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THE OVER-RELAXED $A-$PROXIMAL POINT ALGORITHM AND APPLICATIONS TO NONLINEAR VARIATIONAL INCLUSIONS IN BANACH SPACES

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Abstract. Based on the notion of $A-$maximal relaxed accretiveness, the convergence analysis of the over-relaxed proximal point algorithm in the context of the approximating the solutions of a class of nonlinear variational inclusions is explored. Moreover, some results on the general firm nonexpansiveness are also investigated. The obtained results are general and application-oriented in nature.
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1. Introduction

Let us consider a real Banach space with the duality pairing $\langle,.\rangle$ between the elements of X and the elements of X^*, the dual space of X, and the norm $\|\cdot\|$ on X and X^*. We consider the nonlinear inclusion problem: find a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

Recently Pennanen [10] considered the proximal point algorithm, introduced by Eckstein and Bertsekas [3] in the context of solving (1), while achieving the local convergence of the proximal point algorithm without monotonicity. As a matter of fact, Rockafellar [14] generalized the algorithm of Martinet [9] when working on convex programming, which is referred to as the proximal point algorithm in the literature. This work was followed by the formulation [15] of the proximal point algorithm in conjunction with convex programming duality theory to present a general convergence analysis for the multiplier method in convex programming. Eckstein and Bertsekas [3] introduced the relaxed proximal point algorithm and applied to the solvability of inclusion problems of the form (1).

Recently the author [19-21], introduced and studied the notion of A-maximal relaxed monotonicity in the context of solving variational inclusion problems of the form (1) based on the resolvent operator techniques in Hilbert space settings. The notion of $A-$maximal relaxed monotonicity generalizes the general theory of multivalued maximal monotone mappings, including the notion of $H-$maximal monotonicity introduced by Fang and Huang [5], and provides a general framework to examining variational inclusion problems. Lan, Cho and Verma [8] generalized the notion of $A-$maximal relaxed monotonicity to the case of $A-$maximal relaxed accretiveness in a Banach space setting, and studied the approximation solvability of a class of inclusion problems involving $A-$maximal relaxed accretive mappings based on the resolvent operator technique. There exists an abundance amount of literature on resolvent operator methods and their applications to other fields, for instance equilibria problems, optimization and control theory, operations research, and mathematical programming,management and decision science.

Here, in this paper, we generalize the relaxed proximal point algorithm to the case of $A-$maximal relaxed accretive monotone mappings, and then we apply it to the approximation solvability of a class of nonlinear problems of the form (1) in a real Banach space setting. We have successfully established the linear convergence of the sequence. Furthermore, some results on the generalized firm nonexpansiveness, Lipschitz continuity of the generalized resolvent operator corresponding to $A-$maximal relaxed accretive mappings are included. For more literature, we recommend the reader [1-24].

2. A-MAXIMAL RELAXED ACCRETIVENESS AND APPLICATIONS

In this section we first state some basic properties derived from the notion of $A-$maximal $(m)-$relaxed accretiveness [8], and then we investigate some results involving $A-$maximal $(m)-$relaxed accretiveness and the firm nonexpansiveness. Let X denote a real Banach space with the norm $\|\cdot\|$ on X and X^*, the dual space of X, and with the duality pairing between the elements of X and X^*. Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, which means, the set $\{(x,y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x,y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x,y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$dom(M) = \{x \in X : \exists y \in X : (x,y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

dom(T)=X, shall denote the full domain of M, and the range of M is defined by

$$range(M) = \{y \in X : \exists x \in X : (x,y) \in M\}.$$

The inverse M^{-1} of M is $\{(y,x) : (x,y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{x, \rho y) : (x,y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y+z) : (x,y) \in L, (x,z) \in M\}.$$

Next, we define the generalized duality mapping $J_q : X \to 2^{X^*}$ by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\| \cdot \|f^*\|, \|f^*\| = \|x\|^{q-1}\} \, \forall \, x \in X,$$

where $q > 1$.

For $q = 2$, J_q becomes the usual normalized duality mapping. It clearly follows that $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$, and J_q is single-valued if X^* is strictly convex. The modulus of smoothness function $\rho_X : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_X(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

In this context, a Banach space is uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0,$$

and X is $q-$uniformly smooth if there is a positive constant c such that

$$\rho_X(t) \leq ct^q, \, for \; q > 1.$$

Now we state the following lemma by Xu [23] for $q-$uniformly smooth Banach spaces, which is crucial to accomplishing the proofs.

