Fixed Point Theory, 10(2009), No. 1, 35-71
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

VISCOSITY APPROXIMATION METHODS FOR STRONGLY POSITIVE AND MONOTONE OPERATORS

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Abstract. In this paper, we suggest and analyze both explicit and implicit iterative schemes for two strongly positive operators and a nonexpansive mapping S on a Hilbert space. We also study explicit and implicit versions of iterative schemes for an inverse-strongly monotone mapping T and S by an extragradient-like approximation method. The viscosity approximation methods are employed to establish strong convergence of the iterative schemes to a common element of the set of fixed points of S and the set of solutions of the variational inequality for T. As applications, we consider the problem of finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping which solves some variational inequalities. Our results improve and unify various celebrated results of viscosity approximation methods for fixed-point problems and variational inequality problems.

Key Words and Phrases: General iterative method, viscosity approximation method, hybrid viscosity approximation method, fixed points, inverse-strongly monotone mappings, nonexpansive mappings, variational inequalities, strongly positive operators.
2000 Mathematics Subject Classification: 47J20, 49J43, 47H10.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. Recall that a self-mapping f on C is a contraction if there exists a constant $k \in (0,1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$ for all $x, y \in C$. If $k > 0$ (respectively, $k = 1$), then f is Lipschitz continuous (respectively, nonexpansive). Note that each $f \in \Pi_C$, the set of all contractions on C, has a unique fixed point x_0 (that is, $f(x_0) = x_0$) by the Banach contraction principle. We denote by $\text{Fix}(S) = \{x \in C : S(x) = x\}$, the set of fixed points of a nonexpansive mapping S on C. It is well known that $\text{Fix}(S)$ is closed and convex (cf. [5]). A mapping $T : C \to H$ is called monotone if $\langle Tx - Ty, x - y\rangle \geq 0$ for all $x, y \in C$. The variational inequality problem is to find $x \in C$ such that $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$; see, for example, [1, 7]. The set of solutions of the variational inequality is denoted by $\text{VI}(C, T)$. A mapping $T : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

A bounded linear operator A on H is strongly positive [8] with coefficient $\bar{\gamma}$, if there is a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Marino and Xu [8] have combined the iterative methods in [13] with the viscosity approximation method due to Moudafi [9] and introduced the following general iterative algorithm:

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad \forall n \geq 0, \tag{1.1}$$

where A (respectively, S) is strongly positive (respectively, nonexpansive) on H, f is a contraction on C, $x_0 \in H$ and $\{\alpha_n\}$ in $(0,1)$ satisfies appropriate conditions. They proved strong convergence of $\{x_n\}$ generated by (1.1) to $\tilde{x} \in \text{Fix}(S)$, which solves the variational inequality

$$\langle (A - \gamma f)\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad \forall x \in \text{Fix}(S).$$

Readers interested in the related minimization problems are referred to [13, 14].

The iterative methods of nonexpansive mappings have been extensively applied to solving the convex minimization problems and other problems in [8, 14, 13, 6, 4, 15, 2, 3, 17, 16].

Recently, Chen et al. [4] have introduced the explicit and implicit iterative schemes, respectively, in this context as follows:

$$x_0 \in C, \quad x_{n+1} = (1-\alpha_n)SP_C(x_n - \lambda_n T x_n) + \alpha_n f(x_n), \tag{1.2}$$

$$z_n = (1-\alpha_n)SP_C(z_n - \lambda_n T z_n) + \alpha_n f(z_n), \tag{1.3}$$

where $n \geq 0$, $D := \mathrm{Fix}(S) \cap \mathrm{VI}(C,T) \neq \emptyset$ and $\lambda_n \in [a,b]$ with $0 < a < b < 2\alpha$ and $\{\alpha_n\} \subset [0,1)$.

By imposing suitable conditions on $\{\alpha_n\}$ and $\{\lambda_n\}$, they have proved strong convergence of $\{x_n\}$ in (1.2) and $\{z_n\}$ in (1.3) to the unique $q \in D$ which solves the following variational inequality

$$\langle (I-f)(q), q-p\rangle \leq 0, \quad \forall p \in D \text{ and } \forall f \in \Pi_C, \tag{1.4}$$

where I stands for the identity mapping.

Ceng and Yao [3] introduced explicit and implicit versions of an extra-gradient-like approximation method for $x_0 \in C$ as:

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n T x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n SP_C(x_n - \lambda_n T y_n), \quad \forall n \geq 0 \end{cases} \tag{1.5}$$

$$z_n = (1-\alpha_n-\beta_n)z_n + \alpha_n f((1-\gamma_n)z_n + \gamma_n P_C(z_n - \lambda_n T z_n))$$

$$+\beta_n SP_C[(1-\gamma_n)z_n+\gamma_n P_C(z_n-\lambda_n T z_n)-\lambda_n T((1-\gamma_n)z_n+\gamma_n P_C(z_n-\lambda_n T z_n))], \tag{1.6}$$

where $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\alpha$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$.

In this paper, motivated and inspired by the iterative schemes (1.1)-(1.3), we suggest and analyze a more general iterative method for two strongly positive operators, a nonexpansive self-mapping S (respectively, a contraction) on C and an α-inverse-monotone mapping T. Let $A, B : H \to H$ be strongly positive linear bounded operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$. For an arbitrary initial guess $x_0 \in C$, we define the explicit iterative scheme $\{x_n\}$ as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda_n T x_n), \\ x_{n+1} = P_C\{(I - \alpha_n A)S(y_n) + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))]\}, \end{cases} \tag{1.7}$$

where $n \geq 0$, $\{\alpha_n\} \subset (0,1]$, $\{\beta_n\} \subset \big(0, \min\{1, \|B\|^{-1}\}\big)$, and $\{\lambda_n\} \subset (0, 2\alpha)$.

The implicit iterative scheme $\{z_n\}$ is introduced as:

$$z_n = P_C\{(I - \alpha_n A)SP_C(z_n - \lambda_n T z_n)$$

$$+\alpha_n[SP_C(z_n - \lambda_n T z_n) - \beta_n(BSP_C(z_n - \lambda_n T z_n) - \gamma f(z_n))]\}. \tag{1.8}$$

It is proved that under appropriate conditions, the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an α-inverse-strongly monotone mapping which solves some variational inequality problems in a Hilbert space. Similar results are proved for the extragradient-like approximation schemes given in (1.5) and (1.6). The results presented in this paper improve and extend some recent results in the current literature, including the corresponding results of Marino and Xu [8], Iiduka and Takahashi [6], Chen et al. [4] and Ceng and Yao [3].

2. Preliminaries

For a sequence $\{x_n\}$, $x_n \rightharpoonup x$ (respectively, $x_n \to x$) denotes weak (respectively, strong) convergence to x. Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall x \in C.$$

It is well known that the metric projection P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H, \tag{2.1}$$

Moreover, for all $x \in H$, $y \in C$

$$\langle x - P_C x, P_C x - y \rangle \geq 0 \text{ and } \|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2.$$

In the context of variational inequality problem, this implies that

$$u \in VI(C,T) \quad \Leftrightarrow \quad u = P_C(u - \lambda T u), \quad \text{for all } \lambda > 0. \tag{2.2}$$

If T is an α-inverse-strongly monotone mapping of C into H, then obviously T is monotone and is $\frac{1}{\alpha}$-Lipschitz continuous. For all $x, y \in C$ and $\lambda > 0$,

$$\|(I-\lambda T)x-(I-\lambda T)y\|^2 = \|(x-y)-\lambda(Tx-Ty)\|^2 = \|x-y\|^2$$

$$-2\lambda\langle x-y, Tx-Ty\rangle + \lambda^2\|Tx-Ty\|^2 \leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Tx-Ty\|^2.$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda T$ is a nonexpansive mapping of C into H.

A set-valued mapping $F : H \to 2^H$ is called monotone if for all $x, y \in H$, $u \in Fx$ and $v \in Fy$ imply $\langle x-y, u-v\rangle \geq 0$. A monotone mapping $F : H \to 2^H$ is maximal if $G(F)$, the graph of F, is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping F is maximal if and only if for $(x,u) \in H \times H$, $\langle x-y, u-v\rangle \geq 0$ for every $(y,v) \in G(F)$ implies $u \in Fx$. Let T be an inverse-strongly monotone mapping of C into H and let $N_C x$ be the normal cone to C at $x \in C$, that is, $N_C x = \{w \in H : \langle x-y, w\rangle \geq 0, \text{ for all } y \in C\}$. Define

$$Fx = \begin{cases} Tx + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then F is maximal monotone and $0 \in Fx$ if and only if $x \in VI(C,T)$; see, for example, [11].

The following known lemmas will be used in the proof of our main results.

Lemma 2.1. ([14], Lemma 2.1) Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, or equivalently, $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$, or

(ii)$'$ $\sum_{n=0}^{\infty} a_n\beta_n$ is convergent.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2. [5] Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed (that is, whenever $\{x_n\} \rightharpoonup x$ in C and the sequence $\{(I-T)x_n\} \to y$, it follows that $(I-T)x = y$).

Lemma 2.3. ([12], Lemma 2.3) Let C be a closed convex subset of a Hilbert space H, $f : C \to C$ be a contraction with coefficient $k \in (0,1)$, and B be a strongly positive linear bounded operator with coefficient $\beta > 0$. Then, for $0 < \gamma < \frac{\beta}{k}$,

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y \rangle \geq (\beta - \gamma k)\|x - y\|^2, \quad \text{for all } x, y \in C.$$

That is, $B - \gamma f$ is strongly monotone with coefficient $\beta - \gamma k$.

Lemma 2.4. ([8], Lemma 2.5) Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.5. [10] In an inner product space E, for all $x, y, z \in E$ and α, β, $\gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.6. ([12], Lemma 2) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\varrho_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \varrho_n \leq \limsup_{n\to\infty} \varrho_n < 1$. Suppose that $x_{n+1} = \varrho_n x_n + (1 - \varrho_n) z_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.7. In a real Hilbert space H, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \text{for all } x, y \in H.$$

3. Viscosity methods for strongly positive operators

In this section, we study strong convergence of the iterative schemes (1.7) and (1.8) to the unique $q \in D$ which solves a new variational inequality in Hilbert spaces. First of all, we show that the explicit iterative scheme (1.7) is well defined.

In the sequel, we always assume that $f : C \to C$ is a contraction with coefficient $k \in (0,1)$. Let A, B be strongly positive bounded linear operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$ and $\lim_{n\to\infty} \beta_n = \eta \in \left(\frac{1 - \bar{\gamma}}{\beta - \gamma k}, \frac{2 - \bar{\gamma}}{\beta - \gamma k}\right)$. Then, we may assume without loss of generality that there exists $c \in \left(\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$ such that

$$\frac{1 - \bar{\gamma}}{\beta - \gamma k} < c \leq \beta_n < \frac{2 - \bar{\gamma}}{\beta - \gamma k}, \quad \text{for all } n \geq 0. \tag{3.1}$$

Let $T : C \to H$ be an α-inverse-strongly monotone mapping and let S be a nonexpansive self-mapping on C.

For each $n \geq 0$, consider a mapping $V_n : C \to C$ defined by

$$V_n x := P_C\{(I - \alpha_n A)SP_C(x - \lambda_n Tx) + \alpha_n[SP_C(x - \lambda_n Tx) \\ -\beta_n(BSP_C(x - \lambda_n Tx) - \gamma f(x))]\}, \tag{3.2}$$

for all $x \in C$.

