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COMMON FIXED POINT AND APPROXIMATION RESULTS FOR GENERALIZED ϕ-CONTRACTIONS

N. HUSSAIN[1], V. BERINDE[2] AND N. SHAFQAT[3]

[1]Department of Mathematics
King Abdul Aziz University
P. O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail: nhusain@kau.edu.sa, hussianjam@hotmail.com

[2]Department of Mathematics and Computer Science
Faculty of Sciences, North University of Baia Mare
Victoriei Nr. 76, 430122 Baia Mare, Romania
E-mail: vberinde@ubm.ro

[3]Centre for Advanced Studies in Pure and Applied Mathematics
Bahauddin Zakariya University, Multan, Pakistan

Abstract. We establish common fixed point theorems for weakly compatible generalized ϕ-contractions. As applications, various common fixed point and best approximation results for C_q-commuting and compatible maps are derived. Our results unify, extend and complement various known results existing in the literature.
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1. Introduction and preliminaries

We first review needed definitions. Let M be a subset of a metric space (X, d). We shall use N to denote the set of positive integers, $cl(S)$ to denote the closure of a set S and $wcl(S)$ to denote the weak closure of a set S. A mapping $T : M \to M$ is called an I-contraction if, there exists $0 \leq k < 1$ such that $d(Tx, Ty) \leq k\ d(Ix, Iy)$ for any $x, y \in M$. If $k = 1$, then T is called f-nonexpansive. Let $I, T : M \to M$ be mappings. A point $x \in M$ is a coincidence point (common fixed point) of I and T if $Ix = Tx$ ($x = Ix = Tx$). The set of coincidence points of I and T is denoted by $C(I, T)$. The set

$O_T(x) = \{x, Tx, T^2x, ...\}$ is called the orbit of T relative to x. The pair $\{I, T\}$ is called (1) commuting if $TIx = ITx$ for all $x \in M$; (2) R-weakly commuting [23] if for all $x \in M$ there exists $R > 0$ such that $d(ITx, TIx) \leq \mathrm{R}d(Ix, Tx)$; (3) compatible [17] if $\lim_n d(TIx_n, ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M; (4) weakly compatible if they commute at their coincidence points, i.e.,if $ITx = TIx$ whenever $Ix = Tx$; (5) pointwise R-weakly commuting [23] if given $x \in X$, there exists $R > 0$ such that $d(ITx, TIx) \leq Rd(Ix, Tx)$. The definition implies that pointwise R-weakly commuting maps commute at their coincidence points. The converse is also true. Thus pointwise R-weak commutativity of I and T at their coincidence points is equivalent to weak compatibility of I and T [22]. If I and T are compatible and do have a coincidence point, I and T are called [19] nontrivially compatible. The set M of a normed space X is called q-starshaped with $q \in M$ if the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x, is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called (6) R-subweakly commuting on M (see [21,25]) if for all $x \in M$, there exists a real number $R > 0$ such that $d(ITx, TIx) \leq Rdist(Ix, [q, Tx])$; (7) C_q-commuting [3,15] if $ITx = TIx$ for all $x \in C_q(I, T)$, where $C_q(I, T) = \cup\{C(I, T_k) : 0 \leq k \leq 1\}$ and $T_k x = (1-k)q + kTx$. Clearly, C_q-commuting maps are weakly compatible but not conversely in general (see for details [3,19]). The mapping $T : M \to M$ is called demiclosed at 0 if for every sequence $\{x_n\} \in M$ such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to 0, we have $Tx = 0$. The mapping $T : M \to M$ is said to satisfy condition (C) [19] if $A \cap F(T) \neq \emptyset$ for any nonempty T-invariant closed set $A \subset M$. A Banach space X satisfies Opial's condition if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \neq x$.

