Fixed Point Theory, 10(2009), No. 1, 125-140
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON A D.V. IONESCU PROBLEM FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS

VERONICA ILEA[1] AND DIANA OTROCOL[2]

[1]Department of Applied Mathematics
Babeş-Bolyai University
Cluj-Napoca, Romania
E-mail: vdarzu@math.ubbcluj.ro

[2]Tiberiu Popoviciu Institute of Numerical Analysis
Romanian Academy
Cluj-Napoca, Romania
E-mail: dotrocol@ictp.acad.ro

Abstract. The purpose of this paper is to study a generalization of a D.V. Ionescu's problem. Existence, uniqueness and data dependence (monotony, continuity, differentiability with respect to parameter) results of solution for the Cauchy problem are obtained using weakly Picard operator theory.
Key Words and Phrases: Picard operator, weakly Picard operators, polylocal problem, fixed points, data dependence.
2000 Mathematics Subject Classification: 47H10, 47N20, 34K07.

1. Introduction

We consider the system

$$x'(t) = f(t, x(t), x(\omega(t))),\ t \in [a, b], \tag{1.1}$$

where

$$x := (x_1, x_2, \ldots, x_m), \quad x(\omega) := (x_1(\omega_1), x_2(\omega_2), \ldots, x_m(\omega_m)),$$

$$f := (f_1, f_2, \ldots, f_m)$$

with initial conditions

$$\begin{cases} x_1(a) = 0, \\ x_2(t_2) = x_1(t_2), \\ \dots \\ x_{m-1}(t_{m-1}) = x_{m-2}(t_{m-1}), \\ x_m(b) = 0. \end{cases} \tag{1.2}$$

This kind of problems has been the subject of many works, from which we quote here first of all Ch.J. de la Vallèe Poussin's memorial [15]. From the research upon the polylocal problem in relation with differential equations, research more related to the present approach, we mention the following: improved formulation of Ch.J. de la Vallèe Poussin's theorem regarding second order differential equations have been obtained by Ph. Hartman and A. Wintner [3] and later by D. Ripianu [9], evaluations of differentiable functions with applications on the study of the polylocal problem have been provided by O. Arama [1], the connection between the study of the polylocal problem and the theory of superior order convex functions has been provided by T. Popoviciu, in the case of linear systems of functions [8].

We suppose that:

(C_1) $a = t_1 < t_2 < \ldots < t_{m-1} < t_m = b$;
(C_2) $f \in C([a,b] \times \mathbb{R}^{2m}, \mathbb{R}^m),\ \omega_i \in C([a,b],[a,b]), i = \overline{1,m}$;
(C_3) there exists $S_{i1}, \ldots, S_{i,2m} \in M_{m,2m}(\mathbb{R}_+)$ such that

$$|f_i(t,u_1,\ldots,u_{2m}) - f_i(t,v_1,\ldots,v_{2m})| \le S_{i1}\,|u_1 - v_1| + \ldots + S_{i,2m}\,|u_{2m} - v_{2m}|$$

for all $t \in [a,b],\ u_j, v_j \in \mathbb{R}^{2m},\ i = \overline{1,m}$.

We consider the problem

$$x'(t) = g(t),\ t \in [a,b] \tag{1.3}$$

with the conditions (1.2), where $g : [a,b] \to \mathbb{R}^m,\ g := (g_1, \ldots, g_m)$. The unique solution of this problem has the form

$$x(t) = \int_a^b (K_{ij})_n^n(t,s) g(s) ds. \tag{1.4}$$

The problem (1.1)-(1.2) is equivalent with the fixed point problem (1.4) where $\mathbf{K} :=(K_{ij})_n^n$ has the property that the operator defined by

$$g \to \int_a^{(\cdot)} \mathbf{K}((\cdot), s)g(s)ds$$

is from $C([a,b],\mathbb{R}^m)$ to $C([a,b],\mathbb{R}^m)$.

K_{ij} is given by the below relations

$$K_{11} = \begin{cases} 1, \ t_1 \le s \le t \le t_m, \\ 0, \quad \text{in the rest}, \end{cases} \tag{1.5}$$

$$K_{12}(t,s) = \cdots = K_{1m}(t,s) = 0, \tag{1.6}$$

$$K_{i1} = \begin{cases} 1, \ t_1 \le s \le t_2, \ \text{for all } t, i = \overline{2, n-1}, \\ 0, \quad \text{in the rest}, \end{cases} \tag{1.7}$$

$$K_{i,i-1} = \begin{cases} 1, \ t_{i-1} \le s \le t_i, \ \text{for all } t, i = \overline{2, n-1}, \\ 0, \quad \text{in the rest}, \end{cases} \tag{1.8}$$

$$K_{ii} = \begin{cases} 1, \ t_i \le s \le t \le t_n, i = \overline{2, n-1}, \\ 1, \ t_1 \le t \le s \le t_i, \\ 0, \quad \text{in the rest} \end{cases} \tag{1.9}$$

$$K_{i,i+1}(t,s) = \cdots = K_{im}(t,s) = 0, \tag{1.10}$$

$$K_{m1}(t,s) = \cdots = K_{m,m-1}(t,s) = 0, \tag{1.11}$$

$$K_{mm} = \begin{cases} 1, \ t_1 \le t \le s \le t_m, \\ 0, \quad \text{in the rest}. \end{cases} \tag{1.12}$$

