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# SOME APPLICATIONS OF A COLLECTIVELY FIXED POINT THEOREM FOR MULTIMAPS

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**Abstract.** In this paper, we consider some collectively fixed point theorems for a family of multimaps. As applications, we deduce two intersection theorems for sets with nonempty sections and then we deduce a Nash-Ma type equilibrium theorem.

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## 1. Introduction

Our starting point in this paper consists of some interesting results of L-J. Lin, Z-T. Yu, Q.H. Ansari and L-P. Lai (see [7]) and the seminal paper [8] of S. Park. Actually, the main starting point of this paper (and of [7]) is a theorem of Q.H. Ansari and J.C. Yao [2, Theorem 1], concerning the existence of a fixed point for a family of multimaps in the Hausdorff topological vector spaces setting.

Many mathematicians worked in the fixed point theory of multimaps defined on product spaces. N.C. Yannelis and N.D. Prabhakar (1983), X.P. Ding, W.W. Kim and K.K. Tan (1992) X. Wu and S. Shen (1996), X. Wu (1997) studied fixed point theorems for a family of multimaps defined on product spaces, *with compactness assumptions* on the range sets. By using a partition of unity, in 1998, K.Q. Lan and J. Webb obtained some fixed point theorems for a family of multimaps on product spaces *without compactness assumptions* on the domain and the range sets.

In this paper, using compactness hypothesis on the range sets, we give some results concerning the fixed points of a family of multimaps defined on product spaces. As an application, we solve some problems of nonempty intersection for sets with nonempty sections.

Let $I$ be an index set. Let also $(X_i)_{i\in I}$ be a family of convex sets, each in a Hausdorff topological vector space. Write $X = \prod_{i\in I} X_i$ and let $(A_i)_{i\in I}$ be a family of subsets of $X$. Then, the problem of nonempty intersection for this family is to find suitable conditions on $(A_i)_{i\in I}$ such that the intersection of this family is nonempty.

When $I$ is a finite set and $X$ is compact, Ky Fan (1969) first imposed a convexity and an openness condition on each $A_i$, which assures that the intersection of the family $(A_i)_{i\in I}$ is nonempty. F.E. Browder (1968), using his celebrated fixed point theorem, gave another proof of the above mentioned Fan's result. The result of Fan was improved by E. Tarafdar and H.B. Thompson (1978) and, later, by E. Tarafdar (1982), but their fixed point technique cannot be applied to treat the case when $I$ is infinite. Fan's result was extended for $I$ an infinite set, by T.W. Ma (1969) and, later, by M.H. Shih and K.K. Tan (1985) for noncompact settings.

In 1998, K. Lan and J. Webb and, independently, S. Park gave some results concerning the intersection problem for an arbitrary (possible infinite) family of sets.

It is well known that, for example, the intersection theorems for sets having nonempty sections have applications in the minimax inequalities of von Neumann type and in the game theory (for example the Nash's equilibrium point theorem is an immediate consequence of an intersection theorem). In the last section of our paper, as an application of one of our theorems concerning the intersection for sets with nonempty sections, we give a Nash-Ma type equilibrium theorem, finding a solution for a system of variational inequalities.

## 2. Preliminaries

In this section we recall some definitions and notations.

For a nonempty set $X$, $2^X$ denotes the class of all subsets of $X$. A *multimap* is a function $T : X \longrightarrow 2^Y$ (in another terminology a multimap is also known as a *set-valued* function, a *mapping*, a *map* or a *correspondence*).

The biggest difference between *functions* and *multimaps* has to do with the definitions of an inverse image:

a) the *inverse image* of a set $A$ under a function $f$ is the set $\{x : f(x) \in A\}$;

b) for a multimap $T : X \longrightarrow 2^Y$, the *inverse* of $B$ by $T$ is defined by

$$T^{-1}(B) = \{x \in X : T(x) = B\}.$$

There are two reasonable generalizations for the inverse of a set by a multimap:

- the *upper inverse* of $B$ by $T$ is the set $T^{+}(B) = \{x \in X : T(x) \subset B\}$.
- the *lower inverse* of $B$ by $T$ is the set $T^{-}(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$.

