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SOME RESULTS ON THE CONTINUOUS DEPENDENCE OF THE FIXED POINTS IN NORMED LINEAR SPACE

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Abstract. Recently, in Berinde [3, 4], it has been shown that apart from the Picard iteration process, the continuous dependence of the fixed points has not been studied so far for other fixed point iteration procedures. In this paper, we intend to provide some answers to this challenge by investigating the continuous dependence of the fixed points in normed linear space for both Schaefer and Mann iteration processes using a $(\varphi,\ \psi)-$contractive condition. Our results are new extensions of some of the results of Berinde [3, 4, 5], Rus [14, 16] and Zeidler [22].

Key Words and Phrases: Picard iteration process, continuous dependence of the fixed points, $(\varphi,\ \psi)-$contractive condition, normed linear space.
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1. Introduction

Let (E, d) be a complete metric space and $T : E \to E$ a selfmap of E. Suppose that $F_T = \{\ p \in E \mid Tp = p\ \}$ is the set of fixed points of T.

There are several iteration processes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iteration process $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, \cdots , \tag{1.1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality relation

$$d(Tx, Ty) \leq \alpha d(x, y),\ \text{for all } x,\ y \in E \text{ andforsome } \alpha \in [0, 1). \tag{1.2}$$

Condition (1.2) is called the *Banach's contraction condition.* Any operator satisfying (1.2) is called *contraction.* Also, condition (1.2) is significant in the celebrated Banach's fixed point theorem [1].

In the normed linear space setting, we shall state some of the iteration processes generalizing (1.1) as follows:

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha)x_n + \alpha T x_n,\ \alpha \in [0,1],\ n = 0, 1, \cdots, \tag{1.3}$$

is called the Schaefer's iteration process (see Schaefer [20]). For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n = 0, 1, \cdots, \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$, is called the Mann iteration process (see Mann [11]).

Recently, in Berinde [3, 4], it has been shown that apart from the Picard iteration process, the concept of the continuous dependence of the fixed points of operators has not been investigated so far for other fixed point iteration processes.

In this paper, we intend to provide some answers to this challenge by investigating the continuous dependence of the fixed points in normed linear space for both Schaefer and Mann iteration processes using a general contractive condition. Our results are new extensions of some of the results of Berinde [3, 4, 5], Rus [14, 16] and Zeidler [22].

2. Preliminaries

In Rus [14, 16] and also Berinde [3, 4], the continuous dependence of the fixed points on a parameter has been formulated in the following general context in a metric space.

Let $(E,\ d)$ be a complete metric space, (Y, τ) a topological space and $S_\lambda : E \times Y \to E$ a family of operators depending on the parameter $\lambda \in Y$. Suppose that $S_\lambda := S(.,\ \lambda),\ \lambda \in Y$, has a unique fixed point x_λ^*, for any $\lambda \in Y$.

Define the operator $U : Y \to E$ by

$$U(\lambda) = x_\lambda^*,\ \forall\ \lambda \in Y.$$

We are interested in finding sufficient conditions on T_λ that guarantee the continuity of U. Since metric is induced by norm, we have that $d(x,y) = ||x-y||, \ \forall\ x,\ y \in E$, for the normed linear space setting.

In many applications, the operator T in the Picard iteration of (1.1) depends on an additional parameter $\lambda \in Y$, where Y is a parameter space. Therefore, (1.1) is replaced by the equation

$$x_\lambda = T_\lambda\ x_\lambda,\ x_\lambda \in E,\ \lambda \in Y, \tag{2.1}$$

where $T = S_\lambda$.

We shall require the following definition in the sequel:

Definition 2.1. (a) A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if it satisfies the following conditions:

(i) ψ is monotone increasing;

(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0,\ \forall\ t \geq 0.$

(b) A comparison function satisfying $t - \psi(t) \to \infty$ as t $\to \infty$ is called a *strict comparison function.*

See Berinde [2, 3, 4], Rus [14] and Rus et al [19] for the definition and examples of comparison function.

Remark 2.2. Every comparison function satisfies $\psi(0) = 0$.