Lemma 2.1. [23] *If X is a real uniformly smooth Banach space. Then X q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 2.1. Let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) Accretive if

$$\langle u^* - v^*, J_q(u-v)\rangle \geq 0\, \forall (u, u^*), (v, v^*) \in Graph(M).$$

(ii) $(r)-$ strongly accretive if there exists a positive constant r such that

$$\langle u^* - v^*, J_q(u-v)\rangle \geq r\|u-v\|^q\, \forall (u, u^*), (v, v^*) \in Graph(M).$$

(iii) $(r)-$strongly pseudoaccretive if

$$\langle v^*, J_q(u-v)\rangle \geq 0$$

implies

$$\langle u^*, J_q(u-v)\rangle \geq r\|u-v\|^q\, \forall (u, u^*), (v, v^*) \in Graph(M).$$

(iv) $(m)-$relaxed accretive if there exists a positive constant m such that

$$\langle u^* - v^*, J_q(u-v)\rangle \geq (-m)\|u-v\|^q\, \forall (u, u^*), (v, v^*) \in Graph(M).$$

(v) $(c)-$ cocoercive if there is a positive constant c such that

$$\langle u^* - v^*, J_q(u-v)\rangle \geq c\|u^* - v^*\|^q\, \forall (u, u^*), (v, v^*) \in Graph(M).$$

Definition 2.2. Let $M : X \to 2^X$ be a mapping on X. The map M is said to be:

(i) Nonexpansive if

$$\|u^* - v^*\| \leq \|u-v\|\, \forall (u, u^*), (v, v^*) \in Graph(M).$$

(ii) Firmly $q-$nonexpansive if

$$\|u^* - v^*\|^q \leq \langle u^* - v^*, J_q(u-v)\rangle\, \forall (u, u^*), (v, v^*) \in Graph(M).$$

(iii) $(c)-$firmly $q-$nonexpansive if there exists a constant $c > 0$ such that

$$\|u^* - v^*\|^q \leq c\langle u^* - v^*, J_q(u - v)\rangle \, \forall (u, u^*), (v, v^*) \in Graph(M).$$

Definition 2.3. [8] Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be $A-$maximal $(m)-$relaxed accretive if

(i) M is $(m)-$relaxed accretive

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Proposition 2.1. *Let $A : X \to X$ be an $(r)-$strongly accretive single-valued mapping and let $M : X \to 2^X$ be an $A-$maximal $(m)-$relaxed accretive mapping. Then $(A + \rho M)$ is maximal accretive for $0 < \rho < \frac{r}{m}$.*

Proof. Since A is $(r)-$strongly accretive and M is $A-$maximal $(m)-$relaxed accretive, it implies that $A + \rho M$ is $(r - \rho m)-$strongly accretive. This in turn implies that $A + \rho M$ is pseudoaccretive, and hence $A + \rho M$ is maximal accretive under the given conditions.

Proposition 2.2. *Let $A : X \to X$ be an $(r)-$strongly accretive mapping and let $M : X \to 2^X$ be an $A-$maximal $(m)-$relaxed accretive mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.*

Definition 2.4. Let $A : X \to X$ be an $(r)-$strongly accretive mapping and let $M : X \to 2^X$ be an $A-$maximal $(m)-$relaxed accretive mapping. Then the generalized resolvent operator $J^M_{\rho,A} : X \to X$ is defined by

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Definition 2.5. Let $A, T : X \to X$ be two mappings. Then map T is said to be:

(i) accretive with respect to A if

$$\langle T(x) - T(y), J_q(A(x) - A(y))\rangle \geq 0 \, \forall (x, y) \in X.$$

(ii) $(r) - strongly$ accretive with respect to A if there exists a positive constant r such that

$$\langle T(x) - T(y), J_q(A(x) - A(y))\rangle \geq r\|x - y\|^q \, \forall (x, y) \in X.$$

(iii) (γ, α)-relaxed $q-$cocoercive with respect to A if there exist positive constants γ and α such that

$$\langle T(x) - T(y), J_q(A(x) - A(y))\rangle \geq -\gamma\|T(x) - T(y)\|^q + \alpha\|x - y\|^q$$

$$\forall (x, y) \in X.$$

3. $A-$Proximal point algorithm and application

This section primarily deals with the relaxed $A-$proximal point algorithm and its application to approximation solvability of the inclusion problem (1). Several results connecting the generalized $A-$maximal $(m)-$relaxed accretiveness and corresponding resolvent operator are established, which unify the results on the firm expansiveness from Eckstein and Bertsekas [3]. Furthermore, some auxiliary results on $A-$maximal $(m)-$relaxed accretiveness, and maximal accretiveness are discussed.

