1} - T_s x_{n_j+1}\| = 0$. Hence, by Lemma 2.3 we get $q \in F$. On the other hand, since $Q(f) \in F$ satisfies

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C,\ p \in F,$$

utilizing the weakly sequential continuity of duality mapping $J$, we have

$$\begin{aligned} &\limsup_{n\to\infty}\langle (I-f)Q(f), J(Q(f)-x_{n+1})\rangle \\ &= \lim_{j\to\infty}\langle (I-f)Q(f), J(Q(f)-x_{n_j+1})\rangle \\ &= \langle (I-f)Q(f), J(Q(f)-q)\rangle \le 0. \end{aligned}$$

**Step 5.** We claim that $\lim_{n\to\infty}\|x_n - Q(f)\| = 0$.

Indeed, utilizing Lemma 2.1, we obtain

$$\begin{aligned}
&\|x_{n+1} - Q(f)\|^2 \\
&= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n QT_{r_n} x_n - Q(f)\|^2 \\
&\le \|\beta_n (x_n - Q(f)) + \gamma_n (QT_{r_n} x_n - Q(f))\|^2 \\
&+2\alpha_n \langle f(x_n) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le (\beta_n \|x_n - Q(f)\| + \gamma_n \|QT_{r_n} x_n - Q(f)\|)^2 \\
&+2\alpha_n \langle f(x_n) - f(Q(f)), J(x_{n+1} - Q(f))\rangle \\
&+2\alpha_n \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le (1-\alpha_n)^2 \|x_n - Q(f)\|^2 + 2\alpha\alpha_n \|x_n - Q(f)\| \|x_{n+1} - Q(f)\| \\
&+2\alpha_n \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le (1-\alpha_n)^2 \|x_n - Q(f)\|^2 + \alpha\alpha_n (\|x_n - Q(f)\|^2 + \|x_{n+1} - Q(f)\|^2) \\
&+2\alpha_n \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\|x_{n+1} - Q(f)\|^2 \\
&\le \tfrac{1-(2-\alpha)\alpha_n + \alpha_n^2}{1-\alpha\alpha_n} \|x_n - Q(f)\|^2 \\
&+ \tfrac{2\alpha_n}{1-\alpha\alpha_n} \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle \\
&\le \tfrac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n} \|x_n - Q(f)\|^2 + \tfrac{\alpha_n^2}{1-\alpha\alpha_n} M \\
&+ \tfrac{2\alpha_n}{1-\alpha\alpha_n} \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle,
\end{aligned} \tag{3.14}$$

where $M := \sup_{n\ge 0} \|x_n - Q(f)\|^2$. Put

$$s_n = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$$

and

$$t_n = \frac{M\alpha_n}{2(1-\alpha)} + \frac{1}{1-\alpha} \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f))\rangle.$$

From Step 3 and conditions (C1), (C2) it follows that $s_n \to 0$, $\sum_{n=0}^{\infty} s_n = \infty$ and $\limsup_{n\to\infty} t_n \le 0$. Observe that in this case, (3.14) reduces to

$$\|x_{n+1} - Q(f)\|^2 \le (1-s_n)\|x_n - Q(f)\|^2 + s_n t_n.$$

Therefore, utilizing Lemma 2.4, we have

$$\lim_{n\to\infty} \|x_n - Q(f)\| = 0.$$

This completes the proof. □

The following result is immediate from Theorem 3.2.

**Corollary 3.1.** Let $X$ be a uniformly convex Banach space with both a uniformly Gâteaux differentiable norm and a weakly sequentially continuous duality mapping. Let $C$ be a nonempty closed convex subset of $X$, $f \in \Pi_C$ and $\mathcal{T}= \{T_s : s \in G\}$ be a semigroup of nonexpansive mappings on $C$ such that $F \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0, 1)$ and $\{r_n\}$ be a sequence in $G$. Let $\{\alpha_n\}$ satisfy the control conditions (C1), (C2). Assume that

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $r_n \to \infty$ such that $T_{r_{n+1}}x_n - T_{r_n}x_n \to 0$, $\forall \{x_n\}$ bounded in $C$;

(iv) $\mathcal{T}$ satisfies (ULARC) on bounded subsets of $C$, i.e., for each bounded subset $\widetilde{C}$ of $C$, there holds

$$\lim_{s\in G, s\to\infty} \sup_{x\in\widetilde{C}} \|T_rT_sx - T_sx\| = 0, \quad r \in G. \tag{ULARC}$$

Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{r_n} x_n, \end{cases}$$

converges strongly to $Q(f) \in F$, where $Q(f)$ is a solution of the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

**Proof.** Whenever $\mathcal{T}$ is a semigroup of nonexpansive mappings on $C$, by the careful analysis of the proof of Theorems 3.1 and 3.2, it is easy to see that, the restriction that $C$ is a nonexpansive retract of $X$ can be removed. Therefore, by Theorem 3.2 we obtain the desired conclusion. This completes the proof. □

Note that in the case when $X = H$ a Hilbert space, the nonempty closed convex subset $C$ is a sunny nonexpansive retract of $H$, the nearest point projection $P$ of $C$ onto $F$ is a sunny nonexpansive retraction and the duality mapping $J$ is the identity mapping $I$. It is clear that the following results immediately follow from Theorems 3.1 and 3.2.

**Corollary 3.2.** Let $H$ be a Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $\mathcal{T}=\{T_s : s\in G\}$ be a family of nonexpansive nonself-mappings on $C$. Denote by $P_C$ the nearest point projection of $H$ onto $C$. Assume that for a fixed contraction $f\in \Pi_C$, $\{z_s\}_{s\in G}$ is the net generated by

$$z_s=\alpha_s f(z_s)+(1-\alpha_s)P_C T_s z_s, \quad s\in G,$$

and that $\mathcal{T}$ satisfies the uniformly left asymptotically regular condition on bounded subsets of $C$, i.e., for each bounded subset $\widetilde{C}$ of $C$, there holds

$$\lim_{s\in G, s\to\infty}\sup_{x\in\widetilde{C}}\|T_r P_C T_s x-P_C T_s x\|=0, \quad r\in G. \tag{ULARC}$$

Then $F=\{x\in C : T_s x=x,\ \forall s\in G\}$ is nonempty if and only if $\{z_s\}_{s\in G}$ is bounded and in this case, $z_s$ converges strongly as $s\to\infty$ to a common fixed point of $\mathcal{T}$. If we define $Q:\Pi_C\to F$ by

$$Q(f)=\lim_{s\to\infty} z_s, \quad f\in \Pi_C, \tag{3.15}$$

then $Q(f)$ is the unique solution the variational inequality

$$\langle (I-f)Q(f), Q(f)-p\rangle\le 0, \quad f\in\Pi_C,\ p\in F.$$

In particular, if $f=u\in C$ is a constant, then (3.15) reduces to the nearest point projection $P$ from $C$ onto $F$,

$$\langle P(u)-u, P(u)-p\rangle\le 0, \quad u\in C,\ p\in F.$$

**Corollary 3.3.** Let $H$ be a Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $f\in\Pi_C$ and $\mathcal{T}=\{T_s : s\in G\}$ be a family of nonexpansive nonself-mappings on $C$ such that $F\neq\emptyset$. Denote by $P_C$ the nearest point projection of $H$ onto $C$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{r_n\}$ be a sequence in $G$. Let $\{\alpha_n\}$ satisfy the control conditions (C1), (C2). Assume that

(i) $\alpha_n+\beta_n+\gamma_n=1$;

(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;

(iii) $r_n\to\infty$ such that $T_{r_{n+1}}x_n-T_{r_n}x_n\to 0$, $\forall\{x_n\}$ bounded in $C$;

(iv) $\mathcal{T}$ satisfies (ULARC) on bounded subsets of $C$, i.e., for each bounded subset $\widetilde{C}$ of $C$, there holds

$$\lim_{s\in G, s\to\infty}\sup_{x\in\widetilde{C}}\|T_r P_C T_s x-P_C T_s x\|=0, \quad r\in G. \tag{ULARC}$$

Then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C T_{r_n} x_n, \end{cases}$$

converges strongly to $Q(f) \in F$, where $Q(f)$ is the unique solution of the variational inequality

$$\langle (I - f)Q(f), Q(f) - p \rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

Let $D$ be a subset of a Banach space $X$. Recall that a mapping $T : D \to X$ is said to be firmly nonexpansive if, for each $x, y \in D$, the convex function $\phi : [0, 1] \to [0, \infty)$ defined by

$$\phi(t) = \|(1 - t)x + tTx - ((1 - t)y + tTy)\|,$$

is nonincreasing. Since $\phi$ is convex, it is easy to check that a mapping $T : D \to X$ is firmly nonexpansive if and only if

$$\|Tx - Ty\| \leq \|(1 - t)(x - y) + t(Tx - Ty)\|,$$

for each $x, y \in D$ and $t \in [0, 1]$. It is obvious that every firmly nonexpansive mapping is nonexpansive.