By (3.1)-(3.2) and Lemma 2.4, we have

$$\begin{aligned}
\|V_n x - V_n y\| &= \|P_C\{(I - \alpha_n A)SP_C(x - \lambda_n Tx) \\
&+\alpha_n[SP_C(x - \lambda_n Tx) - \beta_n(BSP_C(x - \lambda_n Tx) - \gamma f(x))]\} \\
&-P_C\{(I - \alpha_n A)SP_C(y - \lambda_n Ty) \\
&+\alpha_n[SP_C(y - \lambda_n Ty) - \beta_n(BSP_C(y - \lambda_n Ty) - \gamma f(y))]\}\| \\
&\leq \|\{(I - \alpha_n A)SP_C(x - \lambda_n Tx) \\
&+\alpha_n[SP_C(x - \lambda_n Tx) - \beta_n(BSP_C(x - \lambda_n Tx) - \gamma f(x))]\} \\
&-\{(I - \alpha_n A)SP_C(y - \lambda_n Ty) + \alpha_n[SP_C(y - \lambda_n Ty) \\
&-\beta_n(BSP_C(y - \lambda_n Ty) - \gamma f(y))]\}\| \\
&\leq \|(I-\alpha_n A)SP_C(x-\lambda_n Tx)-(I-\alpha_n A)SP_C(y-\lambda_n Ty)\|+\|\alpha_n[SP_C(x-\lambda_n Tx) \\
&-\beta_n(BSP_C(x - \lambda_n Tx) - \gamma f(x))] - \alpha_n[SP_C(y - \lambda_n Ty) \\
&-\beta_n(BSP_C(y - \lambda_n Ty) - \gamma f(y))]\| \\
&\leq \|I - \alpha_n A\|\|SP_C(x - \lambda_n Tx) - SP_C(y - \lambda_n Ty)\| \\
&+\alpha_n\|(I - \beta_n B)(SP_C(x - \lambda_n Tx) - SP_C(y - \lambda_n Ty)) + \beta_n\gamma(f(x) - f(y))\| \\
&\leq \|I - \alpha_n A\|\|SP_C(I - \lambda_n T)x - SP_C(I - \lambda_n T)y\| \\
&+\alpha_n[\|I - \beta_n B\|\|SP_C(I - \lambda_n T)x - SP_C(I - \lambda_n T)y\| + \beta_n\gamma\|f(x) - f(y)\|] \\
&\leq (1 - \alpha_n\bar{\gamma})\|x - y\| + \alpha_n[(1 - \beta_n\beta)\|x - y\| + \beta_n\gamma k\|x - y\|] \\
&= [1 - \alpha_n(\bar{\gamma} - 1 + \beta_n(\beta - \gamma k))]\|x - y\| = (1 - \alpha_n\tau_n)\|x - y\|,
\end{aligned}$$

where $\tau_n := \bar{\gamma} - 1 + \beta_n(\beta - \gamma\alpha)$.

Since $c \in \left(\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$, if we denote $\tau := \bar{\gamma} - 1 + c(\beta - \gamma\alpha) \in (0, 1)$, then we have

$$\tau_n = \bar{\gamma} - 1 + \beta_n(\beta - \gamma k) \geq \bar{\gamma} - 1 + c(\beta - \gamma k) = \tau.$$

Hence we get

$$\|V_n x - V_n y\| \leq (1 - \alpha_n\tau)\|x - y\|. \tag{3.3}$$

This shows that V_n is a contraction. Therefore, by the Banach contraction principle, V_n has a unique fixed point $z_n \in C$ such that

$$z_n = P_C\{(I - \alpha_n A)SP_C(z_n - \lambda_n T z_n) + \alpha_n[SP_C(z_n - \lambda_n T z_n) - \beta_n(BSP_C(z_n - \lambda_n T z_n) - \gamma f(z_n))]\}.$$

Hence the scheme (1.7) is well-defined.

Note that z_n indeed depends on f but we will suppress this dependence in the sequel for simplicity of notation.

We now prove our strong convergence theorems for a nonexpansive mapping, an α-inverse-strongly monotone mapping and two strongly positive operators.

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction, $T : C \to H$ an α-inverse-strongly monotone mapping and S a nonexpansive self-mapping on C such that $D \neq \emptyset$. Let A, B be strongly positive bounded linear operators on H. Let $0 < \gamma < \frac{\beta}{k}$. Suppose that $\{x_n\}$ is the sequence in C generated by $x_0 \in C$ as in (1.7). Assume that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\lim_{n\to\infty} \beta_n = \eta \in (\frac{1-\bar{\gamma}}{\beta - \gamma k}, \frac{2-\bar{\gamma}}{\beta - \gamma k})$ and $\sum_{n=0}^{\infty} |\alpha_{n+1}\beta_{n+1} - \alpha_n \beta_n| < \infty$;

(C3) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to the unique $q \in D$ which solves the following variational inequality:

$$\langle [A - I + \eta(B - \gamma f)]q, q - p\rangle \leq 0, \quad \text{for all } p \in D \text{ and } f \in \Pi_C. \tag{3.4}$$

Proof. We may assume, in view of $\lim_{n\to\infty} \alpha_n = 0$, that $\alpha_n < \|A\|^{-1}$.

By Lemma 2.4, we obtain

$$\|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}, \text{ for every } n \in \mathbb{N}.$$

Let $p \in D$. Then we have:

$$\|y_n - p\| = \|P_C(x_n - \lambda_n T x_n) - P_C(p - \lambda_n T p)\|$$

$$= \|P_C(I - \lambda_n T)x_n - P_C(I - \lambda_n T)p\| \leq \|x_n - p\|, \text{ for every } n \in \mathbb{N}.$$

Observe that $V_n p =$

$$P_C\{(I-\alpha_n A)SP_C(p-\lambda_n Tp)+\alpha_n[SP_C(p-\lambda_n Tp)-\beta_n(BSP_C(p-\lambda_n Tp)-\gamma f(p))]\}$$

$$= P_C\{(I-\alpha_n A)p + \alpha_n[p-\beta_n(Bp-\gamma f(p))]\}.$$

Then from (3.3) we have

$$\|x_{n+1}-p\| = \|V_n x_n - V_n p + V_n p - p\| \le \|V_n x_n - V_n p\| + \|V_n p - p\|$$

$$\le (1-\alpha_n\tau)\|x_n - p\| + \|P_C\{(I-\alpha_n A)p + \alpha_n[p-\beta_n(Bp-\gamma f(p))]\} - P_C p\|$$

$$\le (1-\alpha_n\tau)\|x_n - p\| + \|(I-\alpha_n A)p + \alpha_n[p-\beta_n(Bp-\gamma f(p))] - p\|$$

$$\le (1-\alpha_n\tau)\|x_n - p\| + \alpha_n\| - Ap + p - \beta_n(Bp-\gamma f(p))\|$$

$$\le (1-\alpha_n\tau)\|x_n - p\| + \alpha_n[\|A-I\|\|p\| + \|B\|\|p\| + \gamma\|f(p)\|],$$

which implies that

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|A-I\|\|p\| + \|B\|\|p\| + \gamma\|f(p)\|}{\tau}\right\}, \quad \text{for all } n \ge 0.$$

So, $\{x_n\}$ is bounded and hence $\{y_n\}, \{Sy_n\}, \{Tx_n\}$ and $\{f(x_n)\}$ are bounded. Since $I-\lambda_n T$ is nonexpansive and $p = P_C(p-\lambda_n Tp)$, we also have

$$\|y_{n+1}-y_n\| \le \|(x_{n+1}-\lambda_{n+1}Tx_{n+1}) - (x_n - \lambda_n Tx_n)\|$$

$$\le \|(I-\lambda_{n+1}T)x_{n+1} - (I-\lambda_{n+1}T)x_n\| + \|(I-\lambda_{n+1}T)x_n - (I-\lambda_n T)x_n\|$$

$$\le \|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|Tx_n\|, \text{ for every } n \in \mathbb{N}.$$

Denote by M a positive constant such that $M \ge \|(A-I)Sy_n\| + \|BSy_n\| + \gamma\|f(x_n)\|$ for every $n \in \mathbb{N}$. Thus, it follows that:

$$\begin{aligned}
&\|P_C\{(I-\alpha_n A)Sy_n+\alpha_n[Sy_n-\beta_n(BSy_n-\gamma f(x_n))]\}\\
&\ -P_C\{(I-\alpha_n A)Sy_{n-1}+\alpha_n[Sy_{n-1}-\beta_n(BSy_{n-1}-\gamma f(x_{n-1}))]\}\|\\
&\le \|\{(I-\alpha_n A)Sy_n+\alpha_n[Sy_n-\beta_n(BSy_n-\gamma f(x_n))]\}\\
&\ -\{(I-\alpha_n A)Sy_{n-1}+\alpha_n[Sy_{n-1}-\beta_n(BSy_{n-1}-\gamma f(x_{n-1}))]\}\|\\
&= \|(I-\alpha_n A)Sy_n-(I-\alpha_n A)Sy_{n-1}+\alpha_n[(I-\beta_n B)Sy_n\\
&-(I-\beta_n B)Sy_{n-1}+\beta_n\gamma(f(x_n)-f(x_{n-1}))]\|\\
&\le \|(I-\alpha_n A)\|\|Sy_n-Sy_{n-1}\|+\alpha_n[\|I-\beta_n B\|\|Sy_n-Sy_{n-1}\|\\
&\ +\beta_n\gamma\|f(x_n)-f(x_{n-1})\|]\\
&\le (1-\alpha_n\bar{\gamma})\|y_n-y_{n-1}\|+\alpha_n(1-\beta_n\beta)\|y_n-y_{n-1}\|\\
&\ +\alpha_n\beta_n\gamma k\|x_n-x_{n-1}\|\\
&\le [(1-\alpha_n\bar{\gamma})+\alpha_n(1-\beta_n\beta)][\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Tx_{n-1}\|]\\
&\ +\alpha_n\beta_n\gamma k\|x_n-x_{n-1}\|\\
&= [1-\alpha_n(\bar{\gamma}-1+\beta_n(\beta-\gamma k))]\|x_n-x_{n-1}\|+[(1-\alpha_n(\bar{\gamma}\\
&\ -1+\beta_n\beta)]|\lambda_n-\lambda_{n-1}|\|Tx_{n-1}\|\\
&\le [1-\alpha_n(\bar{\gamma}-1+\beta_n(\beta-\gamma k))](\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Tx_{n-1}\|)\\
&= (1-\alpha_n\tau_n)(\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Tx_{n-1}\|)\\
&\le (1-\alpha_n\tau)\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Tx_{n-1}\|.
\end{aligned} \tag{3.5}$$

Furthermore, note that

$$\begin{aligned}
&\|P_C\{(I-\alpha_n A)Sy_{n-1}+\alpha_n[Sy_{n-1}-\beta_n(BSy_{n-1}-\gamma f(x_{n-1}))]\}\\
&\ -P_C\{(I-\alpha_{n-1}A)Sy_{n-1}+\alpha_{n-1}[Sy_{n-1}-\beta_{n-1}(BSy_{n-1}-\gamma f(x_{n-1}))]\}\|\\
&\le \|\{(I-\alpha_n A)Sy_{n-1}+\alpha_n[Sy_{n-1}-\beta_n(BSy_{n-1}-\gamma f(x_{n-1}))]\}\\
&\ -\{(I-\alpha_{n-1}A)Sy_{n-1}+\alpha_{n-1}[Sy_{n-1}-\beta_{n-1}(BSy_{n-1}-\gamma f(x_{n-1}))]\}\|\\
&= \|(I-\alpha_n(A-I))Sy_{n-1}-\alpha_n\beta_n BSy_{n-1}+\alpha_n\beta_n\gamma f(x_{n-1})\\
&\ -(I-\alpha_{n-1}(A-I))Sy_{n-1}+\alpha_{n-1}\beta_{n-1}BSy_{n-1}-\alpha_{n-1}\beta_{n-1}\gamma f(x_{n-1})\|\\
&\le |\alpha_n-\alpha_{n-1}|\|(A-I)Sy_{n-1}\|+|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|\|BSy_{n-1}\|\\
&\ +|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|\gamma\|f(x_{n-1})\|\\
&\le M|\alpha_n-\alpha_{n-1}|+M|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|,
\end{aligned} \tag{3.6}$$

