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NOTES ON BROWDER'S AND HALPERN'S METHODS FOR NONEXPANSIVE MAPPINGS

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Abstract. We show that a projection applied to Browder's and Halpern's methods can find the minimum-norm fixed point of a nonexpansive mapping. This supplements the results in existing literature on iterative methods for finding fixed points of nonexpansive mappings. An application to finding the minimum-norm solution of a convex minimization problem is included.

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1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

A point $x \in C$ is said to be a fixed point of T provided $Tx = x$. We will use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$. Note that $F(T)$ is always closed convex (but may be empty). A sufficient condition that guarantees $F(T) \neq \emptyset$ is that there is a point $x \in C$ for which the trajectory $\{T^n x\}$ is bounded (for more details see the monograph [3].)

Suppose that $F(T) \neq \emptyset$. Then $F(T)$ is a nonempty closed convex subset of a Hilbert space; hence there is a unique point $x^\dagger \in F(T)$ with the minimum-norm property

$$\|x^\dagger\| = \min\{\|x\| : x \in F(T)\}. \tag{1.1}$$

In another word, $x^\dagger$ is the metric (nearest point) projection of the origin onto $F(T)$: $x^\dagger = P_{F(T)}(0)$. We here refer $x^\dagger$ to as the minimum-norm fixed point of T.

An interesting problem is how to find $x^\dagger$ through an iterative method? To the best of our knowledge, there is no direct investigation on this problem, in general. Let us review the existing results which relate to this problem. There are basically two methods: Browder's (implicit) method and Halpern's (explicit) method.

Browder's method [2]. Fix $u \in C$. For each $t \in (0,1)$, let x_t be the unique fixed point in C of the contraction from C into C:

$$T_t x = tu + (1-t)Tx, \quad x \in C. \tag{1.2}$$

Browder proved that

$$s - \lim_{t \downarrow 0} x_t = P_{F(T)} u. \tag{1.3}$$

That is, the strong limit of $\{x_t\}$ as $t \to 0^+$ is the fixed point of T which is closest from $F(T)$ to u.

Halpern's method [4]. This is an iterative method. Again fix $u \in C$. For a given sequence $\{t_n\}$ in (0,1) and an initial guess $x_0 \in C$, define a sequence $\{x_n\}$ by the recursive formula:

$$x_{n+1} = t_n u + (1-t_n)Tx_n, \quad n \geq 0. \tag{1.4}$$

It is now known that this sequence $\{x_n\}$ converges in norm to the same limit $P_{F(T)}u$ as Browder's method provided the sequence $\{t_n\}$ satisfies the conditions $(C1)$, $(C2)$, and either $(C3)$ or $(C4)$ (e.g., $t_n = 1/(n+1)$) as follows:

(C1) $\lim_{n\to\infty} t_n = 0$;

(C2) $\displaystyle\sum_{n=1}^{\infty} t_n = \infty$;

(C3) $\displaystyle\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$;

(C4) $\displaystyle\lim_{n\to\infty} (t_n/t_{n+1}) = 1$.

We notice that the above two methods can be used to find the minimum-norm fixed point $x^\dagger$ of T if $0 \in C$. However, if $0 \notin C$, then neither Browder's nor Halpern's method works any more since the contraction $T_t x = (1-t)Tx$ may be a nonself-mapping of C (so that T_t may fail to have a fixed point) or the point $(1-t_n)Tx_n$ may not belong to C (consequently, x_{n+1} may be undefined).

In order to overcome the difficulties caused by possible exclusion of the origin from C, we introduce the following two remedies:

For Browder's method, we consider the contraction $x \mapsto P_C((1-t)Tx)$. Since this contraction clearly maps C into C, it has a unique fixed point which is still denoted by x_t; thus

$$x_t = P_C((1-t)Tx_t). \tag{1.5}$$

For Halpern's method, we consider the following modification of (1.4)

$$x_{n+1} = P_C((1-t_n)Tx_n), \quad n \geq 0. \tag{1.6}$$

It is easily seen that the sequence $\{x_n\}$ is well-defined (i.e., $x_n \in C$ for all n).

Note that if $0 \in C$, then (1.5) and (1.6) are reduced to (1.2) and (1.4) with $u = 0$, respectively.