**Corollary 3.4.** Let $X$ be a uniformly convex and uniformly smooth Banach space. Let $C$ be a nonempty closed convex subset of $X$ and $\mathcal{T} = \{T_s : s \in G\}$ be a family of firmly nonexpansive nonself-mappings on $C$ such that $F \neq \emptyset$. Suppose that $C$ is a nonexpansive retract of $X$, and that $\mathcal{T}$ satisfies the (ULARC) on bounded subsets of $C$, i.e., for each bounded subset $\widetilde{C}$ of $C$, there holds

$$\lim_{s \in G, s \to \infty} \sup_{x \in \widetilde{C}} \|T_r Q T_s x - Q T_s x\| = 0, \quad r \in G. \tag{ULARC}$$

Then the net $\{z_s\}_{s \in G}$ generated by (3.1) converges strongly to $Q(f) \in F$, where $Q(f)$ is a solution of the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

## References

[1] S. Reich, *Product formulas, nonlinear semigroups and accretive operators*, J. Funct. Anal., **36**(1980), 147-168.

[2] F.E. Browder, *Convergence of approximations to fixed points of nonexpansive mappings in Banach spaces*, Arch. Ration. Mech. Anal., **24**(1967), 82-90.

[3] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc., **73**(1967), 957-961.

[4] P.L. Lions, *Approximation de points fixes de contractions*, C.R. Acad. Sci. Paris Ser. A-B, **284**(1977), 1357-1359.

[5] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **75**(1980), 287-292.

[6] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integers*, J. Math. Anal. Appl., **305**(2005), 227-239.

[7] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., **59**(1992), 486-491.

[8] V. Barbu, Th. Precupanu, *Convexity and Optimization in Banach spaces*, Editura Academiei R.S.R., Bucharest, 1978.

[9] B. Prus, *A characterization of uniform convexity and applications to accretive operators*, Hiroshima J. Math., **11**(1981), 229-234.

[10] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240-256.

[11] T. Dominguez Benavides, G. Lopez Acedo, H.K. Xu, *Construction of sunny nonexpansive retractions in Banach spaces*, Bull. Austral. Math. Soc., **66**(2002), 9-16.

[12] J.S. Jung, C. Morales, *The Mann process for perturbed $m$-accretive operators in Banach spaces*, Nonlinear Anal., **46**(2001), 231-243.

[13] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[14] L.S. Liu, *Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., **194**(1995), 114-125.

[15] W. Takahashi, Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators*, J. Math. Anal. Appl., **104**(1984), 546-553.

[16] L.C. Ceng, J.C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math., **214**(2008), 186-201.

[17] A. Moudafi, *Viscosity approximation methods for fixed point theorems*, J. Math. Anal. Appl., **241**(2000), 46-55.

[18] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279-291.

[19] A. Aleyner, Y. Censor, *Best approximation to common fixed points of a semigroup of nonexpansive operators*, J. Convex Anal., in press.

[20] Y. Yao, M.A. Noor, *On viscosity iterative methods for variational inequalities*, J. Math. Anal. Appl., **325**(2007), 776-787.

[21] M.M. Day, *Normed Linear Spaces*, 3rd ed. Springer-Verlag, Berlin, 1973.

[22] J.P. Gossez, E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math., **40**(1972), 565-573.

[23] J.S. Jung, *Convergence theorems of iterative algorithms for a family of finite nonexpansive mappings*, Taiwan. J. Math., **11**(2007), 883-902.

[24] L.C. Ceng, H.K. Xu, *Strong convergence of a hybrid viscosity approximation method with perturbed mappings for nonexpansive and accretive operators*, Taiwanese J. Math., **11**(2007), 661-682.

[25] L.C. Ceng, H.K. Xu, J.C. Yao, *Strong convergence of an iterative method with perturbed mappings for nonexpansive and accretive operators*, Numer. Funct. Anal. Optim., **29**(2008), 1-22.

[26] L.C. Ceng, H.K. Xu, J.C. Yao, *The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces*, Nonlinear Anal., **69**(2008), 1402-1412.

[27] S. Huang, *Fixed points of a sequence of asymptotically nonexpansive mappings*, Fixed Point Theory, **9**(2008), 465-485.

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