The problem (1.1)-(1.2) is equivalent with the system

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{pmatrix} = \int_a^b \mathbf{K}(t,s) \begin{pmatrix} f_1(s, x_1(s),\ldots, x_m(s), x_1(\omega_1(s)),\ldots, x_m(\omega_m(s))) \\ \vdots \\ f_m(s, x_1(s),\ldots, x_m(s), x_1(\omega_1(s)),\ldots, x_m(\omega_m(s))) \end{pmatrix} ds \tag{1.13}$$

where $\mathbf{K} := (K_{ij})_n^n$ is given by the relations (1.5)-(1.12).

Consider the Banach space $X := (C([a,b],\mathbb{R}^m), \|\cdot\|)$ where $\|\cdot\|$ is the generalized Chebyshev norm,

$$\|u\| := \begin{pmatrix} \|u_1\| \\ \vdots \\ \|u_m\| \end{pmatrix}, \text{ where } \|u_i\| := \max_{a\le t\le b} |u_i(t)|, \ i=\overline{1,m}$$

and the operator

$$B_f : C([a,b],\mathbb{R}^m) \to C([a,b],\mathbb{R}^m)$$

defined by

$$B_f(x)(t) := \text{second part of (1.13)}.$$

In this paper we shall use the Perov's fixed point theorem and the weakly Picard operator theory in the study of existence and uniqueness and data dependence of the solution for the problem (1.1)-(1.2). For a better understanding we need some notions and results from WPO theory, see [10]-[14].

2. Picard and Weakly Picard operators

Let (X,d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subset of A;

$A^{n+1} := A \circ A^n,\ A^0 = 1_X,\ A^1 = A,\ n \in \mathbb{N}$;

Definition 2.1. *Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;

(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 2.2. *Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 2.3. *If A is weakly Picard operator then we consider the operator A^∞ defined by*

$$A^\infty : X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 2.4. *It is clear that* $A^\infty(X) = F_A$.

Lemma 2.5. *Let* $(X, d, \leq)$ *be an ordered metric space and* $A : X \to X$ *an operator. We suppose that:*

(i) *A is WPO;*

(ii) *A is increasing.*

Then, the operator A^∞ *is increasing.*

Lemma 2.6. *Let* $(X, d, \leq)$ *an ordered metric space and* $A, B, C : X \to X$ *be such that:*

(i) *the operator* A, B, C *are WPOs;*

(ii) $A \leq B \leq C$*;*

(iii) *the operator* B *is increasing.*

Then $x \leq y \leq z$ *implies that* $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

Theorem 2.7 (Perov's fixed point theorem)**.** *Let* (X, d) *with* $d(x, y) \in \mathbb{R}^m$*, be a complete generalized metric space and* $A : X \to X$ *an operator. We suppose that there exists a matrix* $Q \in M_{mm}(\mathbb{R}_+)$*, such that*

(i) $d(A(x), A(y)) \leq Qd(x, y)$*, for all* $x, y \in X$;

(ii) $Q^n \to 0$ *as* $n \to \infty$.

Then

(a) $F_A = \{x^*\}$,

(b) $A^n(x) = x^*$ *as* $n \to \infty$ *and*

$$d(A^n(x), x^*) \leq (I - Q)^{-1} Q^n d(x_0, A(x_0)).$$

Theorem 2.8 (Fibre contraction principle)**.** *Let* (X, d) *and* (Y, ρ) *be two metric spaces and* $A : X \times Y \to X \times Y,\ A = (B, C),\ (\ B : X \to X,\ C : X \times Y \to Y\ )$ *a triangular operator. We suppose that*

(i) (Y, ρ) *is a complete metric space;*

(ii) *the operator* B *is Picard operator;*

(iii) *there exists* $l \in [0, 1)$ *such that* $C(x, \cdot) : Y \to Y$ *is a* l*-contraction, for all* $x \in X$*;*

(iv) *if* $(x^*, y^*) \in F_A$*, then* $C(\cdot, y^*)$ *is continuous in* x^*.

Then the operator A *is Picard operator.*

For more details on WPOs theory see [10], [12], [13].

3. Existence and uniqueness

In what follows we consider the problem (1.1)-(1.2) in the conditions (C_1)-(C_3).

The problem (1.1)-(1.2) is equivalent with the fixed point equation

$$B_f(x) = x, \ x \in C([a,b], \mathbb{R}^m)$$

where $B_f =$ the second part of (1.13).