The purpose of this paper is to prove that both implicit and explicit methods (1.5) and (1.6) converge strongly to the minimum-norm fixed point $x^\dagger$ of the nonexpansive mapping T as $t \to 0^+$ and $n \to \infty$, respectively (for (1.6), we assume $(C1)$, $(C2)$, and either $(C3)$ or $(C4)$). For recent convergence studies on Browder's and Halpern's methods, the reader can consult Lions [5], Marino-Xu [6], O'Hara-Pillay-Xu [7, 8], Reich [10], Wittmann [11], and Xu [12, 13, 14, 15, 16, 17, 18].

2. Preliminaries

Let H be a real Hilbert space and let C be a closed convex subset of H. The metric (or nearest point) projection P_C from H onto C is defined as follows: for each $x \in H$, $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

Proposition 2.1. *(Basic properties of projections.)*

(i) $\langle x - P_C x, y - P_C x\rangle \leq 0$ *for all* $x \in H$ *and* $y \in C$;

(ii) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ *for all* $x, y \in H$.
(iii) $\|x - P_C x\|^2 \leq \|x - y\|^2 - \|y - P_K C\|^2$ *for all* $x \in H$ *and* $y \in C$

Lemma 2.2. *(Demiclosedness Principle) (cf.* [3, 9]*) Let* $T : C \to C$ *be a nonexpansive mapping with* $Fix(T) \neq \emptyset$*. If* $\{x_n\}$ *is a sequence in* C *weakly converging to* x *and if* $\{(I - T)x_n\}$ *converges strongly to* y*, then* $(I - T)x = y$*. In particular, if* $y = 0$*, then* $x \in F(T)$*.*

Lemma 2.3. *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) *either* $\sum_{n=1}^{\infty} \gamma_n |\delta_n| < \infty$ *or* $\limsup_{n\to\infty} \delta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

The following notation is employed:

- $x_n \to x$ means that $x_n \to x$ strongly;
- $x_n \rightharpoonup x$ means that $x_n \to x$ weakly;
- $\omega_w(x_n) := \{x : \exists\, x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}$.

3. Convergence results

Recall that we are concerned with the minimum-norm fixed point $x^\dagger$ of a nonexpansive mapping $T : C \to C$, with C a nonempty closed convex subset of a real Hilbert space H, and we assume $F(T) \neq \emptyset$. In this section we present two methods, one implicit and one explicit, to find $x^\dagger$.

3.1. **Implicit method.**

Theorem 3.1. *Let* H *be a real Hilbert space,* C *a nonempty closed convex subset of* H*, and* $T : C \to C$ *a nonexpansive mapping with* $F(T) \neq \emptyset$*. For each*

$t \in (0,1)$, let x_t be the unique fixed point in C of the contraction $P_C((1-t)T)$; that is, x_t is the unique solution of the fixed point equation (1.5). Then

$$s - \lim_{t \downarrow 0} x_t = x^\dagger.$$

Proof. First we prove that (x_t) is bounded. As a matter of fact, taking any point $p \in F(T)$, we derive that

$$\|x_t - p\| = \|P_C((1-t)T)x_t - p\| \leq \|(1-t)Tx_t - p\| = \|(1-t)(Tx_t - Tp) - tp\|$$

$$\leq \|(1-t)\|x_t - p\| + t\|p\|.$$

This implies that, for all $t \in (0,1)$,

$$\|x_t - p\| \leq \|p\|. \tag{3.1}$$

So (x_t) is bounded. Let $M > 0$ satisfy $M \geq \max\{\|x_t\|, \|Tx_t\|\}$ for all $t \in (0,1)$. We then find that

$$\|x_t - Tx_t\| = \|P_C((1-t)T)x_t - Tx_t\| \leq \|(1-t)Tx_t - Tx_t\| = t\|Tx_t\| \leq tM \to 0,$$

as $t \to 0$. By applying Lemma 2.2 we get that $\omega_w(x_t) \subset F(T)$; that is, if (t_n) is a null sequence in $(0,1)$ such that $x_{t_n} \rightharpoonup \bar{x}$, then $\bar{x} \in F(T)$.