Note that in this paper, we will use only the lower inverse of a set $B$ by a multimap $T$, but we will write $T^{-1}(B)$ instead of $T^{-}(B)$. We will denote also $T^{-1}(y)$ instead of $T^{-}(\{y\})$. Therefore, $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

A multimap $T : X \longrightarrow 2^Y$ is *nonempty-valued* (*convex-valued* or *compact-valued*) if the set $T(x)$ is a *nonempty* (respectively *convex*, or *compact*), for each $x \in X$.

The *fiber* of the multimap $T : X \longrightarrow 2^Y$ at the point $y \in Y$ is the set $T^{-1}(\{y\})$.

Recall that a real-valued function $g : X \longrightarrow \mathbb{R}$, on a topological vector space, is *lower* (respectively *upper*) *semicontinuous* if the set $\{x \in X : g(x) > r\}$ (respectively $\{x \in X : g(x) < r\}$) is open for each $r \in \mathbb{R}$.

If $X$ is a convex set in a vector space, then the function $g : X \longrightarrow \mathbb{R}$ is *concave* if $g(\lambda x_1 + (1-\lambda)x_2) \geq \lambda g(x_1) + (1-\lambda) g(x_2)$, for each $x_1, x_2 \in X$ and $\lambda \in [0,1]$.

## 3. Collectively fixed point theorems

Our starting point in this section is a result of Q.H. Ansari and J.C. Yao (see [2, Theorem 1]), concerning the existence of a fixed point for a family of multimaps (see Corollary 3.1 below) in the following framework.

Let $I$ be an index set and for each $i \in I$, let $E_i$ be a Hausdorff topological vector space. Let $(X_i)_{i\in I}$ be a family of nonempty convex subsets with each $X_i$ in $E_i$. Let $X = \prod\limits_{i\in I} X_i$ and $E = \prod\limits_{i\in I} E_i$.

In the result of Ansari and Yao, there are two families of multimaps $(S_i)_{i\in I}$ and $(T_i)_{i\in I}$, related by the condition $coS_i(x) \subseteq T_i(x)$, for each $i \in I$ and $x \in X$.

Also, let us observe that imposing some conditions to the family $(S_i)_{i\in I}$, we obtain a common fixed point for the family $(T_i)_{i\in I}$.

The Ansari and Yao's result was generalized in a certain sense, in 2003, by L-J. Lin, Z-T. Yu, Q.H. Ansari and L.P. Lai (see [7, Theorem 3.1]). Inspired by their result we formulated another generalization of the Ansari and Yao result. In our first theorem, we have only a single family of multimaps, these multimaps being nonempty-valued and convex-valued.

**Theorem 1.** *For each $i \in I$, let $T_i : X \longrightarrow 2^{X_i}$ be a nonempty-valued and convex-valued multimap (that is, for each $x \in X$, the set $T_i(x)$ is a nonempty convex subset of $X$). Suppose that the following conditions hold:*

*1) for each $i \in I$, $X$ can be covered with the interiors of all fibers of $T_i$, i.e.*

$$X = \bigcup \left\{ int_X T_i^{-1}(y_i) : y_i \in X_i \right\};$$

*2) if $X$ is not compact, and $C \subset X$ is a nonempty compact set, assume that for each $i \in I$ and for each finite subset $F_i$ of $X_i$, there exists a nonempty compact convex set $C_{F_i}$ in $X_i$, such that $C_{F_i} \supseteq F_i$ and $X \backslash C$ can be covered with the interiors of all fibers of $T_i$ at the points of $C_{F_i}$, i.e.*

$$X \backslash C \subseteq \bigcup \left\{ int_X T_i^{-1}(y_i) : y_i \in C_{F_i} \right\}.$$

*Then, there exists $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$, such that $\widetilde{x}_i \in T_i(\widetilde{x})$, for each $i \in I$.*

We omit the proof of this theorem, because it is not the subject of this paper. Let us remark only that this proof (very technical but standard) follows the ideea of the original theorem of Ansari and Yao, that is to use the partition of unity subordinated to a finite subcovering of a compact product space and to apply the Tychonoff's fixed point theorem ("if $X$ is a compact set in a locally convex Hausdorff topological vector space and $h : X \longrightarrow X$ is a continuous function, then $h$ has a fixed point" – see for example [9]).

With understanding changes in the proof of Theorem 1 we can also prove the following collectively fixed point theorem which generalizes [2, Theorem 1].