Condition (1.2) was employed in Zeidler [22] to prove a result on the stability of the fixed points (that is, continuous dependence of the fixed points on a parameter) for the Picard iteration. However, in Rus [14], the following contractive condition was used: For a continuous mapping $S_\lambda : E \times Y \to E$, there exists a strict comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, $\forall\ x,\ y \in E$,

$$d(S_\lambda x, S_\lambda y) \leq \psi(d(x,y)),\ \forall\ x,\ y \in E,\ \lambda \in Y. \tag{2.2}$$

We shall establish some new results on the continuous dependence of the fixed points in normed linear space for the Schaefer and Mann iteration processes defined in (1.3) and (1.4) respectively using the following contractive condition: For a continuous mapping $S_\lambda : E \times Y \to E$, there exist a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and a strict comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, $\forall\ x,\ y \in E$,

$$||S_\lambda x - S_\lambda y|| \leq \varphi(||x - S_\lambda x||) + \psi(||x-y||),\ \lambda \in Y. \tag{2.3}$$

For further study of the continuous dependence of the fixed points and various contractive definitions, we refer to Berinde [2, 3], Rhoades [13], Rus [14], Rus et al [19], Rus and Muresan [17], [18], as well as, some other references listed in the reference section of this paper.

3. The main results

Our main results are the following.

Theorem 3.1. *Let* $(E, \ ||.||)$ *be a normed linear space and* (Y, τ) *a topological space. Let* $S : E \times Y \to E$ *be a continuous mapping satisfying (2.3) and suppose that* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a monotone increasing function such that* $\varphi(0) = 0$, and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *a strict comparison function. Let* x^*_λ *be the unique fixed point of* S_λ (where $\mathrm{S}_\lambda \mathrm{x} = \mathrm{S}(\mathrm{x}, \lambda), \ \mathrm{x} \in \mathrm{E}, \ \lambda \in \mathrm{Y}$). *Suppose* $\{x_n\}_{n=0}^{\infty}$ *is the Schaefer's iteration process defined by (1.3), with* $\alpha \in [0, 1]$. *Then, the mapping* $U : Y \to E$, *given by* $U(\lambda) = x^*_\lambda, \ \lambda \in Y$, *is continuous.*

Proof. Let $\lambda_1, \ \lambda_2 \in Y$. Then,

$$\begin{aligned}
||x^*_{\lambda_1} - x^*_{\lambda_2}|| &= ||(1-\alpha)x^*_{\lambda_1} + \alpha S(x^*_{\lambda_1}, \lambda_1) - (1-\alpha)x^*_{\lambda_2} - \alpha S(x^*_{\lambda_2}, \lambda_2)|| \\
&= ||(1-\alpha)(x^*_{\lambda_1} - x^*_{\lambda_2}) + \alpha(S(x^*_{\lambda_1}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_2))|| \\
&\leq (1-\alpha)||x^*_{\lambda_1} - x^*_{\lambda_2}|| + \alpha||S(x^*_{\lambda_1}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_2)||
\end{aligned}$$

$$\leq (1-\alpha)||x^*_{\lambda_1} - x^*_{\lambda_2}|| + \alpha[||S(x^*_{\lambda_1}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_1)|| + ||S(x^*_{\lambda_2}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_2)||]$$

$$\begin{aligned}
&\leq (1-\alpha)||x^*_{\lambda_1} - x^*_{\lambda_2}|| + \alpha[\varphi(||x^*_{\lambda_1} - S(x^*_{\lambda_1}, \lambda_1)||) + \psi(||x^*_{\lambda_1} - x^*_{\lambda_2}||) \\
&\quad + ||S(x^*_{\lambda_2}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_2)||] \\
&= (1-\alpha)||x^*_{\lambda_1} - x^*_{\lambda_2}|| + \alpha[\varphi(||x^*_{\lambda_1} - S_{\lambda_1}x^*_{\lambda_1}||) + \psi(||x^*_{\lambda_1} - x^*_{\lambda_2}||) \\
&\quad + ||S_{\lambda_1}x^*_{\lambda_2} - S_{\lambda_2}x^*_{\lambda_2}||].
\end{aligned}$$

Therefore,

$$||x^*_{\lambda_1} - x^*_{\lambda_2}|| - \psi(||x^*_{\lambda_1} - x^*_{\lambda_2}||) \leq ||S_{\lambda_1}x^*_{\lambda_2} - S_{\lambda_2}x^*_{\lambda_2}||. \tag{3.1}$$

Since S is continuous and ψ is a strict comparison function, we have from (3.1) that $||S_{\lambda_1}x^*_{\lambda_2} - S_{\lambda_2}x^*_{\lambda_2}|| \to 0$ as $\lambda_2 \to \lambda_1$, leading to $||x^*_{\lambda_1} - x^*_{\lambda_2}|| \to 0$ as $\lambda_2 \to \lambda_1$. That is, $||U(\lambda_1) - U(\lambda_2)|| \to 0$ as $\lambda_2 \to \lambda_1$. □