So from (3.5) and (3.6) we derive

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \|P_C\{(I-\alpha_n A)Sy_n+\alpha_n[Sy_n-\beta_n(BSy_n-\gamma f(x_n))]\}\\
&\quad -P_C\{(I-\alpha_n A)Sy_{n-1}+\alpha_n[Sy_{n-1}-\beta_n(BSy_{n-1}-\gamma f(x_{n-1}))]\}\|
\end{aligned}$$

$$
\begin{aligned}
&+\|P_C\{(I-\alpha_n A)Sy_{n-1}+\alpha_n[Sy_{n-1}-\beta_n(BSy_{n-1}-\gamma f(x_{n-1}))]\}\\
&-P_C\{(I-\alpha_{n-1}A)Sy_{n-1}+\alpha_{n-1}[Sy_{n-1}-\beta_{n-1}(BSy_{n-1}-\gamma f(x_{n-1}))]\}\|\\
&\le (1-\alpha_n\tau)\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Tx_{n-1}\|\\
&+M|\alpha_n-\alpha_{n-1}|+M|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|\\
&\le (1-\alpha_n\tau)\|x_n-x_{n-1}\|+L|\lambda_n-\lambda_{n-1}|\\
&+M|\alpha_n-\alpha_{n-1}|+M|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|
\end{aligned}
$$

for every $n=0,1,2,...$, where L is a positive constant such that $L\ge\|Tx_n\|$ for every $n=0,1,2,....$ Since $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ and $\sum_{n=0}^{\infty}|\alpha_{n+1}\beta_{n+1}-\alpha_n\beta_n|<\infty$, so by Lemma 2.1, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Then we obtain $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$.

Now observe that

$$
\begin{aligned}
\|x_n-Sy_n\| &\le \|x_n-Sy_{n-1}\|+\|Sy_{n-1}-Sy_n\|\\
&=\|P_C\{(I-\alpha_{n-1}A)Sy_{n-1}+\alpha_{n-1}[Sy_{n-1}-\beta_{n-1}(BSy_{n-1}-\gamma f(x_{n-1}))]\}\\
&\quad -P_CSy_{n-1}\|+\|Sy_{n-1}-Sy_n\|\\
&\le \|\{(I-\alpha_{n-1}A)Sy_{n-1}+\alpha_{n-1}[Sy_{n-1}-\beta_{n-1}(BSy_{n-1}-\gamma f(x_{n-1}))]\}-Sy_{n-1}\|\\
&\quad +\|y_{n-1}-y_n\|\\
&=\|-\alpha_{n-1}ASy_{n-1}+\alpha_{n-1}[(I-\beta_{n-1}B)Sy_{n-1}+\beta_{n-1}\gamma f(x_{n-1})]\|\\
&\quad +\|y_{n-1}-y_n\|\\
&\le \alpha_{n-1}[\|ASy_{n-1}\|+(1-\beta_{n-1}\beta)\|Sy_{n-1}\|+\beta_{n-1}\gamma\|f(x_{n-1})\|]\\
&\quad +\|y_{n-1}-y_n\|\\
&\le \alpha_{n-1}[\|ASy_{n-1}\|+\|Sy_{n-1}\|+\gamma\|f(x_{n-1})\|]+\|y_{n-1}-y_n\|.
\end{aligned}
$$

Hence we have $\lim_{n\to\infty}\|x_n-Sy_n\|=0$. For $p\in D$,

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P_C\{(I-\alpha_n A)Sy_n+\alpha_n[Sy_n-\beta_n(BSy_n-\gamma f(x_n))]\}-p\|^2\\
&\le \|(I-\alpha_n A)Sy_n+\alpha_n[Sy_n-\beta_n(BSy_n-\gamma f(x_n))]-p\|^2\\
&=\|(I-\alpha_n A)(Sy_n-p)+\alpha_n[Sy_n-\beta_n(BSy_n-\gamma f(x_n))-Ap]\|^2\\
&=\|(I-\alpha_n A)(Sy_n-p)\|^2+\alpha_n^2\|Sy_n-\beta_n(BSy_n-\gamma f(x_n))-Ap\|^2\\
&\quad +2\alpha_n\langle (I-\alpha_n A)(Sy_n-p), Sy_n-\beta_n(BSy_n-\gamma f(x_n))-Ap\rangle\\
&\le (1-\alpha_n\bar{\gamma})^2\|y_n-p\|^2+\alpha_n^2\|(I-\beta_n B)Sy_n+\beta_n\gamma f(x_n)-Ap\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\alpha_n(1-\alpha_n\bar{\gamma})\|y_n-p\|\|(I-\beta_nB)Sy_n+\beta_n\gamma f(x_n)-Ap\| \\
&\le (1-\alpha_n\bar{\gamma})\|y_n-p\|^2+\alpha_n^2\|(I-\beta_nB)Sy_n+\beta_n\gamma f(x_n)-Ap\|^2 \\
&\quad +2\alpha_n\|y_n-p\|\|(I-\beta_nB)Sy_n+\beta_n\gamma f(x_n)-Ap\| \\
&\le (1-\alpha_n\bar{\gamma})\|y_n-p\|^2+\alpha_n^2[(1-\beta_n\beta)\|Sy_n\|+\beta_n\gamma\|f(x_n)\|+\|Ap\|]^2 \\
&\quad +2\alpha_n\|y_n-p\|[(1-\beta_n\beta)\|Sy_n\|+\beta_n\gamma\|f(x_n)\|+\|Ap\|] \\
&\le (1-\alpha_n\bar{\gamma})\|y_n-p\|^2+\alpha_n^2[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]^2 \\
&\quad +2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|] \\
&\le (1-\alpha_n\bar{\gamma})[\|x_n-p\|^2+\lambda_n(\lambda_n-2\alpha)\|Tx_n-Tp\|^2] \\
&\quad +\alpha_n^2[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]^2 \\
&\quad +2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]. \qquad (3.7)
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&(1-\alpha_n\bar{\gamma})a(2\alpha-b)\|Tx_n-Tp\|^2 \\
&\le (1-\alpha_n\bar{\gamma})\lambda_n(2\alpha-\lambda_n)\|Tx_n-Tp\|^2 \\
&\le (1-\alpha_n\bar{\gamma})\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]^2 \\
&\quad +2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|] \\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_n^2[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]^2 \\
&\quad +2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|].
\end{aligned}
$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, and $\{x_n\}, \{y_n\}, \{Sy_n\}$ and $\{f(x_n)\}$ are bounded, so we have $\lim_{n\to\infty} \|Tx_n - Tp\| = 0$. Further, from (2.1) we obtain

$$
\begin{aligned}
\|y_n-p\|^2 &= \|P_C(x_n-\lambda_nTx_n)-P_C(p-\lambda_nTp)\|^2 \\
&\le \langle (x_n-\lambda_nTx_n)-(p-\lambda_nTp), y_n-p\rangle \\
&= \frac{1}{2}\{\|(x_n-\lambda_nTx_n)-(p-\lambda_nTp)\|^2+\|y_n-p\|^2 \\
&\quad -\|(x_n-\lambda_nTx_n)-(p-\lambda_nTp)-(y_n-p)\|^2\} \\
&\le \frac{1}{2}\{\|x_n-p\|^2+\|y_n-p\|^2-\|x_n-y_n\|^2+2\lambda_n\langle x_n-y_n, Tx_n-Tp\rangle \\
&\quad -\lambda_n^2\|Tx_n-Tp\|^2\}.
\end{aligned}
$$

Hence, we get

$$
\|y_n-p\|^2 \le \|x_n-p\|^2-\|x_n-y_n\|^2+2\lambda_n\langle x_n-y_n, Tx_n-Tp\rangle-\lambda_n^2\|Tx_n-Tp\|^2.
$$

Consequently, from (3.7) we derive

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n\bar{\gamma})\|y_n-p\|^2+\alpha_n^2[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]^2\\&\quad+2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]\\&\le \|x_n-p\|^2-(1-\alpha_n\bar{\gamma})\|x_n-y_n\|^2\\&\quad+2(1-\alpha_n\bar{\gamma})\lambda_n\langle x_n-y_n,Tx_n-Tp\rangle\\&\quad-(1-\alpha_n\bar{\gamma})\lambda_n^2\|Tx_n-Tp\|^2+\alpha_n^2[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]^2\\&\quad+2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|],\end{aligned}$$

and hence

$$\begin{aligned}&(1-\alpha_n\bar{\gamma})\|x_n-y_n\|^2\\&\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2(1-\alpha_n\bar{\gamma})\lambda_n\langle x_n-y_n,Tx_n-Tp\rangle\\&\quad-(1-\alpha_n\bar{\gamma})\lambda_n^2\|Tx_n-Tp\|^2+\alpha_n^2[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|]^2\\&\quad+2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Ap\|].\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$, $\|Tx_n-Tp\|\to 0$ and $\{x_n\},\{y_n\},\{Sy_n\}$ and $\{f(x_n)\}$ are bounded, so we obtain $\lim_{n\to\infty}\|x_n-y_n\|=0$.

By Lemma 2.3, it follows that for $0<\gamma<\frac{\beta}{k}$, $B-\gamma f: C\to H$ is strongly monotone with coefficient $\beta-\gamma k>0$. Hence for $\lim_{n\to\infty}\beta_n=\eta\in\left(\dfrac{1-\bar{\gamma}}{\beta-\gamma k},\dfrac{2-\bar{\gamma}}{\beta-\gamma k}\right)$, we have $\bar{\gamma}-1+\eta(\beta-\gamma k)\in(0,1)$ and

$$\begin{aligned}&\langle(A-I+\eta(B-\gamma f))x-(A-I+\eta(B-\gamma f))y,x-y\rangle\\&=\langle A(x-y),x-y\rangle-\|x-y\|^2+\eta\langle(B-\gamma f)x-(B-\gamma f)y,x-y\rangle\\&\ge\bar{\gamma}\|x-y\|^2-\|x-y\|^2+\eta(\beta-\gamma k)\|x-y\|^2\\&=[\bar{\gamma}-1+\eta(\beta-\gamma k)]\|x-y\|^2,\quad \text{for all } x,y\in C,\end{aligned}$$

that is, $A-I+\eta(B-\gamma f): C\to H$ is $\bar{\gamma}-1+\eta(\beta-\gamma k)$-strongly monotone. Furthermore, it is clear that $A-I+\eta(B-\gamma f)$ is Lipschitz continuous with coefficient $\|A-I\|+\eta(\|B\|+\gamma k)>0$. This implies that $A-I+\eta(B-\gamma f)$ is also strongly monotone and Lipschitz continuous on D. In this case, it is well known that the variational inequality

$$\langle[A-I+\eta(B-\gamma f)]q,x-q\rangle\ge 0,\quad \text{for all } x\in D,$$

has the unique solution $q\in D$.