The solvability of the problem (1) depends on the equivalence between (1) and the problem of finding the fixed point of the associated generalized resolvent operator.

Lemma 3.1. *Let X be a real Banach space, let $A : X \to X$ be $(r)-$strongly accretive, and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed accretive. Then the generalized resolvent operator associated with M and defined by*

$$J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u)\,\forall\, u \in X,$$

is $(\frac{1}{r-\rho m})-$Lipschitz continuous $r - \rho m > 0$.

Lemma 3.2. [22] *Let X be a real Banach space, let $A : X \to X$ be $(r)-$strongly accretive, and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed accretive. Then the generalized resolvent operator associated with M and defined by*

$$J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u)\,\forall\, u \in X,$$

is $(\frac{1}{r-\rho m})-$firmly nonexpansive for $r - \rho m > 0$.

Lemma 3.3. *Let X be a real Banach space, let $A : X \to X$ be $(r)-$strongly accretive, and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed accretive. Then $I - J_{\rho,A}^M$ is firmly nonexpansive for $r - \rho m > 1$.*

Theorem 3.1. *Let X be a real Banach space, let $A : X \to X$ be $(r)-$strongly accretive, and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed accretive. Then the following statements are mutually equivalent:*

(i) *An element $u \in X$ is a solution to (1).*

(ii) *For an $u \in X$, we have*

$$u = J_{\rho,A}^M(A(u)).$$

where

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Proof. It follows from the definition of the generalized resolvent operator corresponding to M.

Theorem 3.2. *Let X be a real $q-$uniformly smooth Banach space, let $A : X \to X$ be $(r)-$strongly accretive and $(s)-$Lipschitz continuous, and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed accretive. For an arbitrarily chosen element x^0, let the sequence $\{x^k\}$ be generated by the relaxed $A-$proximal point algorithm*

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \, for \, k \ge 0$$

with

$$\|y^k - J^M_{\rho,A}(A(x^k))\| \le \delta_k \|y^k - x^k\|,$$

where $\delta_k \to 0$,

$$y^{k+1} = (1 - \alpha_k)x^k + \alpha_k J^M_{\rho,A}(A(x^k)),$$

$$\langle J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*)), J_q(x^k - x^*)\rangle \ge \gamma \|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^q, \tag{2}$$

for $\gamma > 0$, $J^M_{\rho,A} = (A + \rho_k M)^{-1}$, and sequences $\{\delta_k\}$, $\{\alpha_k\}$ and $\{\rho_k\}$ satisfy $\alpha_k > 1$, $\sum_{k=0}^{\infty} \delta_k \le \infty$, and $\rho_k \uparrow \rho$.

Then the sequence $\{x^k\}$ converges linearly to a unique solution x^* of (1) with rate

$$\sqrt[q]{(1 - \alpha)^q + [\alpha^q c_q + q\alpha(1 - \alpha)\gamma]\frac{s^q}{(r - \rho m)^q}} < 1,$$

where $\alpha_k^q c_q + q\alpha_k(1 - \alpha_k)\gamma > 0$ and $\alpha = \limsup_{k\to\infty} \alpha_k$.

Proof. Since from Theorem 3.1, x^*, a solution to (1), satisfies the relaxed proximal point algorithm. It further follows from Theorem 3.1 that any solution to (1) is a fixed point of $J^M_{\rho_k,A} oA$ for all $k \ge 0$.

Next, using Lemma 2.1 and (2), we find the estimate

$$\|y^{k+1} - x^*\|^q = \|(1 - \alpha_k)x^k + \alpha_k J^M_{\rho_k,A}(A(x^k)) -$$
$$[(1 - \alpha_k)x^* + \alpha_k J^M_{\rho_k,A}(A(x^*))\|^q$$
$$= \|(1 - \alpha_k)(x^k - x^*) + \alpha_k(J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*)))\|^q$$
$$\le (1-\alpha_k)^q\|x^k - x^*\|^q + q\alpha_k(1-\alpha_k)\langle J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*)), J_q(x^k - x^*)\rangle +$$
$$\alpha_k^q c_q \|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^q$$

$$\leq (1-\alpha_k)^q\|x^k - x^*\|^q + q\alpha_k(1-\alpha_k)\gamma\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^q +$$
$$\alpha_k^q c_q\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^q$$
$$= (1-\alpha_k)^q\|x^k - x^*\|^q + [\alpha_k^q c_q + q\alpha_k(1-\alpha_k)\gamma]\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^q$$
$$\leq (1-\alpha_k)^q\|x^k - x^*\|^q + [\alpha_k^q c_q + q\alpha_k(1-\alpha_k)\gamma]\frac{s^q}{(r-\rho_k m)^q}\|x^k - x^*\|^q$$
$$= \{(1-\alpha_k)^q + [\alpha_k^q c_q + q\alpha_k(1-\alpha_k)\gamma]\frac{s^q}{(r-\rho_k m)^q}\}\|x^k - x^*\|^q,$$

where $\alpha_k^q c_q + q\alpha_k(1-\alpha_k)\gamma > 0$.