Theorem 3.2. *Let* $(E, \ ||.||)$ *be a normed linear space and* (Y, τ) *a topological space. Let* $S : E \times Y \to E$ *be a continuous mapping satisfying (2.3) and suppose that* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a monotone increasing function such that* $\varphi(0) = 0$, and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *a strict comparison function. Let* x^*_λ *be the* *unique fixed point of* S_λ (where $\mathrm{S}_\lambda \mathrm{x} = \mathrm{S}(\mathrm{x}, \lambda)$), $\mathrm{x} \in \mathrm{E}$, $\lambda \in \mathrm{Y}$). *Suppose* $\{x_n\}_{n=0}^{\infty}$ *is the Mann iteration process defined by (1.4), where* $\alpha_n \in [0,1]$. *Then, the mapping* $U : Y \to E$*, given by* $U(\lambda) = x_\lambda^*,\ \lambda \in Y$*, is continuous.*

Proof. We shall employ condition (2.3) to prove this theorem. Let $\lambda_1,\ \lambda_2 \in Y$ and $\alpha_{\lambda_1} > 0$. Suppose that $\lim_{\lambda_2 \to \lambda_1} ||x_{\lambda_2}^*|| + ||S_{\lambda_2} x_{\lambda_2}^*||$ exists. Then,

$$
\begin{aligned}
||x_{\lambda_1}^* - x_{\lambda_2}^*|| &= ||[(1-\alpha_{\lambda_1})x_{\lambda_1}^* - (1-\alpha_{\lambda_2})x_{\lambda_2}^*] + [\alpha_{\lambda_1} S(x_{\lambda_1}^*, \lambda_1) - \alpha_{\lambda_2} S(x_{\lambda_2}^*, \lambda_2)]|| \\
&\leq ||(1-\alpha_{\lambda_1})x_{\lambda_1}^* - (1-\alpha_{\lambda_2})x_{\lambda_2}^*|| + ||\alpha_{\lambda_1} S(x_{\lambda_1}^*, \lambda_1) - \alpha_{\lambda_2} S(x_{\lambda_2}^*, \lambda_2)|| \\
&\leq (1-\alpha_{\lambda_1})||x_{\lambda_1}^* - x_{\lambda_2}^*|| + |\alpha_{\lambda_1} - \alpha_{\lambda_2}|\ ||x_{\lambda_2}^*|| + \alpha_{\lambda_1} ||S(x_{\lambda_1}^*, \lambda_1) - S(x_{\lambda_2}^*, \lambda_2)|| \\
&\quad + |\alpha_{\lambda_1} - \alpha_{\lambda_2}|\ ||S(x_{\lambda_2}^*, \lambda_2)|| \\
&= (1-\alpha_{\lambda_1})||x_{\lambda_1}^* - x_{\lambda_2}^*|| + |\alpha_{\lambda_1} - \alpha_{\lambda_2}| \left[||x_{\lambda_2}^*|| + ||S(x_{\lambda_2}^*, \lambda_2)|| \right] \\
&\quad + \alpha_{\lambda_1} ||S(x_{\lambda_1}^*, \lambda_1) - S(x_{\lambda_2}^*, \lambda_2)|| \\
&\leq (1-\alpha_{\lambda_1})||x_{\lambda_1}^* - x_{\lambda_2}^*|| + |\alpha_{\lambda_1} - \alpha_{\lambda_2}| \left[||x_{\lambda_2}^*|| + ||S(x_{\lambda_2}^*, \lambda_2)|| \right] \\
&\quad + \alpha_{\lambda_1} \left[||S(x_{\lambda_1}^*, \lambda_1) - S(x_{\lambda_2}^*, \lambda_1)|| + ||S(x_{\lambda_2}^*, \lambda_1) - S(x_{\lambda_2}^*, \lambda_2)|| \right] \\
&\leq (1-\alpha_{\lambda_1})||x_{\lambda_1}^* - x_{\lambda_2}^*|| + |\alpha_{\lambda_1} - \alpha_{\lambda_2}| \left[||x_{\lambda_2}^*|| + ||S(x_{\lambda_2}^*, \lambda_2)|| \right] \\
&\quad + \alpha_{\lambda_1} \left[\varphi(||x_{\lambda_1}^* - S(x_{\lambda_1}^*, \lambda_1)||) + \psi(||x_{\lambda_1}^* - x_{\lambda_2}^*||) \right] \\
&\quad + \alpha_{\lambda_1} ||S(x_{\lambda_2}^*, \lambda_1) - S(x_{\lambda_2}^*, \lambda_2)|| \\
&= (1-\alpha_{\lambda_1})||x_{\lambda_1}^* - x_{\lambda_2}^*|| + \alpha_{\lambda_1} \varphi(||x_{\lambda_1}^* - S_{\lambda_1} x_{\lambda_1}^*||) + \alpha_{\lambda_1} \psi(||x_{\lambda_1}^* - x_{\lambda_2}^*||) \\
&\quad + |\alpha_{\lambda_1} - \alpha_{\lambda_2}| \left[||x_{\lambda_2}^*|| + ||S_{\lambda_2} x_{\lambda_2}^*|| \right] + \alpha_{\lambda_1} ||S_{\lambda_1} x_{\lambda_2}^* - S_{\lambda_2} x_{\lambda_2}^*|| \\
&= (1-\alpha_{\lambda_1})||x_{\lambda_1}^* - x_{\lambda_2}^*|| + \alpha_{\lambda_1} \psi(||x_{\lambda_1}^* - x_{\lambda_2}^*||) \\
&\quad + |\alpha_{\lambda_1} - \alpha_{\lambda_2}| \left[||x_{\lambda_2}^*|| + ||S_{\lambda_2} x_{\lambda_2}^*|| \right] + \alpha_{\lambda_1} ||S_{\lambda_1} x_{\lambda_2}^* - S_{\lambda_2} x_{\lambda_2}^*||,
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
||x_{\lambda_1}^* - x_{\lambda_2}^*|| - \psi(||x_{\lambda_1}^* - x_{\lambda_2}^*||) \leq \frac{|\alpha_{\lambda_1} - \alpha_{\lambda_2}|}{\alpha_{\lambda_1}} [||x_{\lambda_2}^*|| \\
+ ||S_{\lambda_2} x_{\lambda_2}^*||] + ||S_{\lambda_1} x_{\lambda_2}^* - S_{\lambda_2} x_{\lambda_2}^*||.
\end{aligned}
$$