Next set $y_t = (1-t)Tx_t$. Then $x_t = P_C y_t$ and for $\tilde{x} \in F(T)$ we deduce that

$$\begin{aligned} x_t - \tilde{x} &= P_C y_t - \tilde{x} = (y_t - \tilde{x}) + P_C y_t - y_t \\ &= (1-t)(Tx_t - \tilde{x}) + t(-\tilde{x}) + (P_C y_t - y_t). \end{aligned}$$

Using $x_t - \tilde{x}$ to make inner product from both sides of the above equation, we get

$$\begin{aligned} \|x_t - \tilde{x}\|^2 &= (1-t)\langle Tx_t - \tilde{x}, x_t - \tilde{x}\rangle + t\langle -\tilde{x}, x_t - \tilde{x}\rangle + \langle P_C y_t - y_t, x_t - \tilde{x}\rangle \\ &\leq (1-t)\|x_t - \tilde{x}\|^2 + t\langle -\tilde{x}, x_t - \tilde{x}\rangle + \langle P_C y_t - y_t, P_C y_t - \tilde{x}\rangle. \end{aligned} \tag{3.2}$$

However, $\langle P_C y_t - y_t, P_C y_t - \tilde{x}\rangle \leq 0$. It then follows from (3.2) that

$$\|x_t - \tilde{x}\|^2 \leq \langle -\tilde{x}, x_t - \tilde{x}\rangle. \tag{3.3}$$

Now if $\bar{x} \in \omega_w(x_t)$ and $x_{t_n} \rightharpoonup \bar{x}$ for some null sequence (t_n) in $(0,1)$. Then, since $\bar{x} \in F(T)$, we may substitute $\bar{x}$ for $\tilde{x}$ and t_n for t in (3.3) to obtain that $x_{t_n} \to \bar{x}$. Hence, (x_t) is indeed relatively compact (as $t \to 0$) in the norm topology.

Note that (3.3) is equivalent to

$$\|x_t\|^2 \le \langle x_t, \tilde{x}\rangle. \tag{3.4}$$

Hence,

$$\|x_t\| \le \|\tilde{x}\|, \quad t \in (0,1),\ \tilde{x} \in F(T). \tag{3.5}$$

This clearly implies that if $\bar{x} \in \omega_w(x_t) = \omega(x_t)$, then

$$\|\bar{x}\| \le \|\tilde{x}\| \quad \forall \tilde{x} \in F(T).$$

Therefore, $\bar{x} = x^\dagger$, where $x^\dagger$ is the minimum-norm fixed point of T (i.e., $x^\dagger = \arg\min_{x\in F(T)} \|x\|$), and we conclude that $x_t \to x^\dagger$ as $t \to 0$. □

Corollary 3.2. *Let H be a real Hilbert space, C a nonempty closed convex subset of H such that $0 \in C$, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. For each $t \in (0,1)$, let x_t be the unique fixed point in C of the contraction $(1-t)T$ mapping C into C; that is, x_t is the unique solution in C of the fixed point equation $x_t = (1-t)Tx_t$. Then*

$$s - \lim_{t\downarrow 0} x_t = x^\dagger.$$

3.2. **Explicit method.**

Theorem 3.3. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0,1)$ satisfying conditions $(C1)$, $(C2)$, and either $(C3)$ or $(C4)$) introduced in Section 1. Define a sequence $\{x_n\}$ according to the recursive formula (1.6). Then*

$$s - \lim_{n\to\infty} x_n = x^\dagger.$$

Proof. We divide the proof into several steps.

(1) We prove that (x_n) is bounded. Indeed, take a $p \in F(T)$ to deduce from (1.6), that:

$$\|x_{n+1} - p\| = \|P_C((1-t_n)Tx_n) - p\| \le \|(1-t_n)Tx_n - p\|$$

$$= \|(1-t_n)(Tx_n - p) - t_n p\| \le (1-t_n)\|x_n - p\| + t_n\|p\| \le \max\{\|x_n - p\|, \|p\|\}.$$

By induction, we get

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \|p\|\}, \text{ for all } n \ge 0.$$

Hence (x_n) is bounded. Let $M > 0$ such that $M \geq \max\{\|x_n\|, \|Tx_n\|\}$ for all $n \in \mathbb{N}$.