**Theorem 2.** *For each $i \in I$, let $S_i, T_i : X \longrightarrow 2^{X_i}$ be two nonempty valued multimaps such that:*

*0) for each $i \in I$ and each $x \in X$, $coS_i(x) \subseteq T_i(x)$;*

*1) for each $i \in I$, $X$ can be covered with the interiors of all fibers of $S_i$, that is*

$$X = \bigcup \left\{ int_X S_i^{-1}(y_i) : y_i \in X_i \right\};$$

*2) if $X$ is not compact, and $C \subset X$ is a nonempty compact set, assume that for each $i \in I$ and each finite subset $F_i$ of $X_i$, there exists a nonempty compact convex set $C_{F_i}$ in $X_i$ such that $C_{F_i} \supseteq F_i$ and*

$$X \backslash C \subseteq \bigcup \left\{ int_X S_i^{-1}(y_i) : y_i \in C_{F_i} \right\}.$$

*Then there exists $\widetilde{x} \in X$ such that $\widetilde{x}_i \in T_i(\widetilde{x})$, for each $i \in I$.*

**Remark 1.** Obviously, according to the condition *0)*, only $S_i$ must be a nonempty-valued multimap.

As a simple consequence of Theorem 2 we obtain the following result:

**Corollary 1.** (see [2, Theorem 1]) *For each $i \in I$, let $S_i, T_i : X \longrightarrow 2^{X_i}$ nonempty-valued multimaps such that:*

*0) for each $i \in I$ and each $x \in X$, $coS_i(x) \subseteq T_i(x)$;*

*1) $X = \bigcup \left\{ int_X S_i^{-1}(y_i) : y_i \in X_i \right\}$;*

*2) if $X$ is not compact, and $C \subset X$ is a nonempty compact set, assume that for each $i \in I$, there exists a nonempty compact convex subset $C_i$ of $X_i$ such that*

$$X \backslash C \subseteq \bigcup \left\{ int_X S_i^{-1}(y_i) : y_i \in C_i \right\}.$$

*Then, there exists $\widetilde{x} \in X$ such that $\widetilde{x}_i \in T_i(\widetilde{x})$, for each $i \in I$.*

**Proof.** For each $i \in I$ and each finite subset $F_i$ of $X_i$, we define $C_{F_i} = co(C_i \cup F_i)$. Therefore, it follows that $C_{F_i} \supseteq F_i$ and the set $C_{F_i}$ is compact and convex. Now we apply Theorem 2. □

**Remark 2.** *If in the above theorems we suppose that $I$ is a singleton set we obtain corresponding results concerning fixed point theorems for a multimap.*

For example, from Theorem 2 we obtain:

**Corollary 2.** *Let $X$ be a nonempty convex set in a Hausdorff topological vector space $E$ and let $C$ be a nonempty compact subset of $X$. Let $S, T : X \longrightarrow 2^X$ be two nonempty-valued multimaps. Suppose that the following conditions hold:*

*0) for each $x \in X$, $coS(x) \subseteq T(x)$;*

*1) $X = \bigcup \left\{ int_X S^{-1}(y) : y \in X \right\}$;*

*2) if $X$ is not compact, and $C \subset X$ is a nonempty compact set, assume that for every finite subset $F$ of $X$, there exists a nonempty compact convex subset $C_F$ of $X$ such that $C_F \supseteq F$ and $X \backslash C \subseteq \bigcup \left\{ int_X S^{-1}(y) : y \in C_F \right\}$.*

*Then, there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$.*

## 4. Intersection theorems for sets with nonempty sections

The collectively fixed point theorems can be reformulated to generalize various Neumann type intersection theorems for sets with nonempty sections.

Let $I$ be an index set, having at least two elements. Let $(X_i)_{i \in I}$ be a family of sets and let $i \in I$ be fixed. Let $X = \prod_{j \in I} X_j$ and $X^i = \prod_{j \in I \backslash \{i\}} X_j$.

For $A \subset X$ a nonempty set and $x^i \in X^i$, let $A\left(x^i\right) = \left\{ y_i \in X_i : \left[x^i, y_i\right] \in A \right\}$ be the set *of the sections of $A$ at $x^i$*, where $\left[x^i, y_i\right] \in X^i \times X_i$ is the element of $X$ having the $i^{th}$ coordinate $y_i$, and, for $j \neq i$, having its $j^{th}$ coordinate $x^i_j$.

**Definition 1.** We say that the set $A \subseteq X$ *has nonempty sections* if for any $i \in I$ and $x \in X$, the set $A\left(x^i\right)$ is nonempty.