Choose a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle q-\beta_n(Bq-\gamma f(q))-Aq, Sy_n-q\rangle\\&=\lim_{i\to\infty}\langle -[A-I+\beta_{n_i}(B-\gamma f)]q, Sy_{n_i}-q\rangle.\end{aligned}\tag{3.8}$$

As $\{y_{n_i}\}$ is bounded, we have a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to z. We may assume without loss of generality that $y_{n_i} \rightharpoonup z$. Since $\|Sy_n - y_n\| \to 0$, we obtain $Sy_{n_i} \rightharpoonup z$. Then we can obtain $z \in D$. Indeed, let us first show that $z \in \mathrm{VI}(C,T)$. Let

$$Fx = \begin{cases} Tx + N_C x, & x \in C,\\ \emptyset, & x \notin C.\end{cases}$$

Then F is maximal monotone. Let $(x,u) \in \mathrm{G}(F)$. Since $u - Tx \in N_C x$ and $y_n \in C$, we have

$$\langle x - y_n, u - Tx\rangle \geq 0.$$

On the other hand, from $y_n = P_C(x_n - \lambda_n T x_n)$, we have

$$\langle x - y_n, y_n - (x_n - \lambda_n T x_n)\rangle \geq 0$$

and hence

$$\left\langle x - y_n, \frac{y_n - x_n}{\lambda_n} + Tx_n\right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}\langle x - y_{n_i}, u\rangle &\geq \langle x - y_{n_i}, Tx\rangle\\ &\geq \langle x - y_{n_i}, Tx\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + Tx_{n_i}\right\rangle\\ &= \left\langle x - y_{n_i}, Tx - Tx_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle\\ &= \langle x - y_{n_i}, Tx - Ty_{n_i}\rangle + \langle x - y_{n_i}, Ty_{n_i} - Tx_{n_i}\rangle\\ &\quad - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle\\ &\geq \langle x - y_{n_i}, Ty_{n_i} - Tx_{n_i}\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle.\end{aligned}$$

Hence as $i \to \infty$, we have $\langle x - z, u\rangle \geq 0$. Since F is maximal monotone, so $z \in F^{-1}0$ and hence $z \in \mathrm{VI}(C,T)$. Also, note that

$$\|x_n - Sx_n\| \leq \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \leq \|x_n - Sy_n\| + \|x_n - y_n\|.$$

So, we have $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. By Lemma 2.2, we obtain $z \in \mathrm{Fix}(S)$. Thus we have that $z \in D$. Therefore, for $z, q \in D$, we conclude from (3.8) and $\beta_n \to \eta$ that

$$\begin{aligned}&\limsup_{n\to\infty}\langle q - \beta_n(Bq - \gamma f(q)) - Aq, Sy_n - q\rangle\\ &\quad = \langle -[A - I + \eta(B - \gamma f)]q, z - q\rangle \le 0.\end{aligned}$$

Finally, we claim that $\lim_{n\to\infty}\|x_n - q\| = 0$. Indeed, observe that

$$\begin{aligned}&\langle Sy_n - q, (I - \beta_n B)Sy_n + \beta_n\gamma f(x_n) - Aq\rangle\\ &= \langle Sy_n - q, (I - \beta_n B)Sy_n + \beta_n\gamma f(x_n) - [(I - \beta_n B)q + \beta_n\gamma f(q)]\rangle\\ &\ + \langle Sy_n - q, (I - \beta_n B)q + \beta_n\gamma f(q) - Aq\rangle\\ &\le \|Sy_n - q\|[\|(I - \beta_n B)Sy_n - (I - \beta_n B)q\| + \beta_n\gamma\|f(x_n) - f(q)\|]\\ &\ + \langle Sy_n - q, (I - \beta_n B)q + \beta_n\gamma f(q) - Aq\rangle\\ &\le \|Sy_n - q\|[\|I - \beta_n B\|\|Sy_n - q\| + \beta_n\gamma k\|x_n - q\|]\\ &\ + \langle Sy_n - q, (I - \beta_n B)q + \beta_n\gamma f(q) - Aq\rangle\\ &\le \|x_n - q\|[(1 - \beta_n\beta)\|x_n - q\| + \beta_n\gamma k\|x_n - q\|]\\ &\ + \langle Sy_n - q, (I - \beta_n B)q + \beta_n\gamma f(q) - Aq\rangle\\ &= (1 - \beta_n(\beta - \gamma k))\|x_n - q\|^2 + \langle -[A - I + \beta_n(B - \gamma f)]q, Sy_n - q\rangle,\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|P_C\{(I - \alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))]\} - q\|^2\\ &\le \|(I - \alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))] - q\|^2\\ &= \|(I - \alpha_n A)(Sy_n - q) + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n)) - Aq]\|^2\\ &= \|(I - \alpha_n A)(Sy_n - q)\|^2 + \alpha_n^2\|(I - \beta_n B)Sy_n + \beta_n\gamma f(x_n) - Aq\|^2\\ &\quad +2\alpha_n\langle (I - \alpha_n A)(Sy_n - q), (I - \beta_n B)Sy_n + \beta_n\gamma f(x_n) - Aq\rangle\\ &= \|(I - \alpha_n A)(Sy_n - q)\|^2 + \alpha_n^2\{[\|(I - \beta_n B)Sy_n + \beta_n\gamma f(x_n) - Aq\|^2\\ &\quad -2\langle A(Sy_n - q), (I - \beta_n B)Sy_n + \beta_n\gamma f(x_n) - Aq\rangle]\}\\ &\quad +2\alpha_n\langle Sy_n - q, (I - \beta_n B)Sy_n + \beta_n\gamma f(x_n) - Aq\rangle\\ &\le (1 - \alpha_n\bar{\gamma})^2\|Sy_n - q\|^2 + \alpha_n^2\{[(1 - \beta_n\beta)\|Sy_n\| + \beta_n\gamma\|f(x_n)\| + \|Aq\|]^2\\ &\quad +2\|A(Sy_n - q)\|[(1 - \beta_n\beta)\|Sy_n\| + \beta_n\gamma\|f(x_n)\| + \|Aq\|]\}\\ &\quad +2\alpha_n\langle Sy_n - q, (I - \beta_n B)Sy_n + \beta_n\gamma f(x_n) - Aq\rangle\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n-q\|^2+\alpha_n^2\{[\|Sy_n\|+\gamma\|f(x_n)\|+\|Aq\|]^2\\
&\quad +2\|A(Sy_n-q)\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Aq\|]\}\\
&\quad +2\alpha_n\langle Sy_n-q,(I-\beta_nB)Sy_n+\beta_n\gamma f(x_n)-Aq\rangle\\
&\leq (1-2\alpha_n\bar{\gamma}+\alpha_n^2\bar{\gamma}^2)\|x_n-q\|^2+\alpha_n^2\{[\|Sy_n\|+\gamma\|f(x_n)\|+\|Aq\|]^2\\
&\quad +2\|A(Sy_n-q)\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Aq\|]\}\\
&\quad +2\alpha_n\{(1-\beta_n(\beta-\gamma k))\|x_n-q\|^2+\langle-[A-I+\beta_n(B-\gamma f)]q,Sy_n-q\rangle\}\\
&= [1-2\alpha_n(\bar{\gamma}-1+\beta_n(\beta-\gamma k))]\|x_n-q\|^2+\alpha_n^2\{[\|Sy_n\|+\gamma\|f(x_n)\|+\|Aq\|]^2\\
&\quad +2\|A(Sy_n-q)\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Aq\|]+\bar{\gamma}^2\|x_n-q\|^2\}\\
&\quad +2\alpha_n\langle-[A-I+\beta_n(B-\gamma f)]q,Sy_n-q\rangle\\
&\leq (1-\bar{\alpha}_n)\|x_n-q\|^2+\bar{\alpha}_n\bar{\beta}_n,
\end{aligned}$$

where $\bar{\alpha}_n=2\alpha_n\tau$ and

$$\begin{aligned}
\bar{\beta}_n=&\frac{\alpha_n}{2\tau}\{[\|Sy_n\|+\gamma\|f(x_n)\|+\|Aq\|]^2\\
&+2\|A(Sy_n-q)\|[\|Sy_n\|+\gamma\|f(x_n)\|+\|Aq\|]+\bar{\gamma}^2\|x_n-q\|^2\}\\
&+\frac{1}{\tau}\langle-[A-I+\beta_n(B-\gamma f)]q,Sy_n-q\rangle.
\end{aligned}$$

It is easily seen that $\bar{\alpha}_n\to 0$, $\sum_{n=0}^{\infty}\bar{\alpha}_n=\infty$, and $\limsup_{n\to\infty}\bar{\beta}_n\leq 0$, so by Lemma 2.1, we obtain that $x_n\to q$. □

Remark 3.1. In the definition of strongly positive operator A, we may assume without loss of generality that $\bar{\gamma}<1$. Consequently, whenever $0<\gamma<\frac{\bar{\gamma}}{k}$, $B=I$ and $\beta=1$, we have

$$\frac{1-\bar{\gamma}}{\beta-\gamma k}=\frac{1-\bar{\gamma}}{1-\gamma k}<1<\frac{2-\bar{\gamma}}{1-\gamma k}=\frac{2-\bar{\gamma}}{\beta-\gamma k}.$$

Thus we can pick $\beta_n=1$ for all $n\geq 0$ and so, as an immediate consequence of Theorem 3.1, we obtain:

Corollary 3.1. Let C, H, f, T, S and D be as in Theorem 3.1. Let A be a strongly positive bounded linear operator on H. Let $0<\gamma<\frac{\bar{\gamma}}{k}$. Suppose that $\{x_n\}$ is a sequence in C generated by $x_0\in C$ as:

$$\begin{cases} y_n=P_C(x_n-\lambda_nTx_n),\\ x_{n+1}=P_C\{(I-\alpha_nA)Sy_n+\alpha_n\gamma f(x_n)\}, \quad \text{for all } n\geq 0.\end{cases}$$

Assume that $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen such that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$ and (C1) and (C3) hold. Then $\{x_n\}$ converges strongly to the unique $q \in D$ which solves the following variational inequality:

$$\langle (A - \gamma f)q, q - p\rangle \leq 0, \quad \text{for all } p \in D \text{ and for all } f \in \Pi_C. \tag{3.9}$$

Remark 3.2. Putting $A = I$ and $\gamma = 1$ in Corollary 3.1, we obtain Proposition 3.1 in [4]. Furthermore, if $f(x_n) = x$, then we obtain Theorem 3.1 in [6].

The well-definedness of the implicit iterative scheme (1.8) can be obtained in the same way as that of the explicit scheme (1.7) on the basis of (3.3) available here as well.

Theorem 3.2. Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction, $T : C \to H$ an α-inverse-strongly monotone mapping, S a nonexpansive self-mapping on C such that $D \neq \emptyset$ and A, B strongly positive bounded linear operators on H. Let $0 < \gamma < \frac{\beta}{k}$. Suppose that $\{z_n\}$ is the sequence in C generated by (1.8). If $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta - \gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$, then $\{z_n\}$ converges strongly to the unique $q \in D$ which solves the variational inequality (3.4).