In 1995, Jungck and Sessa [20] extended the results of Singh [27,28], Smoluk [30] and Subrahmanyam [31] to the pair of commuting maps. More recently, Shahzad [25,26], Hussain and Jungck [12], Hussain et al. [14], O'Regan and Shahzad [21], Jungck and Hussain [19] and O'Regan and Hussain [22] further extended the above mentioned results to R-subweakly commuting and compatible maps. The aim of this paper is to establish common fixed point theorems for weakly compatible generalized I-contractions with respect to a comparison function ϕ (see [4] p. 43-44). As applications, certain common fixed point and invariant approximation results for C_q-commuting and compatible maps are derived. Our results contain properly the recent results of Al-Thagafi and Shahzad [3] and Shahzad [25,26] and unify and extend the results of Agarwal et al. [1], Al-Thagafi [2], Berinde [5], Boyd and Wong [6], Carbone, Rhoades and Singh [7], Ciric [8], Daffer and Kaneko [9], Hussain [10], Hussain and Berinde [11], Hussain and Khan [13], Hussain, O'Regan and Agarwal [14], Hussain and Rhoades [15], Jungck [16,18], Jungck and Sessa [19], O'Regan and Hussain [22], Pant [23], Sahab et al. [24], Singh [27,28], Subrahmanyam [31], and many others.

2. Main results

We begin with a result which extends and improves Theorem 2.2 in [1], Theorems 2.1 in [2,3], Theorem 1 in [7], Theorem 2.4 in [9], Theorem 1 in [23] and contains main results of Boyd and Wong [6], Ciric [8] and Jungck [16] as special cases.

Theorem 2.1. *Let M be a subset of a metric space (X, d), and I and T be weakly compatible self-maps of M. Assume that $clT(M) \subset I(M)$, $clT(M)$ is complete, and there exists a continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in M$, we have*

$$d(Tx,Ty) \leq \phi\left(\max\left\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), \frac{1}{2}[d(Ix,Ty)+d(Iy,Tx)]\right\}\right).$$

Then $F(I) \cap F(T) \neq \emptyset$.

Proof. Fix $x_0 \in M$ arbitrarily. As $T(M) \subset I(M)$, one can choose x_1 in M, such that $Tx_0 = Ix_1$. Consider now Tx_1. Since $Tx_1 \in I(M)$, there exists x_2 in M such that $Tx_1 = Ix_2$. By induction(see proof of Theorem 1[5]), we construct a sequence $\{x_n\}$ of points in M such that $Tx_n = Ix_{n+1}$ for $n = 0, 1, 2, 3, ...$ We claim that $\{Ix_n\}$ is a Cauchy sequence. To prove our claim, we follow arguments of Agarwal et al. [1]. We first show that

$$d(Ix_n, Ix_{n+1}) \leq \phi(d(Ix_{n-1}, Ix_n)) \text{ for } n \in \{1, 2, 3, ...\}. \tag{2.1}$$

Notice that

$$\begin{aligned}
d(Ix_n, Ix_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \phi[\max\{d(Ix_{n-1}, Ix_n), d(Ix_{n-1}, Tx_{n-1}), d(Ix_n, Tx_n), \\
&\quad \frac{1}{2}[d(Ix_{n-1}, Tx_n) + d(Ix_n, Tx_{n-1})]\}] \\
&\leq \phi[\max\{d(Ix_{n-1}, Ix_n), d(Ix_{n-1}, Ix_n), d(Ix_n, Ix_{n+1}), \\
&\quad \frac{1}{2}[d(Ix_{n-1}, Ix_{n+1}) + d(Ix_n, Ix_n)]\}] \\
&\leq \phi[\max\{d(Ix_{n-1}, Ix_n), d(Ix_n, Ix_{n+1}), \\
&\quad \frac{1}{2}[d(Ix_{n-1}, Ix_n) + d(Ix_n, Ix_{n+1})]\}]
\end{aligned}$$

Let

$$\eta_n = \max\left\{d\left(Ix_{n-1}, Ix_n\right), d\left(Ix_n, Ix_{n+1}\right), \frac{1}{2}\left[d\left(Ix_{n-1}, Ix_n\right) + d\left(Ix_n, Ix_{n+1}\right)\right]\right\}.$$

If $\eta_n = d\left(Ix_{n-1}, Ix_n\right)$ then clearly (2.1) holds. If $\eta_n = d\left(Ix_n, Ix_{n+1}\right)$, then $d\left(Ix_n, Ix_{n+1}\right) = 0$ since if not

$$d\left(Ix_n, Ix_{n+1}\right) \leq \phi\left(d\left(Ix_n, Ix_{n+1}\right)\right) < d\left(Ix_n, Ix_{n+1}\right)$$

a contradiction. Thus $d\left(Ix_n, Ix_{n+1}\right) = 0$ and (2.1) is immediate. Thus in all cases (2.1) is true.

