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AN APPLICATION OF A FIXED POINT THEOREM TO A FUNCTIONAL INEQUALITY

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Abstract. We investigate the functional inequality

$$\|f\big(\frac{x-y}{2}+z\big)+f\big(\frac{y-z}{2}+x\big)+f\big(\frac{z-x}{2}+y\big)\| \le \|f(x+y+z)\|$$

and use a fixed point method to prove its stability in the setting of Banach modules over a C^*-algebra.

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1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [24] in 1940 and affirmatively solved by Hyers [11] in the next year. In 1951, Bourgin [3] treated the same problem. The result of Hyers was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [22] for linear mappings by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be bounded by $\varepsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of Th.M. Rassias' theorem was obtained by Găvruta [8], who replaced the bound $\varepsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$. Since then the stability problems of various functional equations and mappings and their pexiderized versions with more general domains and ranges have been investigated by a number of authors (see [6, 9, 12, 13, 23]).

Gilányi [10] and Fechner [7] proved the stability of the the functional inequality $\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\|$ and its stability in Banach spaces. Cho and Kim [4] studied the functional inequalities

$$\left\|f\left(\frac{x-y}{2}-z\right)+f(y)+2f(z)\right\| \le \left\|f\left(\frac{x+y}{2}+z\right)\right\|+\varphi(x,y,z)$$

and

$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\|+\varphi(x,y,z)\,.$$

In addition, Lee, Park and Shin [16] investigated the functional inequality $\|af(x)+bf(y)+cf(z)\| \le \|f(\alpha x+\beta y+\gamma z)\|$, where $a,b,c,\alpha,\beta,\gamma$ are nonzero complex numbers (see also [20]).

Let E be a set. A function $d: E\times E \to [0,\infty]$ is called a *generalized metric* on E if d satisfies

(i) $d(x,y)=0$ if and only if $x=y$;
(ii) $d(x,y)=d(y,x)$ for all $x,y\in E$;
(iii) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.1. [17] *Let (E,d) be a complete generalized metric space and let $J: E\to E$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in E$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a non-negative integer n_0 such that

(1) *$d(J^n x, J^{n+1}x)<\infty$ for all $n\ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $\mathscr{Y} = \{\, y \in E : d(J^{n_0}x, y) < \infty \,\}$;*

(4) *$d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in \mathscr{Y}$.*

Throughout this paper, let A be a unital C^*-algebra with unitary group $U(A)$, unit e and norm $|\cdot|$. Assume that $\mathscr{X}$ and $\mathscr{Y}$ are left Banach A-modules and $\mathscr{Y}$ complete. An additive mapping $T : X \to \mathscr{Y}$ is called *A-linear* if $T(ax) = aT(x)$ for all $a \in A$ and all $x \in X$.

In this paper, we investigate the functional inequality

$$\big\|f\big(\frac{x-y}{2} + z\big) + f\big(\frac{y-z}{2} + x\big) + f\big(\frac{z-x}{2} + y\big)\big\| \le \|f(x+y+z)\| \qquad (1.1)$$

(see also [19]). By using the fixed point method (see [1, 5, 14, 18, 21]) we prove the stability of A-linear mappings in Banach A-modules associated with the functional inequality (1.1).

For convenience, we use the following abbreviation for a given $a \in A$ and a mapping $f : \mathscr{X} \to \mathscr{Y}$

$$D_a f(x, y, z) := f\big(\frac{ax - ay}{2} + az\big) + f\big(\frac{ay - az}{2} + ax\big) + af\big(\frac{z-x}{2} + y\big)$$

for all $x, y, z \in \mathscr{X}$.

2. Functional inequalities in Banach modules

We start our work with the following useful lemma.

Lemma 2.1. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping such that*

$$\|D_a f(x, y, z)\| \le \|f(ax + ay + az)\| \qquad (2.1)$$

for all $x, y, z \in \mathscr{X}$ and all $a \in U(A)$. Then f is A-linear.