From the condition (C_3) we have, for $t \in [a,b]$

$$\begin{aligned}
&|B_f(x)(t) - B_f(y)(t)| \leq \\
&\leq \int_a^b \mathbf{K}(t,s) \left[\begin{pmatrix} |f_1(s, x_1(s), \ldots, x_m(s), x_1(\omega_1(s)), \ldots, x_m(\omega_m(s)))| \\ \vdots \\ |f_m(s, x_1(s), \ldots, x_m(s), x_1(\omega_1(s)), \ldots, x_m(\omega_m(s)))| \end{pmatrix} \right. \\
&\quad \left. - \begin{pmatrix} |f_1(s, y_1(s), \ldots, y_m(s), y_1(\omega_1(s)), \ldots, y_m(\omega_m(s)))| \\ \vdots \\ |f_m(s, y_1(s), \ldots, y_m(s), y_1(\omega_1(s)), \ldots, y_m(\omega_m(s)))| \end{pmatrix} \right] ds \\
&\leq \int_a^b \mathbf{K}(t,s) \begin{pmatrix} S_{11} + S_{1,m+1} & S_{12} + S_{1,m+2} & \cdots & S_{1,m} + S_{1,2m} \\ \vdots & \vdots & \cdots & \vdots \\ S_{m,1} + S_{m,m+1} & S_{m,2} + S_{m,m+2} & \cdots & S_{m,m} + S_{m,2m} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \|x_1 - y_1\| \\ \vdots \\ \|x_m - y_m\| \end{pmatrix} ds \leq Q \begin{pmatrix} \|x_1 - y_1\| \\ \vdots \\ \|x_m - y_m\| \end{pmatrix},
\end{aligned}$$

for all $x, y \in X$ and

$$Q = \max_{a \leq t \leq b} \int_a^b \mathbf{K}(t,s) ds \cdot \begin{pmatrix} S_{11}+S_{1,m+1} & S_{12}+S_{1,m+2} & \cdots & S_{1,m}+S_{1,2m} \\ \vdots & \vdots & \cdots & \vdots \\ S_{m,1}+S_{m,m+1} & S_{m,2}+S_{m,m+2} & \cdots & S_{m,m}+S_{m,2m} \end{pmatrix}.$$

Then

$$\|B_f(x) - B_f(y)\| \leq Q \, \|x - y\|$$

and if $Q^n \to 0$ as $n \to \infty$, the operator B_f is Q-contraction. From the Perov's fixed point theorem we have that the operator B_f is PO and has a unique fixed point

$$\overset{*}{x} = (\overset{*}{x}_1, \ldots, \overset{*}{x}_m) \in X.$$

Since f is continuous, we have that $\overset{*}{x} \in C([a,b], \mathbb{R}^m)$ is the unique solution for the problem (1.1)-(1.2).

So, we have the following existence and uniqueness theorem

Theorem 3.1. *We suppose that:*

(i) *the conditions* (C_1)*-*(C_3) *are satisfied;*
(ii) $Q^n \to 0$ *as* $n \to \infty$.

Then:

(a) *the problem (1.1)-(1.2) has in* $C([a,b], \mathbb{R}^m)$ *a unique solution* $\overset{*}{x} = (\overset{*}{x}_1, \ldots, \overset{*}{x}_m)$
$\in C([a,b], \mathbb{R}^m)$;
(b) *for all* $x^0 \in C([a,b], \mathbb{R}^m)$*, the sequence* $(x^n)_{n \in \mathbb{N}}$ *defined by*

$$x^{n+1} = B_f(x^n),$$

converges uniformly to $\overset{*}{x}$*, for all* $t \in [a,b]$*, and*

$$\begin{pmatrix} \left\| x_1^n - \overset{*}{x}_1 \right\| \\ \vdots \\ \left\| x_m^n - \overset{*}{x}_m \right\| \end{pmatrix} \leq (I - Q)^{-1} Q^n \begin{pmatrix} \| x_1^0 - x_1^1 \| \\ \vdots \\ \| x_m^0 - x_m^1 \| \end{pmatrix}.$$

4. Inequalities of Čaplygin type

In this section we shall study the relation between the solution of the problem (1.1)-(1.2) and the subsolution of the same problem.