Proof. As in the proof of Theorem 3.1, we have

$$\frac{1-\bar{\gamma}}{\beta-\gamma k} < c \leq \beta_n < \frac{2-\bar{\gamma}}{\beta-\gamma k}, \quad \text{for all } n \geq 0.$$

Put $y_n = P_C(z_n - \lambda_n T z_n)$ for every $n = 0, 1, 2, \ldots.$ Let $p \in D$. We have

$$\begin{aligned} \|y_n - p\| &= \|P_C(z_n - \lambda_n T z_n) - P_C(p - \lambda_n T p)\| \\ &= \|P_C(I - \lambda_n T)z_n - P_C(I - \lambda_n T)p\| \\ &\leq \|z_n - p\| \end{aligned}$$

for every $n = 0, 1, 2, \ldots.$ Observe that

$$\begin{aligned} V_n p = P_C\{&(I - \alpha_n A)SP_C(p - \lambda_n T p) \\ &+ \alpha_n[SP_C(p - \lambda_n T p) - \beta_n(BSP_C(p - \lambda_n T p) - \gamma f(p))]\} \\ &= P_C\{(I - \alpha_n A)p + \alpha_n[p - \beta_n(Bp - \gamma f(p))]\}. \end{aligned}$$

Then from (3.3) we have

$$\begin{aligned}\|z_n - p\| &= \|V_n z_n - V_n p + V_n p - p\| \\ &\le \|V_n z_n - V_n p\| + \|V_n p - p\| \\ &\le (1-\alpha_n\tau)\|z_n - p\| \\ &+ \|P_C\{(I-\alpha_n A)p + \alpha_n[p - \beta_n(Bp - \gamma f(p))]\} - P_C p\| \\ &\le (1-\alpha_n\tau)\|z_n - p\| + \|(I-\alpha_n A)p + \alpha_n[p - \beta_n(Bp - \gamma f(p))] - p\| \\ &\le (1-\alpha_n\tau)\|z_n - p\| + \alpha_n\| - Ap + p - \beta_n(Bp - \gamma f(p))\| \\ &\le (1-\alpha_n\tau)\|z_n - p\| + \alpha_n[\|A - I\|\|p\| + \|B\|\|p\| + \gamma\|f(p)\|].\end{aligned}$$

Hence,

$$\|z_n - p\| \le \frac{1}{\tau}[\|A - I\|\|p\| + \|B\|\|p\| + \gamma\|f(p)\|].$$

This implies that $\{z_n\}$ is bounded, and hence so are $\{y_n\}, \{Sy_n\}, \{Tz_n\}$ and $\{f(z_n)\}$. For $p \in D$,

$$\begin{aligned}\|z_n - p\|^2 &= \|P_C\{(I-\alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(z_n))]\} - p\|^2 \\ &\le \|(I-\alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(z_n))] - p\|^2 \\ &= \|(I-\alpha_n A)(Sy_n - p) + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(z_n)) - Ap]\|^2 \\ &= \|(I-\alpha_n A)(Sy_n - p)\|^2 \\ &+\alpha_n^2\|Sy_n - \beta_n(BSy_n - \gamma f(z_n)) - Ap\|^2 \\ &+2\alpha_n\langle(I-\alpha_n A)(Sy_n - p), Sy_n - \beta_n(BSy_n - \gamma f(z_n)) - Ap\rangle \\ &\le (1-\alpha_n\bar{\gamma})^2\|y_n - p\|^2 + \alpha_n^2\|(I-\beta_n B)Sy_n + \beta_n\gamma f(z_n) - Ap\|^2 \\ &+2\alpha_n(1-\alpha_n\bar{\gamma})\|y_n - p\|\|(I-\beta_n B)Sy_n + \beta_n\gamma f(z_n) - Ap\| \\ &\le (1-\alpha_n\bar{\gamma})\|y_n - p\|^2 + \alpha_n^2\|(I-\beta_n B)Sy_n + \beta_n\gamma f(z_n) - Ap\|^2 \\ &+2\alpha_n\|y_n - p\|\|(I-\beta_n B)Sy_n + \beta_n\gamma f(z_n) - Ap\| \\ &\le (1-\alpha_n\bar{\gamma})\|y_n - p\|^2 + \alpha_n^2[(1-\beta_n\beta)\|Sy_n\| \\ &+\beta_n\gamma\|f(z_n)\| + \|Ap\|]^2 \\ &+2\alpha_n\|y_n - p\|[(1-\beta_n\beta)\|Sy_n\| + \beta_n\gamma\|f(z_n)\| + \|Ap\|] \\ &\le (1-\alpha_n\bar{\gamma})\|y_n - p\|^2 + \alpha_n^2[\|Sy_n\| + \gamma\|f(z_n)\| + \|Ap\|]^2 \\ &+2\alpha_n\|y_n - p\|[\|Sy_n\| + \gamma\|f(z_n)\| + \|Ap\|] \\ &\le (1-\alpha_n\bar{\gamma})[\|z_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Tz_n - Tp\|^2] \\ &+\alpha_n^2[\|Sy_n\| + \gamma\|f(z_n)\| + \|Ap\|]^2 \\ &+2\alpha_n\|y_n - p\|[\|Sy_n\| + \gamma\|f(z_n)\| + \|Ap\|].\end{aligned} \tag{3.10}$$

So, we obtain

$$
\begin{aligned}
&(1-\alpha_n\bar{\gamma})a(2\alpha-b)\|Tz_n-Tp\|^2\\
&\le (1-\alpha_n\bar{\gamma})\lambda_n(2\alpha-\lambda_n)\|Tz_n-Tp\|^2\\
&\le \alpha_n^2[\|Sy_n\|+\gamma\|f(z_n)\|+\|Ap\|]^2+2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(z_n)\|+\|Ap\|].
\end{aligned}
$$

Since $\alpha_n \to 0$ $(n\to\infty)$, and $\{y_n\}$, $\{Sy_n\}$ and $\{f(z_n)\}$ are bounded, we derive that

$$\|Tz_n-Tp\|\to 0 \quad (n\to\infty).$$

From (2.1) we have

$$
\begin{aligned}
\|y_n-p\|^2 &= \|P_C(z_n-\lambda_nTz_n)-P_C(p-\lambda_nTp)\|^2\\
&\le \langle (z_n-\lambda_nTz_n)-(p-\lambda_nTp), y_n-p\rangle\\
&= \frac{1}{2}\{\|(z_n-\lambda_nTz_n)-(p-\lambda_nTp)\|^2+\|y_n-p\|^2\\
&\quad -\|(z_n-\lambda_nTz_n)-(p-\lambda_nTp)-(y_n-p)\|^2\}\\
&\le \frac{1}{2}\{\|z_n-p\|^2+\|y_n-p\|^2-\|z_n-y_n\|^2+2\lambda_n\langle z_n-y_n, Tz_n-Tp\rangle\\
&\quad -\lambda_n^2\|Tz_n-Tp\|^2\}.
\end{aligned}
$$

So, we obtain

$$\|y_n-p\|^2\le \|z_n-p\|^2-\|z_n-y_n\|^2+2\lambda_n\langle z_n-y_n, Tz_n-Tp\rangle-\lambda_n^2\|Tz_n-Tp\|^2.$$

Consequently, from (3.10) we derive that

$$
\begin{aligned}
\|z_n-p\|^2 &\le (1-\alpha_n\bar{\gamma})\|y_n-p\|^2+\alpha_n^2[\|Sy_n\|+\gamma\|f(z_n)\|+\|Ap\|]^2\\
&\quad +2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(z_n)\|+\|Ap\|]\\
&\le \|z_n-p\|^2-(1-\alpha_n\bar{\gamma})\|z_n-y_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma})\lambda_n\langle z_n-y_n, Tz_n-Tp\rangle\\
&\quad -(1-\alpha_n\bar{\gamma})\lambda_n^2\|Tz_n-Tp\|^2+\alpha_n^2[\|Sy_n\|+\gamma\|f(z_n)\|+\|Ap\|]^2\\
&\quad +2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(z_n)\|+\|Ap\|],
\end{aligned}
$$

and hence

$$\begin{aligned}
&(1-\alpha_n\bar{\gamma})\|z_n-y_n\|^2\\
&\le 2(1-\alpha_n\bar{\gamma})\lambda_n\langle z_n-y_n, Tz_n-Tp\rangle-(1-\alpha_n\bar{\gamma})\lambda_n^2\|Tz_n-Tp\|^2\\
&\quad+\alpha_n^2[\|Sy_n\|+\gamma\|f(z_n)\|+\|Ap\|]^2+2\alpha_n\|y_n-p\|[\|Sy_n\|+\gamma\|f(z_n)\|+\|Ap\|].
\end{aligned}$$

Since $\alpha_n\to 0$, $\|Tz_n-Tp\|\to 0$, $\{z_n\},\{y_n\},\{Sy_n\}$ and $\{f(z_n)\}$ are bounded, we obtain $\lim_{n\to\infty}\|z_n-y_n\|=0$.

Now we claim that $\lim_{n\to\infty}\|z_n-Sy_n\|=0$. Indeed, observe that

$$\begin{aligned}
\|z_n-Sy_n\| &= \|P_C\{(I-\alpha_nA)Sy_n+\alpha_n[Sy_n-\beta_n(BSy_n-\gamma f(z_n))]\}-Sy_n\|\\
&\le \|\{(I-\alpha_nA)Sy_n+\alpha_n[Sy_n-\beta_n(BSy_n-\gamma f(z_n))]\}-Sy_n\|\\
&= \alpha_n\|-ASy_n+Sy_n-\beta_n(BSy_n-\gamma f(z_n))\|.
\end{aligned}$$

Hence as $\alpha_n\to 0$, it follows that $\lim_{n\to\infty}\|z_n-Sy_n\|=0$. This implies that $\lim_{n\to\infty}\|y_n-Sy_n\|=0$. Note that $\|z_n-Sz_n\|\le\|z_n-Sy_n\|+\|Sy_n-Sz_n\|$. Thus $\lim_{n\to\infty}\|z_n-Sz_n\|=0$.

Repeating the same argument as in the proof of Theorem 3.1, we can deduce that $A-I+\eta(B-\gamma f)$ is strongly monotone and Lipschitz continuous on D. So

$$\langle[A-I+\eta(B-\gamma f)]q, x-q\rangle\ge 0,\qquad \text{for all } x\in D,$$

has the unique solution $q\in D$ and hence, as before, (3.8) holds.

As $\{y_{n_i}\}$ is bounded, we have a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to z. We may assume without loss of generality that $y_{n_i}\rightharpoonup z$. As in the proof of Theorem 3.1, we can show that $z\in D$. Therefore, for $z,q\in D$, we conclude from (3.10) and $\beta_n\to\eta$ that

$$\limsup_{n\to\infty}\langle q-\beta_n(Bq-\gamma f(q))-Aq, Sy_n-q\rangle=\langle -[A-I+\eta(B-\gamma f)]q, z-q\rangle\le 0.$$

Finally, we claim that $\lim_{n\to\infty}\|z_n-q\|=0$. We can now obtain as in the proof of Theorem 3.1 by essentially replacing z_n with x_n that

$$\|z_n-q\|^2\le(1-\bar{\alpha}_n)\|z_n-q\|^2+\bar{\alpha}_n\bar{\beta}_n.$$

Consequently, it follows that

$$\|z_n-q\|^2\le\bar{\beta}_n.$$

Since $\alpha_n \to 0$ $(n \to \infty)$, $\limsup_{n\to\infty} \langle -[A - I + \beta_n(B - \gamma f)]q, Sy_n - q\rangle \leq 0$, and $\{z_n\}$, $\{f(z_n)\}$ and $\{Sy_n\}$ are bounded, so we obtain that $\bar{\beta}_n \to 0$ $(n \to \infty)$. Hence $\lim_{n\to\infty} \|z_n - q\| = 0$. □

In view of Remark 3.1, we can take $B = I$, $\beta = 1$ and $\beta_n = 1$, for all $n \geq 0$ in Theorem 3.2 to get:

Corollary 3.2. Let C, H, f, T, S, D and A be as in Theorem 3.2. Let $0 < \gamma < \frac{\bar{\gamma}}{k}$. Suppose that $\{z_n\}$ is a sequence in C generated by:

$$z_n = P_C\{(I - \alpha_n A)SP_C(z_n - \lambda_n T z_n) + \alpha_n \gamma f(z_n)\}.$$

If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$, and $\lim_{n\to\infty} \alpha_n = 0$, then $\{z_n\}$ converges strongly to the unique $q \in D$ which solves the following variational inequality (3.9).

Putting $A = I$ and $\gamma = 1$ in Corollary 3.2, we obtain Theorem 3.1 in [4].