Therefore,

$$\|y^{k+1} - x^*\| \leq \theta_k\|x^k - x^*\|$$

and

$$\theta_k = \sqrt[q]{(1-\alpha_k)^q + [\alpha_k^q c_q + q\alpha_k(1-\alpha_k)\gamma]\frac{s^q}{(r-\rho_k m)^q}}.$$

Clearly, it follows that

$$\begin{aligned}
&\|x^{k+1} - y^{k+1}\| \\
= \;& \|(1-\alpha_k)x^k + \alpha_k y^k - [(1-\alpha_k)x^k + \alpha_k J^M_{\rho,A}(A(x^k))]\| \\
= \;& \|\alpha_k(y^k - J^M_{\rho,A}(A(x^k)))\| \\
\leq \;& \alpha_k\delta_k\|y^k - x^k\|.
\end{aligned}$$

Since

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^k,$$

it implies that

$$\alpha_k(y^k - x^k) = x^{k+1} - x^k.$$

Now we estimate

$$\begin{aligned}
&\|x^{k+1} - x^*\| = \|y^{k+1} - x^* + x^{k+1} - y^{k+1}\| \\
\leq \;& \|y^{k+1} - x^*\| + \|x^{k+1} - y^{k+1}\| \\
\leq \;& \|y^{k+1} - x^*\| + \alpha_k\delta_k\|y^k - x^k\| \\
= \;& \|y^{k+1} - x^*\| + \delta_k\|x^{k+1} - x^k\| \\
\leq \;& \theta_k\|x^k - x^*\| + \delta_k\|x^{k+1} - x^*\| + \delta_k\|x^k - x^*\|. \qquad (3)
\end{aligned}$$

Therefore, we have

$$\|x^{k+1} - x^*\| \leq \frac{\theta_k + \delta_k}{1-\delta_k}\|x^k - x^*\|, \qquad (4)$$

where

$$\begin{aligned} & \limsup \frac{\theta_k + \delta_k}{1 - \delta_k} = \limsup \theta_k \\ = & \sqrt[q]{(1-\alpha)^q + [\alpha^q c_q + q\alpha(1-\alpha)\gamma]\frac{s^q}{(r-\rho m)^q}} < 1. \end{aligned}$$

Finally, to show the uniqueness of the solution, assume that x_1^* and x_2^* are two distinct solutions of (1). By Theorem 3.1, we have

$$x_1^* = J^M_{\rho_k,A}(A(x_1^*)),$$

and

$$x_2^* = J^M_{\rho_k,A}(A(x_2^*)).$$

Since $J^M_{\rho_k,A}$ is $(\frac{1}{r-\rho_k m})-$ Lipschitz continuous and A is $(s)-$ Lipschitz continuous, we arrive at

$$\begin{aligned} & \|x_1^* - x_2^*\| = \|J^M_{\rho_k,A}(A(x_1^*)) - J^M_{\rho_k,A}(A(x_2^*))\| \\ \le & \frac{1}{r-\rho_k m}\|A(x_1^*) - A(x_2^*)\| \\ \le & \frac{s}{r-\rho_k m}\|x_1^* - x_2^*\|. \end{aligned}$$

Therefore, we find

$$\|x_1^* - x_2^*\| \le \frac{s}{r-\rho_k m}(\|x^k - x_1^*\| + \|x^k - x_2^*\|).$$

It follows from this that

$$\|x_1^* - x_2^*\| = 0. \qquad \square$$

4. Some applications

In this section, based on results from Sections 2 and 3, we derive a special case of Theorem 3.2 for the maximal accretive mapping.