Let $\overset{*}{x}$ the unique solution of the problem (1.1)-(1.2) and y the subsolution of the same problem, i.e.

$$y'(t) \leq f(t, y(t), y(\omega(t))),\ t \in [a,b], \tag{4.1}$$

where $y := (y_1, y_2, \ldots, y_m)$, $y(\omega) := (y_1(\omega_1), y_2(\omega_2), \ldots, y_m(\omega_m))$ and $f := (f_1, f_2, \ldots,$

f_m)) satisfy the conditions (C_1)-(C_3) and

$$\begin{cases} y_1(a) = 0 \\ y_2(t_2) = y_1(t_2) \\ \dots \\ y_{m-1}(t_{m-1}) = y_{m-2}(t_{m-1}) \\ y_m(b) = 0. \end{cases} \tag{4.2}$$

In this section we consider the operator $B_f =$ the second part of (1.13) on the ordered Banach space $X = ((C[a,b], \mathbb{R}^m), \|\cdot\|, \leq)$, where on $\mathbb{R}^m$ we have the ordered relation:

$$x \leq y \Longleftrightarrow x_i \leq y_i, i = \overline{1,m}.$$

We have the following theorem

Theorem 4.1. *We suppose that:*

(a) *the conditions (C_1)-(C_3) are satisfied;*

(b) $Q^n \to 0$ *as* $n \to \infty$*;*

(c) $f(t,\cdot,\cdot) : \mathbb{R}^{2m} \to \mathbb{R}^m$ *is increasing, for all* $t \in [a,b]$.

Let x *be a solution of the system (1.1) and* y *be a solution of the inequality problem (4.1)-(4.2).*

Then $y \leq x$ *for all* $t \in [a,b]$.

Proof. In terms of the operator B_f defined by the relation (1.13), we have

$$x = B_f(x) \text{ and } y \leq B_f(y).$$

On the other hand from condition (c) and Lemma 2.5, we have that the operator B_f^∞ is increasing. Hence

$y \leq B_f(y) \leq B_f^2(y) \leq \cdots \leq B_f^\infty(y) \leq B_f^\infty(x) = x.$

So, $y \leq x$. □

5. Data dependence: monotony

In this section we study the monotony of the system (1.1)-(1.2) with respect to f. For this we use the abstract comparison Lemma from section 2.

Consider the following equations

$$x'(t) = f(t, x_1(t), \dots, x_m(t), x_1(\omega_1(t)), \dots, x_m(\omega_m(t))), \tag{5.1}$$

$$y'(t) = g(t, y_1(t), \dots, y_m(t), y_1(\omega_1(t)), \dots, y_m(\omega_m(t))), \tag{5.2}$$

$$z'(t) = h(t, z_1(t), \ldots, z_m(t), z_1(\omega_1(t)), \ldots, z_m(\omega_m(t))), \tag{5.3}$$

with the polylocal conditions (1.2) for each problem and let $\overset{*}{x}, \overset{*}{y}$ and $\overset{*}{z}$ the unique solutions of these problems. Then we need the operators B_f, B_g and B_h corresponding to the second part of the problems (5.1), (5.2) and (5.3).

Theorem 5.1. *Let $f, g, h \in C([a, b] \times \mathbb{R}^{2m}, \mathbb{R})$, that satisfy the conditions (C_1)-(C_3) from section 1.*

We suppose that we have

(i) $f \leq g \leq h$;

(ii) $g(t, \cdot, \cdot) : \mathbb{R}^{2m} \to \mathbb{R}$ is increasing.

Let $\overset{*}{x}, \overset{*}{y}$ and $\overset{*}{z}$ the solutions of the equations (5.1), (5.2) and (5.3).

Then $\overset{*}{x}(t) \leq \overset{*}{y}(t) \leq \overset{*}{z}(t)$ for all $t \in [a, b]$, meaning that the unique solution of the system (1.1)-(1.2) is increasing with respect to the right hand.

Proof. From Theorem 3.1 the operators B_f, B_g, B_h are POs.

From the condition (ii) it follows that the operator B_g is monotone increasing and from condition (i) we have $B_f \leq B_g \leq B_h$.

But $\overset{*}{x} = {B_f}^{\infty}(\overset{*}{x})$, $\overset{*}{y} = {B_g}^{\infty}(\overset{*}{y})$ and $\overset{*}{z} = {B_h}^{\infty}(\overset{*}{z})$.

By applying the abstract comparison Lemma 2.6 follows that the unique solution of the problem (1.1)-(1.2) is increasing with respect to B_f. □

6. Data dependence: continuity

Consider the problems (1.1)-(1.2) with the dates f, g and suppose that the conditions from Theorem 3.1 are satisfied.

Let $f, g \in C([a, b] \times \mathbb{R}^{2m}, \mathbb{R}^m)$ and

$$S^f_{i1}, \ldots, S^f_{i,2m},\ S^g_{i1}, \ldots, S^g_{i,2m} \in M_{m,2m}(\mathbb{R}_+),\ i = \overline{1, m}$$

as in condition (C_3).