In the following two definitions we introduce two new notions.

**Definition 2.** Let $I$ be an index set, having at least two elements and, for each $i \in I$, let $X_i$ be a set in a Hausdorff topological vector space $E_i$. Let $X = \prod_{i \in I} X_i$ and let $A$ be a subset of $X$. If $i \in I$ is a fixed index, we say that *the nonempty set $D \subseteq X$ can be $(A, i)$ sectioned with a common element of $X_i$* if there exists $y_i \in X_i$ such that

$$y_i \in \bigcap \left\{ A\left(z^i\right) : z \in D \right\}$$

(that is $\left[z^i, y_i\right] \in A$, for each $z \in D$).

**Definition 3.** Let $I$, $E_i$, $X_i$ and $X$ be as in the previous definition. Let $M$ be an arbitrary subset of $X$, and for each $i \in I$, let $A_i$ and $L_i$ subsets of $X$, and respectively of $X_i$. We say that $M$ *can be locally covered with a family* $(D_i)_{i\in I}$ *of open sets of* $X$ *such that the set* $D_i$ *can be* $(A_i, i)$ *sectioned with a common element of* $L_i$*, for each* $i \in I$, if for each $x \in M$, there exists a family of open sets $(D_i)_{i\in I}$ in $X$, such that, for each $i \in I$, $x \in D_i$ and there exists an element $y_i \in L_i$ with

$$y_i \in \bigcap \left\{ A_i \left( z^i \right) : z \in D_i \right\}.$$

**Examples.**

**1.** If $I$,$E_i$, $X_i$, $X$ and $A_i$ are as in the frame of the Definition 3, for $M = X$ and $L_i = X_i$ $(i \in I)$ and if we define the multimap $T_i : X \longrightarrow 2^{X_i}$ by

$$T_i (x) = A_i \left( x^i \right), \text{ for each } x \in X,$$

then:

a) $A_i \left( x^i \right) \neq \varnothing$ if and only if $T_i$ is *nonempty-valued*;

b) $A_i \left( x^i \right)$ is *convex* if and only if $T_i$ is *convex-valued*;

c) Definition 3 is equivalent with the equality

$$X = \bigcup \left\{ intT_i^{-1} (y_i) : y_i \in X_i \right\} \tag{4.1}$$

We will prove "c)". We fix $i \in I$. Then, for each $x \in X$, there exist $y_i \in X_i$ and an open set $D_i \subseteq X$ such that x $\in D_i$ and $y_i \in \bigcap \left\{ A_i \left( z^i \right) : z \in D_i \right\}$, that is $y_i \in T_i (z)$, for all $z \in D_i$. Then, Definition 3 is equivalent with the following: for each $x \in X$, there exist $y_i \in X_i$ and an open set $D_i \subseteq X$ such that $x \in D_i \subseteq T_i^{-1} (y_i)$. It follows that $x \in \bigcup \left\{ intT_i^{-1} (y_i) : y_i \in X_i \right\}$.

**2.** Also, if $I$,$E_i$, $X_i$, $X$, $A_i$ are in the frame of Definition 3, for $C \subset X$ a nonempty compact set, $M = X \backslash C$ and $L_i = C_i$ a nonempty compact convex set in $X_i$, then Definition 3 is equivalent with the inclusion

$$X \backslash C \subseteq \bigcup \left\{ intT_i^{-1} (y_i) : y_i \in C_i \right\} \tag{4.2}$$

Note that the conditions (4.1) and (4.2) are used in the literature and, therefore the notions defined in our Definition 2 and 3 seem to be useful.

Indeed, we will use these notions to solve "the problem of nonempty intersection". The problem of nonempty intersection for the sets $(A_i)_{i\in I}$ (with $A_i$

a subset of $X$, for each $i \in I$), having nonempty sections, asks when the set $\bigcap \{A_i : i \in I\}$ is nonempty.

The following result solves this problem applying Theorem 2 (a collectively fixed point theorem).