Since S is continuous and ψ is a strict comparison function, we have

$$||S_{\lambda_1} x_{\lambda_2}^* - S_{\lambda_2} x_{\lambda_2}^*|| \to 0 \text{ as } \lambda_2 \to \lambda_1$$

and

$$|\alpha_{\lambda_1} - \alpha_{\lambda_2}| \to 0 \text{ as } \lambda_2 \to \lambda_1,$$

leading to $||x_{\lambda_1}^* - x_{\lambda_2}^*|| \to 0$ as $\lambda_2 \to \lambda_1$. That is, $||U(\lambda_1) - U(\lambda_2)|| \to 0$ as $\lambda_2 \to \lambda_1$. Hence, the mapping $U : Y \to E$ is continuous. $\square$

Remark 3.3. Other extensions of Proposition 1.2 of Zeidler [22], Theorem 7.7 of Berinde [3, 4] or Theorem 7.1.2 of Rus [14] from complete metric space to normed linear space are also obtained if condition (2.3) is replaced by the following:

For a continuous mapping $S_\lambda : E \times Y \to E$, there exist a constant $L \geq 0$ and a strict comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, $\forall\ x,\ y \in E$,

$$||S_\lambda x - S_\lambda y|| \leq L||x - S_\lambda x|| + \psi(||x - y||),\ \lambda \in Y. \tag{3.2}$$

Thus, we state the following results without proofs:

Theorem 3.4. *Let* $(E,\ ||.||)$ *be a normed linear space and* (Y, τ) *a topological space. Let* $S : E \times Y \to E$ *be a continuous mapping satisfying (3.2) and suppose that* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *a strict comparison function. Let* x^*_λ *be the unique fixed point of* S_λ (where $\mathrm{S_\lambda x = S(x, \lambda),\ x \in E,\ \lambda \in Y}$). *Suppose* $\{x_n\}_{n=0}^\infty$ *is the Schaefer's iteration process defined by (1.3), with* $\alpha \in [0, 1]$. *Then, the mapping* $U : Y \to E$, *given by* $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, *is continuous.*

Theorem 3.5. *Let* $(E,\ ||.||)$ *be a normed linear space and* (Y, τ) *a topological space. Let* $S : E \times Y \to E$ *be a continuous mapping satisfying (3.2) and suppose that* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *a strict comparison function. Let* x^*_λ *be the unique fixed point of* S_λ (where $\mathrm{S_\lambda x = S(x, \lambda),\ x \in E,\ \lambda \in Y}$). *Suppose* $\{x_n\}_{n=0}^\infty$ *is the Mannn iteration process defined by (1.4), with* $\alpha_n \in [0, 1]$. *Then, the mapping* $U : Y \to E$, *given by* $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, *is continuous.*

Remark 3.6. Theorem 3.1, Theorem 3.2, Theorem 3.4 and Theorem 3.5 are generalizations, extensions and improvements of Proposition 1.2 of Zeidler [22]. Our results also extend and generalize Theorem 7.7 of Berinde [3, 4] (which is Theorem 7.1.2 of Rus [14]) from complete metric space to normed linear space.

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