Consider $S_{ij} \in M_{m,2m}(\mathbb{R}_+)$, $i = \overline{1, m}$, $j = \overline{1, 2m}$ with

$$S_{ij} = max(S^f_{ij}, S^g_{ij}),\ i = \overline{1, m},\ j = \overline{1, 2m}.$$

Let

$$Q_f = \max_{a \leq t \leq b} \int_a^b \mathbf{K}(t, s) ds \begin{pmatrix} S^f_{11} + S^f_{1,m+1} & S^f_{12} + S^f_{1,m+2} & \cdots & S^f_{1,m} + S^f_{1,2m} \\ \vdots & \vdots & \cdots & \vdots \\ S^f_{m,1} + S^f_{m,m+1} & S^f_{m,2} + S^f_{m,m+2} & \cdots & S^f_{m,m} + S^f_{m,2m} \end{pmatrix},$$

Q_g analogously and

$$Q=\max_{a\le t\le b}\int_a^b \mathbf{K}(t,s)ds\cdot\begin{pmatrix} S_{11}+S_{1,m+1} & S_{12}+S_{1,m+2} & \cdots & S_{1,m}+S_{1,2m}\\ \vdots & \vdots & \cdots & \vdots\\ S_{m,1}+S_{m,m+1} & S_{m,2}+S_{m,m+2} & \cdots & S_{m,m}+S_{m,2m}\end{pmatrix},$$

Denote by $\overset{*}{x}(\cdot;f)$ the solution of the problem (1.1)-(1.2).

Theorem 6.1. *Let f,g satisfy the conditions (C_1)-(C_3). Furthermore, we suppose that there exist $\eta\in\mathbb{R}^m_+$ such that*

$$|f(t,x^1,x^2)-g(t,x^1,x^2)|\le\eta,\ \textit{for all } t\in C[a,b] \textit{ and } x^1,x^2\in\mathbb{R}^m.$$

Then

$$\|\overset{*}{x}(t;f)-\overset{*}{x}(t;g)\|\le(I-Q_f)^{-1}\max_{a\le s\le b}\int_a^b\mathbf{K}(t,s)ds\cdot\eta,$$

where $\overset{*}{x}(t;f)$ and $\overset{*}{x}(t;g)$ are the solution of the problem (1.1)-(1.2) with respect to f and g.

Proof. Consider the operators B_f and B_g. From Theorem 3.1 it follows that

$$\|B_f(x)-B_g(y)\|\le Q\|x-y\|\ \text{ for all } x,y\in X.$$

Additionally

$$\|B_f(x)-B_g(x)\|\le\max_{a\le s\le b}\int_a^b\mathbf{K}(t,s)ds\cdot\eta.$$

We have now

$$\begin{aligned}&\|\overset{*}{x}(t;f)-\overset{*}{x}(t;g)\|=\|B_f(\overset{*}{x}(t;f))-B_g(\overset{*}{x}(t;g))\|\le\\ &\le\|B_f(\overset{*}{x}(t;f))-B_f(\overset{*}{x}(t;g))\|+\|B_f(\overset{*}{x}(t;g))-B_g(\overset{*}{x}(t;g))\|\le\\ &\le Q\|\overset{*}{x}(t;f)-\overset{*}{x}(t;g)\|+\max_{a\le s\le b}\int_a^b\mathbf{K}(t,s)ds\cdot\eta.\end{aligned}$$

Because $Q^n\to 0$ as $n\to\infty$ imply that

$$(I-Q)^{-1}\in M_{mm}(\mathbb{R}^+),$$

so we have

$$\|\overset{*}{x}(t;f^1)-\overset{*}{x}(t;f^2)\|\le(I-Q_f)^{-1}\max_{a\le s\le b}\int_a^b\mathbf{K}(t,s)ds\cdot\eta,\qquad\square$$

7. Data dependence: differentiability

In this section we present the dependence by parameter λ of the solution of the problem (1.1)-(1.2).

Consider the following differential system with parameter:

$$x'(t) = f(t, x_1(t), \ldots, x_m(t), x_1(\omega_1(t)), \ldots, x_m(\omega_m(t)); \lambda), \; t \in [a, b], \quad (7.1)$$

$$\begin{cases} x_1(a) = 0, \\ x_2(t_2) = x_1(t_2), \\ \cdots \\ x_{m-1}(t_{m-1}) = x_{m-2}(t_{m-1}), \\ x_m(b) = 0, \end{cases} \quad (7.2)$$

where $x := (x_1, \ldots, x_m)$ and $f := (f_1, \ldots, f_m)$.

We suppose that:

(C_1) $a = t_1 < t_2 < \ldots < t_{m-1} < t_m = b$; $\lambda \in J \subset \mathbb{R}$ a compact interval;
(C_2) $f \in C^1([a, b] \times \mathbb{R}^{2m} \times J, \mathbb{R}^m)$, $\omega_i \in C([a, b], [a, b])$;
(C_3) there exists $S_{ij} \in M_{m,2m}(\mathbb{R}_+)$ such that

$$\left[\left(\left|\frac{\partial f_i(t, u_1, \ldots, u_{2m}; \lambda)}{\partial u_j}\right|\right)_{i,j=\overline{1,m}}\right]_{M_{m,2m}(\mathbb{R})} \leq S_{ij},$$

for all $t \in [a, b]$, $u_j \in \mathbb{R}^{2m}$, $i = \overline{1, m}$, $j = \overline{1, 2m}$;
(C_4) for

$$Q = \max_{a \leq t \leq b} \int_a^b \mathbf{K}(t, s) ds \cdot \begin{pmatrix} S_{11} + S_{1,m+1} & S_{12} + S_{1,m+2} & \cdots & S_{1,m} + S_{1,2m} \\ \vdots & \vdots & \vdots & \vdots \\ S_{m,1} + S_{m,m+1} & S_{m,2} + S_{m,m+2} & \cdots & S_{m,m} + S_{m,2m} \end{pmatrix}$$

we have $Q^n \to 0$ as $n \to \infty$.