Next we show $\{Ix_n\}$ is a Cauchy sequence. Suppose it is not true. Then we can find a $\delta > 0$ and two sequences of integers $\{m\left(k\right)\}, \{n\left(k\right)\}, m(k) > n(k) \geq k$ with

$$r_k = d\left(Ix_{n(k)}, Ix_{m(k)}\right) \geq \delta \text{ for } k \in \{1, 2, 3, 4, \ldots, \}. \tag{2.2}$$

We may also assume that $d\left(Ix_{m(k)-1}, Ix_{n(k)}\right) < \delta$ by choosing $m\left(k\right)$ to be the smallest number exceeding $n(k)$ for which (2.2) holds. Now (2.1) and (2.2) imply

$$\begin{aligned}
\delta &\leq r_k \leq d\left(Ix_{m(k)}, Ix_{m(k)-1}\right) + d\left(Ix_{m(k)-1}, Ix_{n(k)}\right) \\
&\leq \phi^{m(k)-1}(d\left(Ix_1, Ix_0\right)) + \delta
\end{aligned}$$

and so $\lim\limits_{k\to\infty} r_k = \delta$ (note $\lim\limits_{n\to\infty} \phi^n\left(a\right) = 0$ for any $a > 0$, since if we let $a > 0$ and $a_n = \phi^n\left(a\right)$ then $a_n = \phi\left(a_{n-1}\right) \leq a_{n-1}$, thus $a_n \downarrow \beta$ (for some β), and since $\beta = \phi(\beta)$ so $\beta = 0$). Also since

$$\delta \leq r_k \leq d\left(Ix_{n(k)}, Ix_{n(k)+1}\right) + d\left(Ix_{m(k)+1}, Ix_{m(k)}\right) + d(Ix_{n(k)+1}, Ix_{m(k)+1}),$$

we have from (2.1) that

$$\delta \le r_k \le \phi^{n(k)}(d(Ix_0, Ix_1)) + \phi^{m(k)}(d(Ix_0, Ix_1)) + d(Tx_{n(k)}, Tx_{m(k)}).$$

Next notice that

$$\begin{aligned} d\left(Tx_{n(k)}, Tx_{m(k)}\right) \le\ & \phi[\max\{d(Ix_{n(k)}, Ix_{m(k)}), \\ & d(Ix_{n(k)}, Ix_{n(k)+1}), d(Ix_{m(k)}, Ix_{m(k)+1}), \\ & \frac{1}{2}[d(Ix_{n(k)}, Ix_{m(k)+1}) + d\left(Ix_{m(k)}, Ix_{n(k)+1}\right)]\}] \\ & \le \phi[\max\{r_k, \phi^{n(k)}(d(Ix_0, Ix_1)), \phi^{m(k)}(d(Ix_0, Ix_1)), \\ & \frac{1}{2}[2r_k + d(Ix_{n(k)}, Ix_{n(k)+1}) + d(Ix_{m(k)}, Ix_{m(k)+1})]\}] \\ & \le \phi[\max\{r_k, \phi^{n(k)}(d(Ix_0, Ix_1)), \phi^{m(k)}(d(Ix_0, Ix_1)), \\ & r_k + \frac{1}{2}\phi^{n(k)}(d(Ix_0, Ix_1)) + \frac{1}{2}\Phi^{m(k)}(d(Ix_0, Ix_1))\}] \\ & \le \phi\left(r_k + \phi^{n(k)}(d(Ix_0, Ix_1)) + \phi^{m(k)}(d(Ix_0, Ix_1))\right) \end{aligned}$$