4. An extragradient-like approximation method

In this section, we establish strong convergence of the explicit and implicit iterative schemes (1.5) and (1.6) by the extragradient-like approximation method which is based on the so-called extragradient method and viscosity approximation method. The results obtained herein can be viewed as hybrid viscosity approximation results for monotone mappings and nonexpansive mappings.

Let $T : C \to H$ be an α-inverse-strongly monotone mapping and S be a nonexpansive self-mapping on C. As before, we can verify that the schemes (1.5) and(1.6) are well-defined by utilizing nonexpansiveness of $I - \lambda_n T$.

We prove strong convergence of (1.5) when S is nonexpansive mapping and T is an α-inverse-strongly monotone mapping; thereby, we find solution of the variational inequalities (1.4) in the subclass; namely, α-inverse-strongly monotone mappings, of the class of monotone and Lipschitz continuous mappings.

Theorem 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction, $T : C \to H$ an α-inverse-strongly monotone mapping and S a nonexpansive self-mapping on C such that $D \neq \emptyset$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in C generated by $x_0 \in C$ as in (1.5). Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n \leq 0$ for all $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique $q \in D$ which solves the variational inequality (1.4).

Proof. Put $t_n = P_C(y_n - \lambda_n T y_n)$ for every $n = 0, 1, 2, ...$ Then we have $x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n f(y_n) + \beta_n S t_n$ for every $n = 0, 1, 2, ...$ We divide the proof of the theorem into several steps.

Step 1. $\{x_n\}$ is bounded. Indeed, let $u \in D$. We have

$$\begin{aligned}
\|y_n - u\| &= \|(1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n T x_n) - (1 - \gamma_n)u + \gamma_n P_C(u - \lambda_n T u)\| \\
&\leq (1 - \gamma_n)\|x_n - u\| + \gamma_n \|P_C(I - \lambda_n T)x_n - P_C(I - \lambda_n T)u\| \\
&\leq (1 - \gamma_n)\|x_n - u\| + \gamma_n \|x_n - u\| \\
&= \|x_n - u\|
\end{aligned}$$

and hence

$$\|t_n - u\| = \|P_C(y_n - \lambda_n T y_n) - P_C(u - \lambda_n T u)\| \leq \|y_n - u\| \leq \|x_n - u\|$$

for every $n = 0, 1, 2, ...$ Then we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \|(1 - \alpha_n - \beta_n)(x_n - u) + \alpha_n(f(y_n) - u) + \beta_n(S t_n - u)\| \\
&\leq (1 - \alpha_n - \beta_n)\|x_n - u\| + \alpha_n \|f(y_n) - u\| + \beta_n \|S t_n - u\| \\
&\leq (1 - \alpha_n - \beta_n)\|x_n - u\| + \alpha_n \|f(y_n) - f(u)\| \\
&+ \alpha_n \|f(u) - u\| + \beta_n \|t_n - u\| \\
&\leq (1 - \alpha_n - \beta_n)\|x_n - u\| + \alpha_n k \|y_n - u\| \\
&+ \alpha_n \|f(u) - u\| + \beta_n \|x_n - u\| \\
&\leq (1 - \alpha_n)\|x_n - u\| + \alpha_n k \|x_n - u\| + \alpha_n \|f(u) - u\| \\
&= (1 - (1 - k)\alpha_n)\|x_n - u\| + \alpha_n \|f(u) - u\| \\
&\leq \max\left\{\|x_n - u\|, \frac{1}{1-k}\|f(u) - u\|\right\}.
\end{aligned}$$

By induction,

$$\|x_n - u\| \leq \max\left\{\|x_0 - u\|, \frac{1}{1-k}\|f(u) - u\|\right\}, \quad \text{for all } n \geq 0.$$

Therefore, $\{x_n\}$ is bounded and hence so are

$$\{y_n\}, \{t_n\}, \{St_n\}, \{Ax_n\}, \{Ay_n\}, \{f(y_n)\}.$$

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Indeed, observe that

$$\begin{aligned}
&\|P_C(x_{n+1} - \lambda_{n+1}Tx_{n+1}) - P_C(x_n - \lambda_n Tx_n)\| \\
&\le \|P_C(x_{n+1} - \lambda_{n+1}Tx_{n+1}) - P_C(x_n - \lambda_{n+1}Tx_n)\| \\
&+ \|P_C(x_n - \lambda_{n+1}Tx_n) - P_C(x_n - \lambda_n Tx_n)\| \\
&\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Tx_n\|,
\end{aligned}$$

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&= \|(1 - \gamma_{n+1})x_{n+1} + \gamma_{n+1}P_C(x_{n+1} - \lambda_{n+1}Tx_{n+1}) \\
&- (1 - \gamma_n)x_n - \gamma_n P_C(x_n - \lambda_n Tx_n)\| \\
&= \|(1 - \gamma_{n+1})(x_{n+1} - x_n) - (\gamma_{n+1} - \gamma_n)x_n \\
&\ + \gamma_{n+1}(P_C(x_{n+1} - \lambda_{n+1}Tx_{n+1}) - P_C(x_n - \lambda_n Tx_n)) \\
&+ (\gamma_{n+1} - \gamma_n)P_C(x_n - \lambda_n Tx_n)\| \\
&= \|(1 - \gamma_{n+1})(x_{n+1} - x_n) + (\gamma_{n+1} - \gamma_n)(P_C(x_n - \lambda_n Tx_n) - x_n) \\
&\ + \gamma_{n+1}(P_C(x_{n+1} - \lambda_{n+1}Tx_{n+1}) - P_C(x_n - \lambda_n Tx_n))\| \\
&\le (1 - \gamma_{n+1})\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\lambda_n\|Tx_n\| \\
&\ + \gamma_{n+1}(\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Tx_n\|) \\
&\le \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\lambda_n\|Tx_n\| + |\lambda_{n+1} - \lambda_n|\|Tx_n\|,
\end{aligned}$$

and hence

$$\begin{aligned}
&\|t_{n+1} - t_n\| \\
&= \|P_C(y_{n+1} - \lambda_{n+1}Ty_{n+1}) - P_C(y_n - \lambda_n Ty_n)\| \\
&\le \|P_C(y_{n+1} - \lambda_{n+1}Ty_{n+1}) - P_C(y_n - \lambda_{n+1}Ty_n)\| \\
&+ \|P_C(y_n - \lambda_{n+1}Ty_n) - P_C(y_n - \lambda_n Ty_n)\| \\
&\le \|y_{n+1} - y_n\| + |\lambda_{n+1} - \lambda_n|\|Ty_n\| \\
&\le \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\lambda_n\|Tx_n\| + |\lambda_{n+1} - \lambda_n|\|Tx_n\| \\
&+ |\lambda_{n+1} - \lambda_n|\|Ty_n\| \\
&\le \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\lambda_n\|Tx_n\| + |\lambda_{n+1} - \lambda_n|(\|Tx_n\| + \|Ty_n\|).
\end{aligned}$$

Now we define a sequence $\{z_n\}$ by

$$x_{n+1} = \varrho_n x_n + (1-\varrho_n) z_n, \quad \text{for all } n \geq 0,$$

where $\varrho_n = 1 - \alpha_n - \beta_n,$ for all $n \geq 0$. Then we have

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \varrho_{n+1} x_{n+1}}{1-\varrho_{n+1}} - \frac{x_{n+1} - \varrho_n x_n}{1-\varrho_n} \\
&= \frac{\alpha_{n+1} f(y_{n+1}) + \beta_{n+1} S t_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n f(y_n) + \beta_n S t_n}{1-\varrho_n} \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}} f(y_{n+1}) - \frac{\alpha_n}{1-\varrho_n} f(y_n) \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} (S t_{n+1} - S t_n) + \left(\frac{\alpha_n}{1-\varrho_n} - \frac{\alpha_{n+1}}{1-\varrho_{n+1}}\right) S t_n \\
&= \frac{\alpha_{n+1}}{1-\varrho_{n+1}} (f(y_{n+1}) - f(y_n)) + \left(\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\right) f(y_n) \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} (S t_{n+1} - S t_n) + \left(\frac{\alpha_n}{1-\varrho_n} - \frac{\alpha_{n+1}}{1-\varrho_{n+1}}\right) S t_n.
\end{aligned}$$

Hence from $\varrho_n = 1 - \alpha_n - \beta_n$ it follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1-\varrho_{n+1}} \|f(y_{n+1}) - f(y_n)\| + \left|\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\right| \|f(y_n)\| \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} \|S t_{n+1} - S t_n\| + \left|\frac{\alpha_n}{1-\varrho_n} - \frac{\alpha_{n+1}}{1-\varrho_{n+1}}\right| \|S t_n\| \\
&\leq \frac{k\alpha_{n+1}}{1-\varrho_{n+1}} \|y_{n+1} - y_n\| + \left|\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\right| (\|f(y_n)\| + \|S t_n\|) \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} \|t_{n+1} - t_n\| \\
&\leq \frac{k\alpha_{n+1}}{1-\varrho_{n+1}} [\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n| \lambda_n \|T x_n\| + |\lambda_{n+1} - \lambda_n| \|T x_n\|] \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\right| (\|f(y_n)\| + \|S t_n\|) \\
&\quad + \frac{\beta_{n+1}}{1-\varrho_{n+1}} [\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n| \lambda_n \|T x_n\| \\
&\quad + |\lambda_{n+1} - \lambda_n| (\|T x_n\| + \|T y_n\|)] \\
&\leq \frac{k\alpha_{n+1} + \beta_{n+1}}{1-\varrho_{n+1}} [\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n| \lambda_n \|T x_n\| \\
&\quad + |\lambda_{n+1} - \lambda_n| (\|T x_n\| + \|T y_n\|)] \\
&\quad + \left|\frac{\alpha_{n+1}}{1-\varrho_{n+1}} - \frac{\alpha_n}{1-\varrho_n}\right| (\|f(y_n)\| + \|S t_n\|) \\
&\leq \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n| \lambda_n \|T x_n\|
\end{aligned}$$

$$+|\lambda_{n+1}-\lambda_n|(\|Tx_n\|+\|Ty_n\|)$$
$$+|\frac{\alpha_{n+1}}{1-\varrho_{n+1}}-\frac{\alpha_n}{1-\varrho_n}|(\|f(y_n)\|+\|St_n\|),$$

which implies that

$$\begin{aligned}&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\\&\le|\gamma_{n+1}-\gamma_n|\lambda_n\|Tx_n\|+|\lambda_{n+1}-\lambda_n|(\|Tx_n\|+\|Ty_n\|)\\&+|\frac{\alpha_{n+1}}{1-\varrho_{n+1}}-\frac{\alpha_n}{1-\varrho_n}|(\|f(y_n)\|+\|St_n\|).\end{aligned}$$

Note that

$$\limsup_{n\to\infty}\frac{\alpha_n}{1-\varrho_n}=\limsup_{n\to\infty}\frac{\alpha_n}{\alpha_n+\beta_n}\le\limsup_{n\to\infty}\frac{\alpha_n}{\beta_n}=0.$$

Since $\lim_{n\to\infty}|\gamma_{n+1}-\gamma_n|=0$, $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=0$, $\{Ax_n\}$, $\{Ay_n\}$, $\{f(y_n)\}$ and $\{St_n\}$ are bounded, so we deduce that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)=0.$$

Since $\varrho_n=1-\alpha_n-\beta_n$, we know from the conditions (ii) and (iii) that

$$0<\liminf_{n\to\infty}\varrho_n\le\limsup_{n\to\infty}\varrho_n<1.$$

Thus by Lemma 2.6, we obtain $\lim_{n\to\infty}\|z_n-x_n\|=0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\varrho_n)\|z_n-x_n\|=0.$$