In the above conditions, from Theorem 3.1 we have that the problem (1.1)-(1.2) has a unique solution, $\overset{*}{x}(\cdot; \lambda)$, for any $\lambda \in \mathbb{R}$.

We prove that $\overset{*}{x}(t; \cdot) \in C^1(J, \mathbb{R}^m)$, $\forall\, t \in [a, b]$.

For this we consider the system

$$x'(t; \lambda) = f(t, x_1(t; \lambda), \ldots, x_m(t; \lambda), x_1(\omega_1(t); \lambda), \ldots, x_m(\omega_m(t); \lambda); \lambda), \quad (7.3)$$

$t \in [a, b]$, $\lambda \in J$, $x \in C([a, b] \times J, \mathbb{R}^m)$.

The system (7.3) is equivalent with

$$x_i(t;\lambda)=\int_a^b \mathbf{K}(t,s)f_i(s,x_1(s;\lambda),...,x_m(s;\lambda),x_1(\omega_1(s);\lambda),...,x_m(\omega_m(s);\lambda);\lambda)ds, \tag{7.4}$$

where $i=\overline{1,m}$.

Let $X:=(C([a,b]\times J,\mathbb{R}^m),\|\cdot\|)$ with the Chebyshev norm,

$$\|x\|_C:=\begin{pmatrix}\|x_1\|\\ \vdots\\ \|x_m\|\end{pmatrix}\in\mathbb{R}^m_+.$$

Now we consider the operator

$$B:C([a,b]\times J,\mathbb{R}^m)\to C([a,b]\times J,\mathbb{R}^m)$$

where

$$B(x)(t;\lambda):=\text{second part of (7.4)}.$$

It is clear, from the proof of the Theorem 3.1, that in the conditions (C_1)-(C_4), the operator B is Picard operator, since

$$\|B(y)-B(z)\|_C\leq Q\,\|y-z\|_C\,.$$

Let $\overset{*}{x}=(\overset{*}{x}_1,\dots,\overset{*}{x}_m)$ be the unique fixed point of B.

We suppose that there exists $\dfrac{\partial\overset{*}{x}_i}{\partial\lambda}$, $i=\overline{1,m}$. From relation (7.4) and condition (C_3) we have

$$\frac{\partial\overset{*}{x}_i(t;\lambda)}{\partial\lambda}=$$

$$=\int_a^b \mathbf{K}(t,s)\left(\frac{\partial f_i(s,x_1(s;\lambda),\dots,x_m(s;\lambda),x_1(\omega_1(s);\lambda),\dots,x_m(\omega_m(s);\lambda);\lambda)}{\partial u_1}\right)_{i,j}$$

$$\cdot\frac{\partial\overset{*}{x}_1(s;\lambda)}{\partial\lambda}ds+\cdots+$$

$$+\int_a^b \mathbf{K}(t,s)\left(\frac{\partial f_i(s,x_1(s;\lambda),\dots,x_m(s;\lambda),x_1(\omega_1(s);\lambda),\dots,x_m(\omega_m(s);\lambda);\lambda)}{\partial u_m}\right)_{i,j}$$

$$\cdot\frac{\partial\overset{*}{x}_m(s;\lambda)}{\partial\lambda}ds+$$

$$+\int_a^b \mathbf{K}(t,s)\left(\frac{\partial f_i(s,x_1(s;\lambda),\dots,x_m(s;\lambda),x_1(\omega_1(s);\lambda),\dots,x_m(\omega_m(s);\lambda);\lambda)}{\partial u_{m+1}}\right)_{i,j}$$

$$\cdot \frac{\partial \overset{*}{x}_1(h_1(s);\lambda)}{\partial \lambda} ds + \cdots +$$

$$+ \int_a^b \mathbf{K}(t,s) \left(\frac{\partial f_i(s, x_1(s;\lambda), \ldots, x_m(s;\lambda), x_1(\omega_1(s);\lambda), \ldots, x_m(\omega_m(s);\lambda);\lambda)}{\partial u_{2m}} \right)_{i,j}$$

$$\cdot \frac{\partial \overset{*}{x}_m(h_m(s);\lambda)}{\partial \lambda} ds +$$

$$+ \int_a^b \mathbf{K}(t,s) \left(\frac{f_i(s, x_1(s;\lambda), \ldots, x_m(s;\lambda), x_1(\omega_1(s);\lambda), \ldots, x_m(\omega_m(s);\lambda);\lambda)}{\partial \lambda} \right)_{i,j} ds$$

for $t \in [a,b]$, $\lambda \in J$, $i = \overline{1,m}, j = \overline{1,2m}$.