Proof. Letting $x = y = z = 0$ and $a = e \in U(A)$ in (2.1), we get that $f(0) = 0$. Letting $z = -x - y$ and $a = e \in U(A)$ in (2.1), we get

$$\big\|f\big(\frac{-x-3y}{2}\big) + f\big(\frac{3x+2y}{2}\big) + f\big(\frac{-2x+y}{2}\big)\big\| \le \|f(0)\| = 0$$

for all $x, y \in \mathscr{X}$. Hence

$$f(-x-3y) + f(3x+2y) + f(-2x+y) = 0 \qquad (2.2)$$

for all $x, y \in \mathscr{X}$.

Replacing x and y by $\frac{x+3y}{7}$ and $\frac{2x-y}{7}$ respectively, in (2.2), we get

$$f(-x) + f(x+y) + f(-y) = 0 \tag{2.3}$$

for all $x, y \in \mathscr{X}$. Since $f(0) = 0$, letting $y = 0$ in (2.3), we infer that f is odd. It follows from (2.3) that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathscr{X}$. Hence $f(rx) = rf(x)$ for all $x \in \mathscr{X}$ and all $r \in \mathbb{Q}$. By letting $z = -x$ and $y = 0$ in (2.1) and using the oddness of f, we get

$$f(ax) = af(x) \tag{2.4}$$

for all $a \in U(A)$ and all $x \in \mathscr{X}$. It is clear that (2.4) holds for $a = 0$.

Now let $a \in A$ ($a \neq 0$) and m an integer greater than $4|a|$. Then $|\frac{a}{m}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By Theorem 1 of [15], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $\frac{3}{m}a = u_1 + u_2 + u_3$. Hence by (2.4) we have

$$\begin{aligned} f(ax) &= \frac{m}{3} f\big(\frac{3}{m}ax\big) = \frac{m}{3} f(u_1x + u_2x + u_3x) \\ &= \frac{m}{3}[f(u_1x) + f(u_2x) + f(u_3x)] \\ &= \frac{m}{3}(u_1 + u_2 + u_3)f(x) = \frac{m}{3}.\frac{3}{m}af(x) = af(x) \end{aligned}$$

for all $x \in \mathscr{X}$. So $f : X \to \mathscr{Y}$ is A-linear, as desired. □

Now we prove the stability of A-linear mappings in Banach A-modules.

Theorem 2.2. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping for which there exists a function $\varphi : \mathscr{X}^3 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} 2^n \varphi\big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\big) = 0, \tag{2.5}$$

$$\|D_a f(x, y, z)\| \leq \|f(ax + ay + az)\| + \varphi(x, y, z) \tag{2.6}$$

for all $x, y, z \in \mathscr{X}$ and all $a \in U(A)$. If there exists a constant $L < 1$ such that the function

$$x \mapsto \psi(x) := 2\varphi\big(\frac{x}{7}, \frac{2x}{7}, \frac{-3x}{7}\big) + \varphi\big(\frac{4x}{7}, \frac{x}{7}, \frac{-5x}{7}\big)$$

has the property

$$2\psi(x) \leq L\psi(2x)$$

for all $x \in \mathscr{X}$, then there exists a unique A-linear mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{1-L}\psi(x) \tag{2.7}$$

for all $x \in \mathscr{X}$.