Thus we have

$$\begin{aligned} \delta \le r_k \le\ & \phi^{n(k)}\left(d\left(Ix_0, Ix_1\right)\right) + \phi^{m(k)}(d\left(Ix_0, Ix_1\right)) \\ & + \phi\left(r_k + \phi^{n(k)}(d(Ix_0, Ix_1)) + \phi^{m(k)}(d(Ix_0, Ix_1))\right) \end{aligned}$$

and let $k \to \infty$ to obtain (use $\lim_{k\to\infty} r_k = \delta$ and $\lim_{n\to\infty} \phi^n(a) = 0$ for any $a > 0$) $\delta \le \phi(\delta)$. This is a contradiction since $\phi(z) < z$ for $z > 0$. Thus $\{Ix_n\}$ is a Cauchy sequence and hence $\{Tx_n\}$ is a Cauchy sequence. It follows from the completeness of $clT(M)$ that $Tx_n \to w$ for some $w \in M$ and hence $Ix_n \to w$ as $n \to \infty$. Consequently, $\lim_n Ix_n = \lim_n Tx_n = w \in clT(M) \subset I(M)$. Thus $w = Iy$ for some $y \in M$. Notice that for all $n \ge 1$, we have

$$d(w, Ty) \le d(w, Tx_n) + d(Tx_n, Ty) \le d(w, Tx_n)$$
$$+\phi(\max\{d(Ix_n, Iy), d(Tx_n, Ix_n), d(Ty, Iy), \frac{1}{2}[d(Ty, Ix_n) + d(Tx_n, Iy)]\}).$$

Letting $n \to \infty$, we obtain $Iy = w = Ty$. We now show that Ty is a common fixed point of I and T. Since I and T are weakly compatible and $Iy = Ty$, we obtain by the definition of weak compatibility that $ITy = TIy$.

Thus we have $T^2y = TIy = ITy$ and then we get successively $d(TTy, Ty) \le$

$$\phi(\max\{d(ITy, Iy), d(ITy, TTy), d(Iy, Ty), \frac{1}{2}[d(ITy, Ty) + d(Iy, TTy)]\})$$

$$\leq \phi(d(ITy, Ty)).$$

Hence $TTy = Ty$ and so $Ty = TTy = ITy$. This implies that Ty is a common fixed point of T and I. Hence $F(I) \cap F(T) \neq \emptyset$. □

In certain circumstances, it is possible to remove the condition that ϕ is nondecreasing in Theorem 2.1. We prove the following extension of Theorem 2.3 [1], Corollary 2.2 [18] and Theorem 1 [23].

Theorem 2.2. *Let M be a subset of a metric space (X, d), and I and T be weakly compatible self-maps of M. Assume that $clT(M) \subset I(M)$, $clT(M)$ is complete, and there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\Phi(z) < z$ for $z > 0$ such that*

$$d(Tx, Ty) \leq \phi(\max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty)\}).$$

Then $F(I) \cap F(T) \neq \emptyset$.

Proof. Fix $x_0 \in M$. As in the proof of Theorem 2.1, we construct a sequence $Tx_n = Ix_{n+1}$ for $n = 0, 1, 2, 3, \ldots$.

We claim that $\{Ix_n\}$ is a Cauchy sequence. To prove our claim we will need to prove

$$\alpha_n = d\left(Ix_{n+1}, Ix_n\right) = d\left(Tx_n, Tx_{n-1}\right) \to 0 \text{ as } n \to \infty \tag{2.3}$$

To see (2.3) notice that

$$\alpha_n = d\left(Ix_{n+1}, Ix_n\right) = d\left(Tx_n, Tx_{n-1}\right)$$

$$\leq \phi\left(\max\left\{d\left(Ix_n, Ix_{n-1}\right), d\left(Ix_n, Tx_n\right), d\left(Ix_{n-1}, Tx_{n-1}\right)\right\}\right)$$

$$= \phi\left(\max\left\{d\left(Ix_n, Ix_{n-1}\right), d\left(Ix_n, Ix_{n+1}\right), d\left(Ix_n, Ix_{n-1}\right)\right\}\right)$$

$$= \phi\left(\max\left\{d\left(Ix_n, Ix_{n-1}\right), d\left(Ix_n, Ix_{n+1}\right)\right\}\right) = \phi\left(\max\left\{\alpha_{n-1}, \alpha_n\right\}\right).$$

We now show that

$$\alpha_n \leq \phi\left(\alpha_{n-1}\right). \tag{2.4}$$

If $\max\{\alpha_{n-1}, \alpha_n\} = \alpha_{n-1}$ then clearly (2.4) is true, whereas if we have $\max\{\alpha_{n-1}, \alpha_n\} = \alpha_n$, then