This relation suggest us to consider the following operator

$$C : X \times X \to X,\ (x_1, \ldots, x_m, y_1, \ldots, y_m) \to C(x_1, \ldots, x_m, y_1, \ldots, y_m),$$

where

$$C(x_1, \ldots, x_m, y_1, \ldots, y_m)(t;\lambda) :=$$

$$= \int_a^b \mathbf{K}(t,s) \left(\frac{\partial f_i(s, x_1(s;\lambda), \ldots, x_m(s;\lambda), x_1(\omega_1(s);\lambda), \ldots, x_m(\omega_m(s);\lambda);\lambda)}{\partial u_1} \right)_{i,j}$$

$$\cdot\, y_1(s;\lambda) ds + \cdots +$$

$$+ \int_a^b \mathbf{K}(t,s) \left(\frac{\partial f_i(s, x_1(s;\lambda), \ldots, x_m(s;\lambda), x_1(\omega_1(s);\lambda), \ldots, x_m(\omega_m(s);\lambda);\lambda)}{\partial u_m} \right)_{i,j}$$

$$\cdot\, y_m(s;\lambda) ds +$$

$$+ \int_a^b \mathbf{K}(t,s) \left(\frac{\partial f_i(s, x_1(s;\lambda), \ldots, x_m(s;\lambda), x_1(\omega_1(s);\lambda), \ldots, x_m(\omega_m(s);\lambda);\lambda)}{\partial u_{m+1}} \right)_{i,j}$$

$$\cdot\, y_1(h_1(s);\lambda) ds + \cdots +$$

$$+ \int_a^b \mathbf{K}(t,s) \left(\frac{\partial f_i(s, x_1(s;\lambda), \ldots, x_m(s;\lambda), x_1(\omega_1(s;\lambda)), \ldots, x_m(\omega_m(s);\lambda);\lambda)}{\partial u_{2m}} \right)_{i,j}$$

$$\cdot\, y_m(h_m(s);\lambda) ds$$

$$+ \int_a^b \mathbf{K}(t,s) \left(\frac{\partial f_i(s, x_1(s;\lambda), \ldots, x_m(s;\lambda), x_1(\omega_1(s;\lambda)), \ldots, x_m(\omega_m(s);\lambda);\lambda)}{\partial \lambda} \right)_{i,j} ds$$

for $t \in [t_0, b], \lambda \in J, i = \overline{1,m}, j = \overline{1,2m}$.

In this way we have the triangular operator

$$A : X \times X \to X \times X,$$

$$A(x_1, \ldots, x_m, y_1, \ldots, y_m)$$
$$= (B(x_1, \ldots, x_m)(t;\lambda), C(x_1, \ldots, x_m, y_1, \ldots, y_m)(t;\lambda))$$

where B is a Picard operator and $C(x_1,\ldots,x_m,\cdot):X\to X$ is Q-contraction.

Indeed we have

$$\left\|C(\overset{*}{x},u)(t;\lambda)-C(\overset{*}{x},v)(t;\lambda)\right\|_{\mathbb{R}^m}\le Q\left\|u-v\right\|_C,\ \forall t\in[t_0,b],\forall\lambda\in J$$

which implies that

$$\left\|C(\overset{*}{x},u)-C(\overset{*}{x},v)\right\|_C\le Q\left\|u-v\right\|_C,\ \forall u,v\in X.$$

Since $Q^n\to 0$ as $n\to\infty$, from the Theorem of fibre contraction (see [12], [13]) follows that the operator A is Picard operator and has a unique fixed point $(\overset{*}{x},\overset{*}{y})\in X\times X$. So the sequences

$$(x^{n+1},y^{n+1})=(B(x^n),C(x^n,y^n)),n\in\mathbb{N}$$

converges uniformly (with respect to $t\in[a,b],\ \lambda\in J$) to $(\overset{*}{x},\overset{*}{y})\in F_A$, for any $x^0\in X,\ y^0\in X$.

If we take

$$x_i^0=0,y_i^0=\frac{\partial x_i^0}{\partial\lambda}=0,\ \text{then}\ y_i^1=\frac{\partial x_i^1}{\partial\lambda},i=\overline{1,m}.$$

By induction we prove that

$$y_i^n=\frac{\partial x_i^n}{\partial\lambda},\ \forall n\in\mathbb{N},i=\overline{1,m}.$$

Thus

$$x_i^n\overset{unif}{\to}\overset{*}{x}_i,\ \text{as}\ n\to\infty,i=\overline{1,m}$$

$$\frac{\partial x_i^n}{\partial\lambda}\overset{unif}{\to}\overset{*}{y}_i,\ \text{as}\ n\to\infty,i=\overline{1,m}.$$