**Theorem 3.** *Let $I$ be an index set, and, for each $i \in I$, let $X_i$ be a nonempty convex subset of a Hausdorff topological vector space $E_i$ and let $X = \prod_{i\in I} X_i$. Let also $C$ be a nonempty compact subset of $X$, and, for each $i \in I$, let $A_i$ and $B_i$ be subsets of $X$, having nonempty sections. Suppose that:*

*0) for each $i \in I$, and each $x^i \in X^i$, $co\, B_i\left(x^i\right) \subseteq A_i\left(x^i\right)$;*

*1) $X$ can be locally covered with open sets which can be $(B_i, i)$ sectioned with a common element of $X_i$ ($i \in I$), that is, for each $i \in I$, and $x \in X$, there exists an element $y_i \in X_i$ and an open set $D_i \subseteq X$, such that $x \in D_i$ and $y_i \in \bigcap \left\{B_i\left(z^i\right) : z \in D_i\right\}$;*

*2) if $X$ is not compact, assume that $X \backslash C$ can be locally covered with open sets which can be $(B_i, i)$ sectioned with a common element in a nonempty compact convex set $C_{F_i}$, associated to a finite set $F_i \subseteq X_i$ ($i \in I$), that is, for each $i \in I$ and for each finite subset $F_i$ of $X_i$, there exists a nonempty compact convex subset $C_{F_i}$ of $X_i$ such that $C_{F_i} \supseteq F_i$ and for each $x \in X \backslash C$, there exists $y_i \in C_{F_i}$ and an open subset $D_i$ of $X$, such that $x \in D_i$ and $y_i \in \bigcap \left\{B_i\left(z^i\right) : z \in D_i\right\}$.*

*Then, $\bigcap \{A_i : i \in I\} \neq \varnothing$.*

**Proof.** We apply Theorem 2 with $S_i, T_i : X \longrightarrow 2^{X_i}$, the nonempty-valued multimaps given by $S_i(x) = B_i\left(x^i\right)$ and $T_i(x) = A_i\left(x^i\right)$, for each $x \in X (i \in I)$.

Then, for each $i \in I$, we have:

(0) for each $x \in X$, $coS_i(x) \subseteq T_i(x)$;

(1) for each $i \in I$, $X = \bigcup \left\{intS_i^{-1}(y_i) : y_i \in X_i\right\}$;

(2) for each $i \in I$ and for each finite subset $F_i$ of $X_i$, there exists a nonempty compact convex subset $C_{F_i}$ of $X_i$ such that $C_{F_i} \supseteq F_i$ and

$$X \backslash C \subseteq \bigcup \left\{int_X S_i^{-1}(y_i) : y_i \in C_{F_i}\right\}.$$

Therefore, by the Theorem 2, there exists $\widetilde{x} \in X$ such that $\widetilde{x}_i \in T_i(\widetilde{x}) = A_i\left(\widetilde{x}^i\right)$, for each $i \in I$. So, $\widetilde{x} = \left[\widetilde{x}^i, \widetilde{x}_i\right] \in \bigcap \{A_i : i \in I\}$. This completes the proof. □

Similarly, applying Corollary 1 instead of Theorem 2, we can prove the following intersection type theorem.

**Theorem 4.** *In the Theorem 3 we change the condition 2) as follows:*

*2') if $X$ is not compact, assume that $X\backslash C$ can be locally covered with open sets which can be $(B_i, i)$ sectioned with a common element in a nonempty compact convex subset $C_i$ of $X_i$ ($i \in I$), that is, for each $i \in I$, there exists a nonempty compact convex subset $C_i$ of $X_i$ such that for each $x \in X$, there exist $y_i \in C_i$ and an open subset $D_i$ of $X$, such that $x \in D_i$ and $y_i \in \bigcap \left\{B_i\left(z^i\right) : z \in D_i\right\}$.*

*Then, $\bigcap \{A_i : i \in I\} \neq \varnothing$.*

## 5. An equilibrium theorem

Some results concerning the fixed point theory can be applied in the equilibrium theory. For example, using our first intersection theorem and an idea of S. Park (see [8, Theorem 8.2]), we can obtain a Nash-Ma type equilibrium theorem (which, in its analitic form, gives a solution for a system of variational inequalities).