Theorem 4.1. *Let X be a real $q-$uniformly smooth Banach space, and let $M : X \to 2^X$ be maximal accretive. For an arbitrarily chosen element x^0, let the sequence $\{x^k\}$ be generated by the relaxed proximal point algorithm*

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^k) \, for \, k \ge 0$$

with

$$\|y^k - J^M_\rho(x^k)\| \le \delta_k \|y^k - x^k\|,$$

where $\delta_k \to 0$,

$$y^{k+1} = (1-\alpha_k)x^k + \alpha_k J_{\rho}^{M}(x^k),$$

$J_{\rho}^{M} = (I+\rho_k M)^{-1}$, *and sequences* $\{\delta_k\}$, $\{\alpha_k\}$ *and* $\{\rho_k\}$ *satisfy* $\alpha_k > 1$, $\sum_{k=0}^{\infty} \delta_k \leq \infty$, *and* $\rho_k \uparrow \rho$.

Then the sequence $\{x^k\}$ *converges linearly to a unique solution* x^* *of (1) with rate*

$$\sqrt{1-\alpha^*[2-\alpha^*]} < 1 \, for \, \alpha^* < 2,$$

where

$$\alpha^* = \lim \sup_{k\to\infty} \alpha_k.$$

References

[1] Y.J. Cho and H.Y. Lan, *A new class of generalized nonlinear multi-valued quasi-variational-like inclusions with $H-$ monotone mappings*, Math. Ineq. & Appl., (in press).

[2] J. Eckstein, *Splitting methods for monotone operators with applications to parallel optimization*, Doctoral dissertation, Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, MA, 1989.

[3] J. Eckstein and D.P. Bertsekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Math. Programming, **55**(1992), 293-318.

[4] Y.P. Fang and N.J. Huang, *$H-$accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces*, Applied Math. Letters, **17**(2004), 647-653.

[5] Y.P. Fang and N.J. Huang, *$H-$monotone operators and system of variational inclusions*, Communications on Applied Nonlinear Analysis, **11**(1)(2004), 93-101.

[6] H.Y. Lan, *A class of nonlinear $(A, \eta)-$ monotone operator inclusion problems with relaxed cocoercive mappings*, Advances in Nonlinear Variational Inequalities, **9**(2)(2006), 1-11.

[7] H.Y. Lan, Y.J. Cho and R.U. Verma, *Nonlinear relaxed cocoercive variational inclusions involving $(A, \eta)-$accretive mappings in Banach spaces*, Computers & Mathematics with Applications, **51**(2006), 1529-1538.

[8] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev. Francaise Inform. Rech. Oper. Ser. R-3, **4**(1970), 154-158.

[9] T. Pennanen, *Local convergence of the proximal point algorithm and multiplier methods without monotonicity*, Mathematics of Operations Research, **27**(2002), no.1, 170-191.

[10] S.M. Robinson, *Composition duality and maximal monotonicity*, Math. Programming, **85**(1990), 1-13.

[11] S.M. Robinson, *Linear convergence of epsilon-subgradient descent methods for a class of convex functions*, Mathematical Programming, **86**(1990), 41-50.

[12] R.T. Rockafellar, *On the maximal monotonicity of subdifferential mappings,* Pacific J. Math., **33**(1970), 209-216.

[13] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM Journal of Control and Optimization, **14**(1976), 877-898.

[14] R.T. Rockafellar, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming*, Mathematics of Operations Research, **1**(1976), 97-116.

[15] R.T. Rockafellar and R.J-B. Wets, *Variational Analysis,* Springer-Verlag, Berlin, 1998.

[16] M.V. Solodov and B.F. Svaiter, *An inexact hybrid generalized proximal point algorithm and some new results on the theory of Bregman functions*, Mathematics of Operations Research, **25**(2)(2000), 214-230.

[17] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM Journal of Control and Optimization, **38**(2000), 431-446.

[18] R.U. Verma, *$A-$monotonicity and its role in nonlinear variational inclusions*, J. Optim. Theor. Appl., **129**(2006), no.3, 457-467.

[19] R.U. Verma, *Nonlinear A- monotone mixed variational inclusion problems based on resolvent operator techniques*, Mathematical Sciences Research Journal, **9**(2005), no.10, 255-267.

[20] R.U. Verma, *Approximation solvability of a class of set-valued variational inclusions involving $(A, \eta)-$ monotone mappings*, J. Math. Anal. Appl., **337**(2008), 969-975.

[21] R.U. Verma, *Rockafellar's theorem on linear convergence and $A-$proximal point algorithm*, DCDIS (to appear).

[22] H.K. Xu, *Inequalities in Banach spaces with applications,* Nonlinear Anal., **16**(1991), 1127-1138.

[23] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, II/B, Springer-Verlag, New York, New York, 1990.

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