$\alpha_n \leq \phi(\alpha_n)$ and so $\alpha_n = 0$, so (2.4) is immediate. Now since $\alpha_n \leq \phi(\alpha_{n-1}) \leq \alpha_{n-1}$, there exists $\alpha \geq 0$ with $\alpha_n \downarrow \alpha$. Now $\alpha_n \leq \phi(\alpha_{n-1})$ implies $\alpha \leq \phi(\alpha)$ so $\alpha = 0$, and this establishes (2.3). Suppose our claim is false then we can find a $\delta > 0$ and two sequences of integers $\{m(k)\}, \{l(k)\}, m(k) > l(k)$ with

$$r_k = d(Ix_{l(k)}, Ix_{m(k)}) \geq \delta \text{ for } k \in \{1, 2, 3, 4, ...\} \tag{2.5}$$

We may also assume $d\left(Ix_{m(k)-1}, Ix_{l(k)}\right) < \delta$ by choosing $m(k)$ to be smallest number exceeding $l(k)$ for which (2.5) holds. Now

$$\delta \leq r_k \leq d(Ix_{m(k)-1}, Ix_{l(k)}) + d(Ix_{m(k)}, Ix_{m(k)-1}) < \delta + \alpha_{m(k)-1},$$

from which with (2.3) we get

$$\lim_{k \to \infty} r_k = \delta. \tag{2.6}$$

Note that

$$\begin{aligned} \delta \leq r_k &\leq d\left(Ix_{l(k)}, Ix_{l(k)+1}\right) + d\left(Ix_{l(k)+1}, Ix_{m(k)+1}\right) + d\left(Ix_{m(k)+1}, Ix_{m(k)}\right) \\ &= \alpha_{l(k)} + \alpha_{m(k)} + d\left(Tx_{l(k)}, Tx_{m(k)}\right) \\ &\leq \alpha_{l(k)} + \alpha_{m(k)} + \phi(\max\{d\left(Ix_{l(k)}, Ix_{m(k)}\right), d\left(Ix_{l(k)}, Tx_{l(k)}\right), \\ &\quad d(Ix_{m(k)}, Tx_{m(k)})\}) \\ &= \alpha_{l(k)} + \alpha_{m(k)} \\ &\quad + \phi\left(\max\left\{d\left(Ix_{l(k)}, Ix_{m(k)}\right), d\left(Ix_{l(k)}, Ix_{l(k)+1}\right), d\left(Ix_{m(k)}, Ix_{m(k)+1}\right)\right\}\right) \\ &= \alpha_{l(k)} + \alpha_{m(k)} + \phi\left(\max\left\{r_k, \alpha_{l(k)}, \alpha_{m(k)}\right\}\right), \end{aligned}$$

and let $k \to \infty$ to obtain (using 2.3 and 2.6) $\delta \leq \phi(\delta)$. Thus $\delta = 0$, which is a contradiction. As a result our claim is true. It follows from the completeness of $clT(M)$ that $Tx_n \to w$ for some $w \in M$ and hence $Ix_n \to w$ as $n \to \infty$. Consequently, $\lim_n Ix_n = \lim_n Tx_n = w \in clT(M) \subset I(M)$. Thus $w = Iy$ for some $y \in M$. The analysis similar to the proof of Theorem 2.1 implies that $F(I) \cap F(T) \neq \emptyset$.

The first part of the proof of Theorem 2.2 establishes the following corollary.

Corollary 2.3. *Let M be a subset of a metric space (X, d), and I and T be self-maps of M. Assume that $clT(M) \subset I(M)$, $clT(M)$ is complete, and there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in M$, we have*

$$d(Tx, Ty) \leq \phi(\max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty)\}).$$

Then $C(I, T) \neq \emptyset$.

The following result extends and improves Theorem 2.1 of [2,3,15,22], Theorem 1 [5] and Lemma 2.1 of [21,25].

Theorem 2.4. *Let M be a subset of a metric space (X, d), and I and T be weakly compatible self-maps of M with bounded orbits. Assume that $clT(M) \subset I(M)$, $clT(M)$ is complete, and there exists a continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi^n(z) \to 0$ for each $z > 0$ such that*

$$d(Tx, Ty) \le \phi(\max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Tx, Iy), d(Ty, Ix)\}).$$

Then $F(I) \cap F(T) \neq \emptyset$.