**Theorem 5.** (Nash-Ma type equilibrium theorem) *Let $I$ be an index set having at least two elements and, for each $i \in I$, let $X_i$ be a nonempty compact convex subset in a Hausdorff topological vector space. Let also $X = X^i \times X_i$, where $X^i = \prod\limits_{j \in I\backslash\{i\}} X_j$, and, for each $i \in I$, let $f_i, g_i : X \longrightarrow \mathbb{R}$ be two functions. Suppose that:*

*(1) $g_i(x) \leqslant f_i(x)$, for each $i \in I$ and $x \in X$;*

*(2) for each $x^i \in X^i$, $f_i\left[x^i, \cdot\right]$ is concave on $X_i$;*

*(3) for each $x^i \in X^i$, $g_i\left[x^i, \cdot\right]$ is upper semicontinuous on $X_i$;*

*(4) for each $x_i \in X_i$, $g_i[\cdot, x_i]$ is lower semicontinuous on $X^i$;*

*(5) for any $\varepsilon > 0$, $x \in X$ and $y_i \in X_i$, if $z \in X$ is such that $g_i\left[z^i, y_i\right] > M_i(x) - \varepsilon$, then $g_i\left[z^i, y_i\right] > M_i(z) - \varepsilon$, where, for any $x \in X$, $M_i(x) = \max\limits_{y_i \in X_i} g_i\left[x^i, y_i\right]$; is the "value function" corresponding to $g_i\left[x^i, \cdot\right]$.*

*Then, there exists $\widetilde{x} \in X$ such that $f_i(\widetilde{x}) \geqslant M_i(\widetilde{x})$ for any $i \in I$.*

**Proof.** Firstly, we remark that the "value function" $M_i$ exists, according to *(3)*, because $X_i$ is compact and we can apply [1, Theorem 2.43, p.44].

Now, for all $\varepsilon > 0$ and $i \in I$, we define the following sets:

$$A_{\varepsilon,i} = \{x \in X : f_i(x) > M_i(x) - \varepsilon\}$$
$$B_{\varepsilon,i} = \{x \in X : f_i(x) > M_i(x) - \varepsilon\}$$

We will prove that the hypothesis of Theorem 3 are fulfilled. Indeed, the sets $A_{\varepsilon,i}$ and $B_{\varepsilon,i}$ have nonempty sections, that is, for each $x \in X$,

$$A_{\varepsilon,i}\left(x^i\right) = \left\{y_i \in X_i : \left[x^i, y_i\right] \in A_{\varepsilon,i}\right\} \neq \varnothing$$
$$B_{\varepsilon,i}\left(x^i\right) = \left\{y_i \in X_i : \left[x^i, y_i\right] \in B_{\varepsilon,i}\right\} \neq \varnothing$$

(To prove this, we remark that there exists $y_i \in X_i$ such that

$$M_i(x) - \varepsilon < g_i\left[x^i, y_i\right] \leqslant f_i\left[x^i, y_i\right],$$

that is $y_i \in B_{\varepsilon,i}\left(x^i\right)$ and $y_i \in A_{\varepsilon,i}\left(x^i\right)$ , too.) Also, we have that $coB_{\varepsilon,i}\left(x^i\right) \subseteq A_{\varepsilon,i}\left(x^i\right)$, because $B_{\varepsilon,i} \subseteq A_{\varepsilon,i}$ (from *(1)*) and the set $A_{\varepsilon,i}$ is convex (from *(2)*).

Let now $i \in I$ and $x \in X$, fixed. There exists $y_i \in X_i$ such that

$$g_i\left[x^i, y_i\right] > M_i(x) - \varepsilon.$$

Denote by $D_i$ the set $\left\{z^i \in X^i : g_i\left[z^i, y_i\right] > M_i(x) - \varepsilon\right\} \times X_i$. This set is open, according to *(4)*. Obviously, $x \in D$ and for each $z \in D_i$, it follows that

$$g_i\left[z^i, y_i\right] > M_i(x) - \varepsilon.$$

Then, using *(5)*, we obtain that

$$g_i\left[z^i, y_i\right] > M_i(z) - \varepsilon, \text{ that is } y_i \in \bigcap\left\{B_{\varepsilon,i}\left(z^i\right) \ : \ z \in D_i\right\}.$$

Because $X = \prod_{i \in I} X_i$ is compact, we can apply Theorem 3, obtaining that $\bigcap_{i \in I} A_{\varepsilon,i} \neq \varnothing$, for each $\varepsilon > 0$. If $\widetilde{x}_\varepsilon \in A_{\varepsilon,i}$, for each $i \in I$, it follows that $f_i(\widetilde{x}_\varepsilon) \geqslant M_i(\widetilde{x}_\varepsilon) - \varepsilon$. Because $X$ is compact, there exists $\widetilde{x} \in X$ such that $f_i(\widetilde{x}) \geqslant M_i(\widetilde{x})$ (see the same argument in the proof of Theorem 8.2 in [8]; see also [3, p.56]). □

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