(2) We show that $\|x_{n+1} - x_n\| \to 0$. We compute, using the fact that P_C is nonexpansive,

$$\|x_{n+1} - x_n\| = \|P_C((1-t_n)Tx_n) - P_C((1-t_{n-1})Tx_{n-1})\|$$

$$\leq \|(1-t_n)Tx_n - (1-t_{n-1})Tx_{n-1}\| = \|(1-t_n)(Tx_n - Tx_{n-1}) + (t_{n-1} - t_n)Tx_{n-1}\|$$

$$\leq (1-t_n)\|x_n - x_{n-1}\| + M|t_{n-1} - t_n| \tag{3.6}$$

$$= (1-t_n)\|x_n - x_{n-1}\| + Mt_n|1 - (t_{n-1}/t_n)|. \tag{3.7}$$

Now applying Lemma 2.3 to either (3.6) (if $(C3)$ holds) or (3.7) (if $(C4)$ holds), we find that $\|x_{n+1} - x_n\| \to 0$.

(3) We show that $\omega_n(x_n) \subset F(T)$. Indeed, we have

$$\|x_n - Tx_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\| = \|x_{n+1} - x_n\| + \|P_C((1-t_n)Tx_n) - Tx_n\|$$

$$= \|x_{n+1} - x_n\| + t_n\|Tx_n\| \leq \|x_{n+1} - x_n\| + t_n M \to 0, \text{ as } n \to +\infty.$$

It follows from Lemma 2.2 that $\omega_n(x_n) \subset F(T)$.

(4) We prove that $\limsup_{n\to\infty}\langle x^\dagger - x_n, x^\dagger\rangle \leq 0$. To see this, we take a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^\dagger - x_n, x^\dagger\rangle = \lim_{n'\to\infty}\langle x^\dagger - x_{n'}, x^\dagger\rangle.$$

Since (x_n) is bounded, we may assume without loss of generality that $x_{n'} \rightharpoonup x' \in F(T)$. Since $x^\dagger = P_{F(T)}(0)$, we obtain by Proposition 2.1(i) and from the last equation

$$\limsup_{n\to\infty}\langle x^\dagger - x_n, x^\dagger\rangle = \langle x^\dagger - x', x^\dagger\rangle \leq 0.$$

(5) Finally we prove that $x_n \to x^\dagger$. We compute

$$\|x_{n+1} - x^\dagger\|^2 = \|P_C((1-t_n)Tx_n) - x^\dagger\|^2 \leq \|(1-t_n)Tx_n - x^\dagger\|^2$$

$$= \|(1-t_n)(Tx_n - x^\dagger) - t_n x^\dagger\|^2 = (1-t_n)^2\|Tx_n - x^\dagger\|^2$$

$$+2t_n(1-t_n)\langle Tx_n - x^\dagger, -x^\dagger\rangle + t_n^2\|x^\dagger\|^2 \leq (1-t_n)\|x_n - x^\dagger\|^2 + t_n\delta_n \tag{3.8}$$

where

$$\begin{aligned}
\delta_n &= 2(1-t_n)\langle Tx_n - x^\dagger, -x^\dagger\rangle + t_n\|x^\dagger\|^2 \\
&= 2(1-t_n)(\langle Tx_n - x_n, -x^\dagger\rangle + \langle x_n - x^\dagger, -x^\dagger\rangle) + t_n\|x^\dagger\|^2 \\
&\leq 2\|x_n - Tx_n\|\|x^\dagger\| + 2(1-t_n)\langle x^\dagger - x_n, x^\dagger\rangle + t_n\|x^\dagger\|^2.
\end{aligned}$$

From steps 3 and 4, we get $\limsup_{n\to\infty} \delta_n \le 0$. Then we apply Lemma 2.3 to (3.8) to get $\|x_n - x^\dagger\| \to 0$. This completes the proof. □

Corollary 3.4. *Let H be a real Hilbert space, C a nonempty closed convex subset of H such that $0 \in C$, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0,1)$ satisfying conditions $(C1)$, $(C2)$, and either $(C3)$ or $(C4)$) introduced in Section 1. Define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - t_n)Tx_n, \quad n \ge 0.$$

Then

$$s - \lim_{n\to\infty} x_n = x^\dagger.$$

4. An application

Consider the minimization problem

$$\min_{x\in C} \varphi(x) \tag{4.1}$$

where C is a closed convex subset of a real Hilbert space H and $\varphi : C \to \mathbb{R}$ is a continuously Frechet differentiable convex function. Denote by S the solution set of (4.1); that is,

$$S = \{z \in C : \varphi(z) = \min_{x\in C} \varphi(x)\}.$$

Assume $S \neq \emptyset$.