Proof. Fix $x_0 \in M$. As in the proof of Theorem 2.1, we construct a sequence $Tx_n = Ix_{n+1}$ for $n \in \mathbb{N}$.

Then by following the proof of Theorem 1 [5], we get that $\{Ix_n\}$ is a Cauchy sequence. It follows from the completeness of $clT(M)$ that $Tx_n \to w$ for some $w \in M$ and hence $Ix_n \to w$ as $n \to \infty$. Consequently, $\lim_n Ix_n = \lim_n Tx_n = w \in clT(M) \subset I(M)$. Thus $w = Iy$ for some $y \in M$. The analysis similar to the proof of Theorem 2.1 implies that $F(I) \cap F(T) \neq \emptyset$. □

As an application of Theorem 2.2, we obtain the following generalization of the corresponding results in [2,3,14,24-28].

Theorem 2.5. *Let I and T be self-maps on a q-starshaped subset M of a normed space X where $q \in F(I)$ and I is affine. Suppose that there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in M$, we have*

$$\|Tx - Ty\| \le \frac{1}{k}\phi(\|Ix - Iy\|) \tag{2.7}$$

for each $k \in (0, 1)$. If I and T are C_q-commuting, then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds;

(i) $cl(T(M)) \subset I(M)$, $cl(T(M))$ is compact and T is continuous,

(ii) X is complete, $wcl(T(M)) \subset I(M)$, $wcl(T(M))$ is weakly compact, I is weakly continuous and either $I - T$ is demiclosed at 0 or X satisfies Opial's condition.

Proof. Define $T_n : M \to M$ by

$$T_n x = (1 - k_n)q + k_n Tx$$

for some q and all $x \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Then, for each n, $cl(T_n(M)) \subset I(M)$ as M is q-starshaped, $cl(T(M)) \subset I(M)$, I is affine and $Iq = q$. As I and T are C_q-commuting and I is affine with $Iq = q$, then for each $x \in C_q(I, T)$ we have:

$$IT_nx = (1 - k_n)q + k_nITx = (1 - k_n)q + k_nTIx = T_nIx.$$

Thus $IT_nx = T_nIx$ for each $x \in C(I, T_n) \subset C_q(I, T)$. Hence I and T_n are weakly compatible for all n. Also by (2.7),

$$\|T_nx - T_ny\| = k_n\|Tx - Ty\| \le k_n(\frac{1}{k_n}\phi(\|Ix - Iy\|)) = \phi(\|Ix - Iy\|)$$

for each $x, y \in M$.

(i) Since $cl(T(M))$ is compact, $cl(T_n(M))$ is also compact. By Theorem 2.3, for each $n \ge 1$, there exists $x_n \in M$ such that $x_n = Ix_n = T_nx_n$. The compactness of $cl(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Then the definition of T_mx_m implies $x_m \to y$, so by the continuity of T we have $Ty = y$. Since $T(M) \subseteq I(M)$, there exists $z \in M$ such that $Iz = Ty = y$. Also, for each m, we have

$$\|Tx_m - Tz\| \le \frac{1}{k_m}\phi(\|Ix_m - Iz\|) = \frac{1}{k_m}\phi(\|x_m - y\|),$$

which, on letting $m \to \infty$, implies on the basis of continuity of ϕ that $Iz = Ty = y = Tz$. This implies that $C(I, T)$ is nonempty. Since I and T are weakly compatible, we obtain $Iy = ITz = TIz = Ty = y$. Thus $F(I) \cap F(T) \neq \emptyset$.

(ii) The analysis in (i), and the completeness of $wcl(T_n(M))$ guarantee that there exists $x_n \in M$ such that $x_n = Ix_n = T_nx_n$. The weak compactness of $wcl(T(M))$ implies that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y$ weakly as $m \to \infty$. As I is weakly continuous, $Iy = y$. Since $\{x_m\}$ is bounded, $k_m \to 1$, and

$$\begin{aligned}\|x_m - Tx_m\| &= \|Ix_m - Tx_m\| = \|((1 - k_m)q + k_mTx_m) - Tx_m\| \\ &\le (1 - k_m)(\|q\| + \|Tx_m\|),\end{aligned}$$

then $\|x_m - Tx_m\| \to 0$ as $m \to \infty$. If $I - T$ is demiclosed at 0, $(I - T)y = 0$ and hence $y = Iy = Ty$.