Step 3. $\lim_{n\to\infty}\|St_n-t_n\|=\lim_{n\to\infty}\|Sx_n-x_n\|=0$. Indeed, observe that

$$x_{n+1}-x_n=\alpha_n(f(y_n)-x_n)+\beta_n(St_n-x_n).$$

Hence we have

$$\beta_n\|St_n-x_n\|\le\|x_{n+1}-x_n\|+\alpha_n\|f(y_n)-x_n\|.$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\alpha_n=0$ and $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, so by the boundedness of $\{f(y_n)-x_n\}$, we obtain

$$\lim_{n\to\infty}\|St_n-x_n\|=0.$$

Also, observe that for $u \in D$,

$$\begin{aligned}
\|x_{n+1}-u\|^2 &= \|(1-\alpha_n-\beta_n)(x_n-u)+\alpha_n(f(y_n)-u)+\beta_n(St_n-u)\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n-u\|^2+\alpha_n\|f(y_n)-u\|^2+\beta_n\|t_n-u\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n-u\|^2+\alpha_n\|f(y_n)-u\|^2 \\
&\quad +\beta_n[\|y_n-u\|^2+\lambda_n(\lambda_n-2\alpha)\|Ty_n-Tu\|^2] \\
&\le (1-\alpha_n-\beta_n)\|x_n-u\|^2+\alpha_n\|f(y_n)-u\|^2+\beta_n[\|x_n-u\|^2 \\
&\quad +\lambda_n(\lambda_n-2\alpha)\|Ty_n-Tu\|^2] \\
&\le \|x_n-u\|^2+\alpha_n\|f(y_n)-u\|^2+\beta_n\lambda_n(\lambda_n-2\alpha)\|Ty_n-Tu\|^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\beta_n a(2\alpha-b)\|Ty_n-Tu\|^2 \\
&\le \beta_n\lambda_n(2\alpha-\lambda_n)\|Ty_n-Tu\|^2 \\
&\le \alpha_n\|f(y_n)-u\|^2+(\|x_n-u\|+\|x_{n+1}-u\|)(\|x_n-u\|-\|x_{n+1}-u\|) \\
&\le \alpha_n\|f(y_n)-u\|^2+(\|x_n-u\|+\|x_{n+1}-u\|)\|x_n-x_{n+1}\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\alpha_n=0$ and $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, we have $\lim_{n\to\infty}\|Ty_n-Tu\|=0$. Further, from (2.1), we obtain

$$\begin{aligned}
\|t_n-u\|^2 &= \|P_C(y_n-\lambda_nTy_n)-P_C(u-\lambda_nTu)\|^2 \\
&\le \langle (y_n-\lambda_nTy_n)-(u-\lambda_nTu),t_n-u\rangle \\
&= \frac{1}{2}\{\|(y_n-\lambda_nTy_n)-(u-\lambda_nTu)\|^2+\|t_n-u\|^2 \\
&\quad -\|(y_n-\lambda_nTy_n)-(u-\lambda_nTu)-(t_n-u)\|^2\} \\
&\le \frac{1}{2}\{\|y_n-u\|^2+\|t_n-u\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,Ty_n-Tu\rangle \\
&\quad -\lambda_n^2\|Ty_n-Tu\|^2\}.
\end{aligned}$$

So, we get

$$\begin{aligned}
\|t_n-u\|^2 &\le \|y_n-u\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,Ty_n-Tu\rangle \\
&\quad -\lambda_n^2\|Ty_n-Tu\|^2 \\
&\le \|x_n-u\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,Ty_n-Tu\rangle \\
&\quad -\lambda_n^2\|Ty_n-Tu\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le (1-\alpha_n-\beta_n)\|x_n-u\|^2+\alpha_n\|f(y_n)-u\|^2\\ &+\beta_n\|St_n-u\|^2\\ &\le (1-\alpha_n-\beta_n)\|x_n-u\|^2+\alpha_n\|f(y_n)-u\|^2\\ &+\beta_n\|t_n-u\|^2\\ &\le (1-\alpha_n-\beta_n)\|x_n-u\|^2+\alpha_n\|f(y_n)-u\|^2\\ &+\beta_n[\|x_n-u\|^2-\|y_n-t_n\|^2\\ &+2\lambda_n\langle y_n-t_n, Ty_n-Tu\rangle-\lambda_n^2\|Ty_n-Tu\|^2]\\ &\le \|x_n-u\|^2+\alpha_n\|f(y_n)-u\|^2-\beta_n\|y_n-t_n\|^2\\ &+2\beta_n\lambda_n\langle y_n-t_n, Ty_n-Tu\rangle-\beta_n\lambda_n^2\|Ty_n-Tu\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\beta_n\|y_n-t_n\|^2 \le (\|x_n-u\|+\|x_{n+1}-u\|)\|x_n-x_{n+1}\|+\alpha_n\|f(y_n)-u\|^2\\ +2\beta_n\lambda_n\langle y_n-t_n, Ty_n-Tu\rangle-\beta_n\lambda_n^2\|Ty_n-Tu\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\|Ty_n-Tu\|=0$, $\lim_{n\to\infty}\alpha_n=0$ and $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, we obtain $\lim_{n\to\infty}\|y_n-t_n\|=0$. Note that $\lim_{n\to\infty}\gamma_n=0$ so we have

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\gamma_n\|P_C(x_n-\lambda_nTx_n)-x_n\|=0.$$

Since

$$\|St_n-t_n\|\le\|St_n-x_n\|+\|x_n-y_n\|+\|y_n-t_n\|,$$

so we get $\lim_{n\to\infty}\|St_n-t_n\|=0$. Also, observe that

$$\begin{aligned}\|Sx_n-x_n\| &\le \|Sx_n-Sy_n\|+\|Sy_n-St_n\|+\|St_n-x_n\|\\ &\le \|x_n-y_n\|+\|y_n-t_n\|+\|St_n-x_n\|.\end{aligned}$$

Consequently, we have $\lim_{n\to\infty}\|Sx_n-x_n\|=0$.

Step 4. $\limsup_{n\to\infty}\langle f(q)-q, x_n-q\rangle\le 0$. Indeed, pick a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle f(q)-q, St_n-q\rangle=\lim_{i\to\infty}\langle f(q)-q, St_{n_i}-q\rangle.$$

Now using the same argument as in the proof of Theorem 3.1, we can show that

$$\limsup_{n\to\infty}\langle f(q)-q, St_n-q\rangle = \lim_{i\to\infty}\langle f(q)-q, St_{n_i}-q\rangle = \langle f(q)-q, z-q\rangle \le 0.$$

Consequently, from $\lim_{n\to\infty}\|St_n - x_n\| = 0$ we get that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q)-q, x_n-q\rangle &\le \limsup_{n\to\infty}\langle f(q)-q, x_n-St_n\rangle \\ &\quad + \limsup_{n\to\infty}\langle f(q)-q, St_n-q\rangle \le 0.\end{aligned}$$

Step 5. $\lim_{n\to\infty}\|x_n-q\| = 0$ where $q = P_{\mathrm{Fix}(S)\cap VI(C,T)}f(q)$. Indeed, by Lemma 2.7 we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|(1-\alpha_n-\beta_n)(x_n-q)+\alpha_n(f(y_n)-q)+\beta_n(St_n-q)\|^2 \\ &\le \|(1-\alpha_n-\beta_n)(x_n-q)+\beta_n(St_n-q)\|^2 \\ &+ 2\alpha_n\langle f(y_n)-q, x_{n+1}-q\rangle \\ &\le [(1-\alpha_n-\beta_n)\|x_n-q\|+\beta_n\|t_n-q\|]^2 \\ &+ 2\alpha_n\langle f(y_n)-q, x_{n+1}-q\rangle \\ &\le [(1-\alpha_n-\beta_n)\|x_n-q\|+\beta_n\|x_n-q\|]^2 \\ &+ 2\alpha_n\langle f(y_n)-q, x_{n+1}-q\rangle \\ &\le (1-\alpha_n)^2\|x_n-q\|^2 \\ &+ 2\alpha_n[\langle f(y_n)-f(q), x_{n+1}-q\rangle + \langle f(q)-q, x_{n+1}-q\rangle] \\ &\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n k\|y_n-q\|\|x_{n+1}-q\| \\ &+ 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\ &\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n k\|x_n-q\|\|x_{n+1}-q\| \\ &+ 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle \\ &\le (1-\alpha_n)^2\|x_n-q\|^2 + k\alpha_n(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\ &+ 2\alpha_n\langle f(q)-q, x_{n+1}-q\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \frac{(1-\alpha_n)^2+k\alpha_n}{1-k\alpha_n}\|x_n-q\|^2 + \frac{2\alpha_n}{1-k\alpha_n}\langle f(q)-q, x_{n+1}-q\rangle \\ &\le (1-2(1-k)\alpha_n + \frac{\alpha_n^2}{1-k\alpha_n})\|x_n-q\|^2 + \frac{2\alpha_n}{1-k\alpha_n}\langle f(q)-q, x_{n+1}-q\rangle\end{aligned}$$

$$= (1-2(1-k)\alpha_n)\|x_n-q\|^2$$

$$+2(1-k)\alpha_n \cdot \frac{1}{(1-k)(1-k\alpha_n)}\left[\frac{\alpha_n}{2}\|x_n-q\|^2 + \langle f(q)-q, x_{n+1}-q\rangle\right].$$

Note that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} 2(1-k)\alpha_n = \infty$. Since

$$\limsup_{n\to\infty}\langle f(q)-q, x_{n+1}-q\rangle \leq 0,$$

and $\{x_n - q\}$ is bounded, we have

$$\limsup_{n\to\infty} \frac{1}{(1-k)(1-k\alpha_n)}\left[\frac{\alpha_n}{2}\|x_n-q\|^2 + \langle f(q)-q, x_{n+1}-q\rangle\right] \leq 0.$$

Therefore, by Lemma 2.1, we conclude that $\|x_n - q\| \to 0$ as $n \to \infty$. Further, from $\|y_n - x_n\| \to 0$ $(n \to \infty)$, we get $\|y_n - q\| \to 0$ $(n \to \infty)$. □

Putting $\gamma_n = 0,$ for all $n \geq 0$ in Theorem 4.1, we obtain:

Corollary 4.1. Let C, H, f, T, S and D be as in Theorem 4.1. Suppose that $\{x_n\}$ in C is generated by $x_0 \in C$ as:

$$x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(x_n) + \beta_n SP_C(x_n - \lambda_n Tx_n), \quad \text{for all } n \geq 0$$

where $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n \leq 1$ for all $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0.$

Then $\{x_n\}$ converges strongly to the unique $q \in D$ which solves variational inequality (1.4).

Remark 4.1. Our Theorem 4.1 improves and develops ([4], Proposition 3.1) in the following aspects:

(i) Our explicit iterative scheme is very much different and contains a two step iterative scheme with four parameteric sequences;

(ii) The technique of proof of Theorem 4.1 is based on an alternative result by Suzuki (Lemma 2.6);

(iii) Our Theorem 4.1 removes the restrictions $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ and $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$ from Proposition 3.1 [4].

Theorem 4.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f:C\to C$ be a contraction, $T:C\to H$ an α-inverse-strongly monotone mapping and S a nonexpansive self-mapping on C such that $D\neq\emptyset$. Suppose that $\{z_n\}$ in C is generated by (1.6) with $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ such that $\alpha_n+\beta_n\le 1$ for all $n\ge 0$, $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\gamma_n=0$ and $\liminf_{n\to\infty}\beta_n>0$. Then $\{z_n\}$ converges strongly to the unique $q\in D$ which solves the variational inequality (1.4).