These imply that there exists $\dfrac{\partial\overset{*}{x}_i}{\partial\lambda}$ and

$$\frac{\partial x_i(t;\lambda)}{\partial\lambda}=\overset{*}{y}_i(t;\lambda),\ i=\overline{1,m}.$$

So, we have

Theorem 7.1. *Suppose that conditions* (C$_1$)-(C$_3$) *hold. Then,*

(i) *the problem (7.1)-(7.2) has a unique solution* $\overset{*}{x}=(\overset{*}{x}_1,\ldots,\overset{*}{x}_m)\in C([a,b]\times J,\mathbb{R}^m)$;

(ii) $\overset{*}{x}(t;\cdot)\in C^1(J,\mathbb{R}^m),\forall t\in[a,b]$.

8. Remarks

Remark 8.1. *The problem (1.1)-(1.2) is a generalization of a problem studied by D.V. Ionescu in* [2].

D.V. Ionescu's problem. *Let* $t_k \in [a,b]$ *with* $t_1 < t_2 < \ldots < t_n$ $(n \in \mathbb{N}, n \geq 3)$ *and* $f = (f_1, \ldots, f_n) \in C([a,b] \times \mathbb{R}^n, \mathbb{R}^n)$. *We suppose that* $t_1 = a$ *and* $t_n = b$. *The problem is to study the existence of* $x = (x_1, \ldots, x_n) \in C^1([a,b], \mathbb{R}^n)$ *such that*

$$x'(t) = f(t, x(t)),\ t \in [a,b]$$

and

$$\begin{cases} x_1(t_1) = 0, \\ x_2(t_2) = x_1(t_2), \\ \ldots \\ x_{n-1}(t_{n-1}) = x_{n-2}(t_{n-1}), \\ x_n(t_n) = 0. \end{cases}$$

D.V. Ionescu proved that if the interval $[a,b]$ *is sufficiently small and the functions* f_i *are Lipschitz with respect to* x*, then this problem has a unique solution,* [7].

Remark 8.2. *Some problems concerning equation (1.1) were study in the following particular cases (see* [10], [11]*)*

$$\omega_i(t) = t - \tau_i,\ i = \overline{1,m},\ \tau > 0$$

and

$$\omega_1(t) = \lambda t,\ \omega_2(t) = \frac{1}{\lambda} t,\ 0 < \lambda < 1 \textit{ (see [4])}.$$

For other considerations on the functional-differential equations we mention: [6], [12], [13], [14].

References

[1] O. Arama, *Contributions to the study of polylocal problems relative to differential equations*, Ph.D. Thesis (in Romanian), Cluj, 1965.

[2] D.V. Ionescu, *Quelques théorems d'existence des intégrales des systèmes d'équations différentielles*, C.R. de l'Acad. Sci. Paris, **186**(1929), 1262-1263.

[3] Ph. Hartman, A. Wintner, *On an oscillation criterion of de la Vallée Poussin*, Quaterly of Applied Mathematics, **13**(1955), no. 3, 330-332.

[4] V. Mureşan, *Differential equation with linear modification of arguments*, Transilvania Press, Cluj-Napoca, 1997.

[5] A.I. Perov, A.V. Kibenko, *On a general method to study boundary value problems*, Iz. Akad. Nauk., **30**(1966), 249-264.

[6] A. Petruşel, I.A. Rus, *Fixed point theorems in L-spaces*, Proc. Amer. Math. Soc., **134**(2006), 411-418.

[7] A. Petruşel, I.A. Rus, *Mathematical contributions of Professor D.V. Ionescu*, Notices from the ISMS, January, 2008, 1-11.

[8] T. Popoviciu, *Sur quelques propriètès des fonctions d'une ou deux variables rèelles*, Mathematica (Cluj), **8**(1934), 1-86.

[9] D. Ripianu, *On Vallée Poussin inequality in the case of second order differential equations* (in Romanian), St. Cerc. Mat. (Cluj), **8**(1963), 123-150.

[10] I.A. Rus, *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58** (2003), no.1, 191-219.

[11] I.A. Rus, *Functional differential equations of mixed type, via weakly Picard operators*, Sem. on Fixed Point Theory Cluj-Napoca, **3**(2002), 335-346.

[12] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[13] I.A. Rus, *Weakly Picard operators and applications*, Sem. on Fixed Point Theory, Cluj-Napoca, **2**(2001), 41-58.

[14] I.A. Rus, A. Petruşel, M. Şerban, *Weakly Picard operators: equivalent definitions, applications and open problems*, Fixed Point Theory, **7**(2006), no. 1, 3-22.

[15] Ch.J. de la Vallée Poussin, *Sur l'équation différentialle du second ordre. Détermination d'une intégrale par deux valeurs assignée. Extension aux équations d'ordre n*, J. Math. Pures Appl., **8**(1929), 125-144.

Received: June 5, 2008; Accepted: October 10, 2008.