Suppose that X satisfies Opial's condition. If $y \neq Ty$, then

$$\begin{aligned}\liminf_{m\to\infty} \|x_m - y\| &< \liminf_{m\to\infty} \|x_m - Ty\| \\ &\leq \liminf_{m\to\infty} \|x_m - Tx_m\| + \liminf_{m\to\infty} \|Tx_m - Ty\| \\ &= \liminf_{m\to\infty} \|Tx_m - Ty\| \leq \liminf_{m\to\infty} \frac{1}{k_m}\phi(\|Ix_m - Iy\|) \\ &= \phi(\liminf_{m\to\infty} \|x_m - y\|)\end{aligned}$$

which is a contradiction to the property $\phi(z) < z$ for $z > 0$. Thus $Iy = y = Ty$ and hence $F(I) \cap F(T) \neq \emptyset$. □

For $\phi(t) = kt$, $t \in [0, \infty)$, $0 < k < 1$, from Theorem 2.5 we obtain:

Corollary 2.6 [3, Theorem 2.2-Theorem 2.4]. *Let I and T be self-maps on a q-starshaped subset M of a normed space X where $q \in F(I)$ and I is affine. Suppose that I and T are C_q-commuting and T is I-nonexpansive, then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds;*

(i) $cl(T(M)) \subset I(M)$, $cl(T(M))$ is compact and T is continuous,

(ii) X is complete, $wcl(T(M)) \subset I(M)$, $wcl(T(M))$ is weakly compact, I is weakly continuous and either $I - T$ is demiclosed at 0 or X satisfies Opial's condition.

The classes of compatible and C_q-commuting maps are mutually disjoint. For this let $X = R$ with usual norm and $M = [1, \infty)$. Let $I(x) = 2x - 1$ and $T(x) = x^2$, for all $x \in M$. Let $q = 1$. Then M is q-starshaped with $Iq = q$ and $C_q(I, T) = [1, \infty)$. Note that I and T are compatible maps and T satisfies condition (C) but I and T are not C_q-commuting and hence not R-subweakly commuting maps.

An application of Corollary 2.3 provides the following result for compatible maps.

Theorem 2.7. *Let I and T be self-maps on a q-starshaped subset M of a normed space X where $q \in F(I)$ and I is affine. Suppose that there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that I and T satisfy (2.7). If I and T are continuous and compatible and T satisfies condition (C), then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds;*

(i) $cl(T(M)) \subset I(M)$ and $cl(T(M))$ is compact,

(ii) X is complete, $wcl(T(M)) \subset I(M)$, $wcl(T(M))$ is weakly compact and either $I - T$ is demiclosed at 0 or X satisfies Opial's condition.

Proof. (i) Let $\{k_n\}$ and $\{T_n\}$ be defined as in Theorem 2.5. The analysis in Theorem 2.5 (using Corollary 2.3 above) guarantees that there exists $x_n \in M$ such that $Ix_n = T_n x_n$ (see for details [19]). The compactness of $cl(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Since $k_m \to 1$, $Ix_m = (1 - k_m)q + k_m Tx_m$ converges to y. Now since I and T are continuous, $TIx_m \to Ty$ and $ITx_m \to Iy$ as $m \to \infty$. By the compatibility of I and T, we obtain $Iy = Ty$. Hence the pair $\{I, T\}$ is nontrivially compatible. Theorem 1.3 [19] guarantees that, $F(I) \cap F(T) \neq \emptyset$.

(ii) As in (i) there exists $x_n \in M$ such that $Ix_n = T_n x_n$. The weak compactness of $wcl(T(M))$ implies that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y$ weakly as $m \to \infty$. Since $\{x_m\}$ is bounded and $k_m \to 1$, so $\|Ix_m - Tx_m\| = \|((1 - k_m)q + k_m Tx_m) - Tx_m\| \leq (1 - k_m)(\|q\| + \|Tx_m\|)$ converges to 0. If $(I - T)$ is demiclosed at 0 then $(I - T)y = 0$ and hence $Iy = Ty$. Thus the pair $\{I, T\}$ is nontrivially compatible and the conclusion follows from Theorem 1.3 [19].