Proof. It follows from (2.5) that $\varphi(0,0,0) = 0$. Letting $x = y = z = 0$ and $a = e \in U(A)$ in (2.6), we get that $f(0) = 0$. Letting $z = -x-y$ in (2.6), we get

$$\left\|f\left(\frac{-x-3y}{2}\right) + f\left(\frac{3x+2y}{2}\right) + f\left(\frac{-2x+y}{2}\right)\right\| \leq \varphi(x, y, -x-y)$$

for all $x, y \in \mathscr{X}$. So

$$\|f(-x-3y) + f(3x+2y) + f(-2x+y)\| \leq \varphi(2x, 2y, -2x-2y) \tag{2.8}$$

for all $x, y \in \mathscr{X}$. Replacing x and y by $\frac{x+3y}{7}$ and $\frac{2x-y}{7}$, respectively, in (2.8), we get

$$\|f(-x) + f(x+y) + f(-y)\| \leq \varphi\left(\frac{2x+6y}{7}, \frac{4x-2y}{7}, \frac{-6x-4y}{7}\right) \tag{2.9}$$

for all $x, y \in \mathscr{X}$. Letting $y = 0$ and $y = x$ in (2.9), respectively, we get

$$\|f(-x) + f(x)\| \leq \varphi\left(\frac{2x}{7}, \frac{4x}{7}, \frac{-6x}{7}\right), \tag{2.10}$$

$$\|f(2x) + 2f(-x)\| \leq \varphi\left(\frac{8x}{7}, \frac{2x}{7}, \frac{-10x}{7}\right) \tag{2.11}$$

for all $x \in \mathscr{X}$. It follows from (2.10) and (2.11) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \psi(x) \tag{2.12}$$

for all $x \in \mathscr{X}$. Let E be the set of all mappings $g : \mathscr{X} \to \mathscr{Y}$ with $g(0) = 0$ and introduce a generalized metric on E as follows:

$$d(g,h) := \inf\{\, C \in [0,\infty] : \|g(x) - h(x)\| \leq C\psi(x) \text{ for all } x \in X \,\}.$$

It is easy to show that (E, d) is a generalized complete metric space [5].

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = 2g\left(\frac{x}{2}\right), \quad \text{for all } g \in E \text{ and } x \in \mathscr{X}.$$

Let $g, h \in E$ and let $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) \leq C$. From the definition of d, we have

$$\|g(x) - h(x)\| \leq C\psi(x)$$

for all $x \in \mathscr{X}$. By the assumption and last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = 2\Big\|g\big(\frac{x}{2}\big) - h\big(\frac{x}{2}\big)\Big\| \leq 2C\psi\big(\frac{x}{2}\big) \leq CL\psi(x)$$

for all $x \in \mathscr{X}$. So

$$d(\Lambda g, \Lambda h) \leq Ld(g,h)$$

for any $g, h \in E$. It follows from (2.12) that $d(\Lambda f, f) \leq 1$. Therefore according to Theorem 1.1, the sequence $\{\Lambda^n f\}$ converges to a fixed point T of Λ, i.e.,

$$T : \mathscr{X} \to \mathscr{Y}, \quad T(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} 2^n f\big(\frac{x}{2^n}\big)$$

and $T(2x) = 2T(x)$ for all $x \in \mathscr{X}$. Also T is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f,g) < \infty\}$ and

$$d(T, f) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{1}{1-L},$$

i.e., inequality (2.7) holds true for all $x \in \mathscr{X}$. It follows from the definition of T, (2.5) and (2.6) that

$$\begin{aligned}\|D_a T(x,y,z)\| &= \lim_{n\to\infty} 2^n \Big\|D_a f\big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\big)\Big\| \\ &\leq \lim_{n\to\infty} 2^n \Big\|f\big(\frac{ax+ay+az}{2^n}\big)\Big\| + \lim_{n\to\infty} 2^n \varphi\big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\big) \\ &= \|T(ax+ay+az)\|\end{aligned}$$

for all $x, y, z \in \mathscr{X}$ and all $a \in U(A)$. By Lemma 2.1, the mapping $T : X \to \mathscr{Y}$ is A-linear. Finally it remains to prove the uniqueness of T. Let $P : X \to \mathscr{Y}$ be another A-linear mapping satisfying (2.7). Since $d(f, P) \leq \frac{1}{1-L}$, and P is additive, then $P \in E^*$ and $(\Lambda P)(x) = 2P(x/2) = P(x)$ for all $x \in X$, i.e., P is a fixed point of Λ. Since T is the unique fixed point of Λ in E^*, then $P = T$. □