It is known that a point $z \in C$ is a solution of (4.1) if and only if the following optimality condition holds:

$$z \in C, \quad \langle \nabla\varphi(z), x - z\rangle \ge 0, \quad x \in C. \tag{4.2}$$

(Here $\nabla\varphi(x)$ denotes the gradient of φ at $x \in C$.) It is also known that the optimality condition (4.2) is equivalent to the following fixed point problem

$$z = T_\gamma z, \quad T_\gamma = P_C(I - \gamma\nabla\varphi) \tag{4.3}$$

where P_C is the metric projection onto C and $\gamma > 0$ is any positive number.

Suppose now that the gradient $\nabla\varphi$ is Lipschitz continuous on C; that is, there is a constant $L > 0$ such that

$$\|\nabla\varphi(x) - \nabla\varphi(y)\| \le L\|x - y\|, \quad x, y \in C. \tag{4.4}$$

It is then not hard to see that if $0 < \gamma < 2/L$, the mapping T_γ is nonexpansive. As a matter of fact, noticing the fact that the Lipschitz condition (4.4) implies (see [1]) that the gradient $\nabla\varphi$ satisfies the inequality

$$\langle x - y, \nabla\varphi(x) - \nabla\varphi(y)\rangle \geq \frac{1}{L}\|\nabla\varphi(x) - \nabla\varphi(y)\|^2, \quad x, y \in C, \tag{4.5}$$

we derive that

$$\begin{aligned}\|T_\gamma x - T_\gamma y\|^2 &= \|P_C(I-\gamma\nabla\varphi)x - P_C(I-\gamma\nabla\varphi)y\|^2 \leq \|(x-y) - \gamma(\nabla\varphi(x) - \nabla\varphi(y))\|^2 \\ &= \|x - y\|^2 - 2\gamma\langle x - y, \nabla\varphi(x) - \nabla\varphi(y)\rangle + \gamma^2\|\nabla\varphi(x) - \nabla\varphi(y)\|^2 \\ &\leq \|x - y\|^2 - \gamma\left(\frac{2}{L} - \gamma\right)\|\nabla\varphi(x) - \nabla\varphi(y)\|^2 \leq \|x - y\|^2.\end{aligned}$$

Hence, T_γ is nonexpansive.

Using Theorems 3.1 and 3.3, we immediately arrive at the following result.

Theorem 4.1. *Assume the objective function φ is continuously Frechet differentiable and convex and its gradient $\nabla\varphi$ satisfies the Lipschitz condition (4.4). Assume the solution set S of the minimization (4.1) is nonempty. Fix γ such that $0 < \gamma < 2/L$.*

(i) *For each $t \in (0, 1)$, let x_t be the unique solution to the fixed point equation*

$$x_t = P_C((1 - t)P_C(I - \gamma\nabla\varphi)x_t). \tag{4.6}$$

Then $\{x_t\}$ converges in norm to the minimum-norm solution of the minimization (4.1).

(ii) *Define a sequence $\{x_n\}$ via the recursive algorithm:*

$$x_{n+1} = P_C((1 - t_n)P_C(I - \gamma\nabla\varphi)x_n), \tag{4.7}$$

where the sequence $\{t_n\}$ satisfies the conditions $(C1)$, $(C2)$, and either $(C3)$ or $(C4)$ introduced in Section 1. Then $\{x_n\}$ converges in norm to the minimum-norm solution of the minimization (4.1).

Proof. Replacing the mapping T in both (1.5) and (1.6) with the mapping T_γ and noticing that the solution set S of (4.1) is precisely $F(T_\gamma)$, we see that the conclusions of Theorem 4.1 are immediate consequences of Theorems 3.1 and 3.3. □

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