Suppose X satisfies Opial's condition. If $Iy \neq Ty$, then

$$\begin{aligned}
\liminf_{m\to\infty} \|Ix_m - Iy\| &< \liminf_{m\to\infty} \|Ix_m - Ty\| \\
&\leq \liminf_{m\to\infty} \|Ix_m - Tx_m\| + \liminf_{m\to\infty} \|Tx_m - Ty\| \\
&= \liminf_{m\to\infty} \|Tx_m - Ty\| \leq \liminf_{m\to\infty} \frac{1}{k_m} \phi(\|Ix_m - Iy\|) \\
&= \phi(\liminf_{m\to\infty} \|Ix_m - Iy\|)
\end{aligned}$$

which is a contradiction to the property $\phi(z) < z$ for $z > 0$. Thus $Iy = Ty$. Thus the pair $\{I, T\}$ is nontrivially compatible and the conclusion follows from Theorem 1.3 [19]. □

3. Concluding remarks

(3.1) Following the arguments as above and those in [1], we can prove Theorem 2.1 and Theorem 2.2 in the setup of gauge spaces.

(3.2) Theorem 2.1 can be used to find the solution of an operator equation of the form Tx=Gx, under suitable conditions on G; consequently, Theorem 2 [7] is extended and improved.

(3.3) Results similar to Theorems 2.5 and 2.7 for more general inequalities,

$$\|Tx - Ty\| \leq \phi\left(\max\left\{\begin{array}{c} \|Ix - Iy\|, dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\ \frac{1}{2}[dist(Ix, [Ty, q]) + dist(Iy, [Tx, q])] \end{array}\right\}\right) \tag{2.8}$$

and

$$\|Tx - Ty\| \leq \phi\left(\max\left\{\begin{array}{c} \|Ix - Iy\|, dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\ dist(Ix, [Ty, q]), dist(Iy, [Tx, q]) \end{array}\right\}\right) \tag{2.9}$$

can be obtained by applying our Theorems 2.1 and 2.4 respectively; which in turn generalize the corresponding results in [12,19,21,22].

(3.4) As an application of Theorem 2.5, the analogue of all recent approximation results (Theorem 3.1-Theorem 4.4), due to Al-Thagafi and Shahzad [3] and (Corollary 2.3-Corollary 2.10) due to O'Regan and Shahzad [21] can be established for C_q-commuting pair $\{I, T\}$ satisfying more general inequality (2.7), or (2.8) or (2.9).

(3.5) As an application of Theorem 2.7, the analogue of all recent approximation results (Theorem 3.1-Theorem 4.4), due to Al-Thagafi and Shahzad [3] and (Theorem 2.7-Theorem 2.10) due to Jungck and Hussain [19], can be established for the compatible pair $\{I, T\}$ satisfying inequality (2.7), or (2.8) or (2.9).

(3.6) A subset M of a linear space X is said to have property (N) with respect to T [11,14] if,

i) $T : M \to M$,

(ii) $(1 - k_n)q + k_n Tx \in M$, for some $q \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in M$.

A mapping I is said to be affine on a set M with property (N) if

$$I((1 - k_n)q + k_n Tx) = (1 - k_n)Iq + k_n ITx \text{ for each } x \in M \text{ and } n \in N.$$

Theorem 2.5-Theorem 2.7 remain valid, provided I is assumed to be surjective and the q-starshapedness of the set M is replaced by the property (N), in the setup of p-normed space [11] and metrizable locally convex topological vector space (X, d) [14] where d is translation invariant and $d(ax, ay) \leq ad(x, y)$, for each a with $0 < a < 1$ and $x, y \in X$. Consequently, all of the results of Hussain [10], Hussain and Berinde [11], Hussain, O'Regan and Agarwal

[14], Hussain and Rhoades [15] and Theorem 2.2-Theorem 3.3 due to Hussain and Khan [13] are improved and extended to the pair of C_q-commuting and compatible maps. We leave details to the reader.

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