Corollary 2.3. *Let $r > 1$ and θ be non-negative real numbers and let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping such that*

$$\|D_a f(x,y,z)\| \leq \|f(ax+ay+az)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in \mathscr{X}$ and all $a \in U(A)$. Then there exists a unique A-linear mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x) - T(x)\| \leq \frac{2^r(3 + 2.2^r + 2.3^r + 4^r + 5^r)}{7^r(2^r - 2)} \theta \|x\|^r$$

for all $x \in \mathscr{X}$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x,y,z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x,y,z \in \mathscr{X}$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.4. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping with $f(0) = 0$ for which there exists a function $\Phi : \mathscr{X}^3 \to [0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{2^n}\Phi(2^n x, 2^n y, 2^n z) = 0,$$

$$\|D_a f(x,y,z)\| \le \|f(ax+ay+az)\| + \Phi(x,y,z)$$

for all $x,y,z \in \mathscr{X}$ and all $a \in U(A)$. If there exists a constant $L < 1$ such that the function

$$x \mapsto \Psi(x) := 2\varphi(\frac{2x}{7}, \frac{4x}{7}, \frac{-6x}{7}) + \varphi(\frac{8x}{7}, \frac{2x}{7}, \frac{-10x}{7})$$

has the property

$$\Psi(2x) \le 2L\Psi(x)$$

for all $x \in \mathscr{X}$, then there exists a unique A-linear mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x) - T(x)\| \le \frac{L}{1-L}\Psi(x) \tag{2.13}$$

for all $x \in \mathscr{X}$.

Proof. Using the same method as in the proof of Theorem 2.2, we have

$$\left\|\frac{1}{2}f(2x) - f(x)\right\| \le \frac{1}{2}\Psi(2x) \le L\Psi(x) \tag{2.14}$$

for all $x \in \mathscr{X}$. We introduce the same definitions for E and d as in the proof of Theorem 2.2 such that (E,d) becomes a generalized complete metric space. Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda g)(x) = \frac{1}{2}g(2x), \quad \text{for all } g \in E \text{ and } x \in \mathscr{X}.$$

One can show that $d(\Lambda g, \Lambda h) \le Ld(g,h)$ for any $g,h \in E$. It follows from (2.14) that $d(\Lambda f, f) \le L$. Due to Theorem 1.1, the sequence $\{\Lambda^n f\}$ converges to a fixed point T of Λ, i.e.,

$$T : \mathscr{X} \to \mathscr{Y}, \quad T(x) = \lim_{n\to\infty}(\Lambda^n f)(x) = \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

and $T(2x) = 2T(x)$ for all $x \in \mathscr{X}$. Also

$$d(T, f) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{L}{1-L},$$

i.e., inequality (2.13) holds true for all $x \in \mathscr{X}$.

The rest of the proof is similar to the proof of Theorem 2.4 and we omit the details. □

Corollary 2.5. *Let $0 < r < 1$ and θ, δ be non-negative real numbers and let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\|D_a f(x, y, z)\| \leq \|f(ax + ay + az)\| + \delta + \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in \mathscr{X}$ and all $a \in U(A)$. Then there exists a unique A-linear mapping $T : \mathscr{X} \to \mathscr{Y}$ such that

$$\|f(x) - L(x)\| \leq \frac{3.2^r}{2 - 2^r}\delta + \frac{4^r(3 + 2.2^r + 2.3^r + 4^r + 5^r)}{7^r(2 - 2^r)}\theta\|x\|^r$$

for all $x \in \mathscr{X}$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x, y, z) := \delta + \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in \mathscr{X}$. Then we can choose $L = 2^{r-1}$ and we get the desired result. □

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