Proof. Put $y_n=(1-\gamma_n)z_n+\gamma_nP_C(z_n-\lambda_nTz_n)$ and $t_n=P_C(y_n-\lambda_nTy_n)$ for every $n=0,1,2,....$ Let $u\in D$. We have

$$\begin{aligned}\|y_n-u\|&=\|(1-\gamma_n)(z_n-u)+\gamma_n(P_C(z_n-\lambda_nTz_n)-P_C(u-\lambda_nTu))\|\\&\le(1-\gamma_n)\|z_n-u\|+\gamma_n\|P_C(I-\lambda_nT)z_n-P_C(I-\lambda_nT)u\|\\&\le\|z_n-p\|\end{aligned}$$

for every $n=0,1,2,....$ Observe that

$$\begin{aligned}V_nu&=(1-\alpha_n-\beta_n)u+\alpha_nf((1-\gamma_n)u+\gamma_nP_C(u-\lambda_nTu))\\&\quad+\beta_nSP_C[(1-\gamma_n)u+\gamma_nP_C(u-\lambda_nTu)\\&\quad-\lambda_nT((1-\gamma_n)u+\gamma_nP_C(u-\lambda_nTu))]\\&=(1-\alpha_n-\beta_n)u+\alpha_nf(u)+\beta_nSu\\&=(1-\alpha_n)u+\alpha_nf(u).\end{aligned}$$

Then from (3.3) we have

$$\begin{aligned}\|z_n-u\|&=\|V_nz_n-V_nu+V_nu-u\|\\&\le\|V_nz_n-V_nu\|+\|V_nu-u\|\\&\le(1-(1-k)\alpha_n)\|z_n-u\|+\alpha_n\|f(u)-u\|\\&=\alpha_nk\|z_n-u\|+(1-\alpha_n)\|z_n-u\|+\alpha_n\|f(u)-u\|.\end{aligned}$$

Hence,

$$\|z_n-u\|\le\frac{1}{1-k}\|f(u)-u\|.$$

This implies that $\{z_n\}$ is bounded, and hence so are $\{Tz_n\}$, $\{y_n\}$, $\{Ty_n\}$, $\{t_n\}$ and $\{f(y_n)\}$. For $u \in D$,

$$\begin{aligned}
\|z_n - u\|^2 &= \|(1-\alpha_n-\beta_n)z_n + \alpha_n f(y_n) + \beta_n S t_n - u\|^2 \\
&\le (1-\alpha_n-\beta_n)\|z_n-u\|^2 + \alpha_n\|f(y_n)-u\|^2 + \beta_n\|t_n-u\|^2 \\
&\le (1-\alpha_n-\beta_n)\|z_n-u\|^2 + \alpha_n\|f(y_n)-u\|^2 \\
&+\beta_n[\|y_n-u\|^2 + \lambda_n(\lambda_n-2\alpha)\|Ty_n-Tu\|^2] \\
&\le (1-\alpha_n-\beta_n)\|z_n-u\|^2 + \alpha_n\|f(y_n)-u\|^2 \\
&+\beta_n[\|z_n-u\|^2 + \lambda_n(\lambda_n-2\alpha)\|Ty_n-Tu\|^2] \\
&\le (1-\alpha_n)\|z_n-u\|^2 + \alpha_n\|f(y_n)-u\|^2 \\
&+\beta_n\lambda_n(\lambda_n-2\alpha)\|Ty_n-Tu\|^2 \\
&\le (1-\alpha_n)\|z_n-u\|^2 + \alpha_n\|f(y_n)-u\|^2 \\
&+\beta_n a(b-2\alpha)\|Ty_n-Tu\|^2.
\end{aligned} \tag{4.1}$$

So, we obtain

$$\beta_n a(2\alpha - b)\|Ty_n - Tu\|^2 \le \alpha_n(\|f(y_n)-u\|^2 - \|z_n-u\|^2).$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n > 0$, and $\{f(y_n)\}, \{z_n\}$ are bounded, we derive that

$$\|Ay_n - Au\| \to 0 \quad (n \to \infty).$$

From (2.1) we have

$$\begin{aligned}
\|t_n - u\|^2 &= \|P_C(y_n - \lambda_n Ty_n) - P_C(u - \lambda_n Tu)\|^2 \\
&\le \langle (y_n - \lambda_n Ty_n) - (u - \lambda_n Tu), t_n - u\rangle \\
&= \frac{1}{2}\{\|(y_n - \lambda_n Ty_n) - (u - \lambda_n Tu)\|^2 + \|t_n - u\|^2 \\
&- \|(y_n - \lambda_n Ty_n) - (u - \lambda_n Tu) - (t_n - u)\|^2\} \\
&\le \frac{1}{2}\{\|y_n - u\|^2 + \|t_n - u\|^2 - \|y_n - t_n\|^2 + 2\lambda_n\langle y_n - t_n, Ty_n - Tu\rangle \\
&- \lambda_n^2\|Ty_n - Tu\|^2\}.
\end{aligned}$$

So, we get

$$\begin{aligned}
\|t_n - u\|^2 &\le \|y_n - u\|^2 - \|y_n - t_n\|^2 \\
&+2\lambda_n\langle y_n - t_n, Ty_n - Tu\rangle - \lambda_n^2\|Ty_n - Tu\|^2 \\
&\le \|z_n - u\|^2 - \|y_n - t_n\|^2 \\
&+2\lambda_n\langle y_n - t_n, Ty_n - Tu\rangle - \lambda_n^2\|Ty_n - Tu\|^2.
\end{aligned}$$

Consequently, from (4.1) we derive that

$$\begin{aligned}\|z_n - u\|^2 &\le (1-\alpha_n-\beta_n)\|z_n-u\|^2 + \alpha_n\|f(y_n)-u\|^2 + \beta_n\|St_n-u\|^2 \\ &\le (1-\alpha_n-\beta_n)\|z_n-u\|^2 + \alpha_n\|f(y_n)-u\|^2 + \beta_n\|t_n-u\|^2 \\ &\le (1-\alpha_n-\beta_n)\|z_n-u\|^2 + \alpha_n\|f(y_n)-u\|^2 + \beta_n[\|z_n-u\|^2 \\ &\quad - \|y_n-t_n\|^2 + 2\lambda_n\langle y_n-t_n, Ty_n-Tu\rangle - \lambda_n^2\|Ty_n-Tu\|^2] \\ &\le \alpha_n\|f(y_n)-u\|^2 + (1-\alpha_n)\|z_n-u\|^2 - \beta_n\|y_n-t_n\|^2 \\ &\quad + 2\beta_n\lambda_n\langle y_n-t_n, Ty_n-Tu\rangle - \beta_n\lambda_n^2\|Ty_n-Tu\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\beta_n\|y_n-t_n\|^2 &\le \alpha_n\|f(y_n)-u\|^2 - \alpha_n\|z_n-u\|^2 \\ &\quad + 2\beta_n\lambda_n\langle y_n-t_n, Ty_n-Tu\rangle - \beta_n\lambda_n^2\|Ty_n-Tu\|^2.\end{aligned}$$

As $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n > 0$, $\lim_{n\to\infty}\|Ty_n - Tu\| = 0$, $\{z_n\}, \{y_n\}, \{t_n\}$ and $\{f(y_n)\}$ are bounded, so we obtain $\lim_{n\to\infty}\|y_n - t_n\| = 0$.

Now we claim that $\lim_{n\to\infty}\|t_n - St_n\| = 0$ and $\lim_{n\to\infty}\|z_n - Sz_n\| = 0$. Note that

$$z_n - St_n = (1-\alpha_n-\beta_n)(z_n - St_n) + \alpha_n(f(y_n) - St_n),$$

so we have

$$\|z_n - St_n\| \le (1-\alpha_n-\beta_n)\|z_n - St_n\| + \alpha_n\|f(y_n) - St_n\|,$$

and hence

$$\beta_n\|z_n - St_n\| \le \alpha_n\|f(y_n) - St_n\| - \alpha_n\|z_n - St_n\|.$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\liminf_{n\to\infty}\beta_n > 0$, it follows that $\lim_{n\to\infty}\|z_n - St_n\| = 0$. Also, note that

$$\|y_n - z_n\| = \gamma_n\|P_C(z_n - \lambda_n Tz_n) - z_n\| \le \gamma_n\lambda_n\|Tz_n\|,$$

from $\lim_{n\to\infty}\gamma_n = 0$ we obtain $\lim_{n\to\infty}\|y_n - z_n\| = 0$. Now observe that

$$\|t_n - St_n\| \le \|t_n - y_n\| + \|y_n - z_n\| + \|z_n - St_n\|.$$

Consequently, we have $\lim_{n\to\infty}\|t_n - St_n\| = 0$. Furthermore,

$$\begin{aligned}\|z_n - Sz_n\| &\le \|z_n - St_n\| + \|St_n - Sy_n\| + \|Sy_n - Sz_n\| \\ &\le \|z_n - St_n\| + \|t_n - y_n\| + \|y_n - z_n\|,\end{aligned}$$

so we have $\lim_{n\to\infty} \|z_n - Sz_n\| = 0$.

Define a mapping $Q = P_{\mathrm{Fix}(S)\cap VI(C,T)}f$. Then it is clear that Q is a contraction on D. Hence there exists a unique fixed point $q \in D$ such that $Qq = q$, that is, $P_{\mathrm{Fix}(S)\cap VI(C,T)}f(q) = q$. It is easy to see that $P_{\mathrm{Fix}(S)\cap VI(C,T)}f(q) = q$ if and only if q is the unique solution in D to the variational inequality (1.4)

As before, for $z, q \in D$ we conclude that

$$\limsup_{n\to\infty}\langle f(q) - q, St_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, St_{n_i} - q\rangle = \langle f(q) - q, z - q\rangle \le 0.$$

Finally, we claim that $\lim_{n\to\infty} \|z_n - q\| = 0$. Indeed, utilizing Lemma 2.7 and (3.3) we have

$$\begin{aligned}\|z_n - q\|^2 &= \|V_n z_n - V_n q + V_n q - q\|^2 \\ &\le \|V_n z_n - V_n q\|^2 + 2\langle V_n q - q, z_n - q\rangle \\ &\le (1 - (1-k)\alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\langle f(q) - q, z_n - q\rangle \\ &\le (1 - (1-k)\alpha_n)\|z_n - q\|^2 + 2\alpha_n\langle f(q) - q, z_n - q\rangle,\end{aligned}$$

which implies that

$$\|z_n - q\|^2 \le \frac{2}{1-k}\langle f(q) - q, z_n - q\rangle. \tag{4.2}$$

From $\|z_n - St_n\| \to 0$ $(n \to \infty)$, it now follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q) - q, z_n - q\rangle &\le \limsup_{n\to\infty}\langle f(q) - q, z_n - St_n\rangle \\ &\quad + \limsup_{n\to\infty}\langle f(q) - q, St_n - q\rangle \le 0,\end{aligned}$$

and so from (4.2), we obtain

$$\lim_{n\to\infty} \|z_n - q\| = 0. \qquad \square$$

Putting $\gamma_n = 0$, for all $n \ge 0$ in Theorem 4.2, we obtain:

Corollary 4.2. Let C, H, f, T, S and D be as in Theorem 4.1. Suppose that $\{z_n\}$ in C is generated by:

$$z_n = (1 - \alpha_n - \beta_n)z_n + \alpha_n f(z_n) + \beta_n SP_C(z_n - \lambda_n Tz_n)$$

If $\alpha_n + \beta_n \le 1$ for all $n \ge 0$, $\lim_{n\to\infty}\alpha_n = 0$ and $\liminf_{n\to\infty}\beta_n > 0$, then $\{z_n\}$ converges strongly to the unique $q \in D$ which solves the variational inequality (1.4).

Remark 4.2. Our Theorem 4.2 improves and extends ([4], Theorem 3.1) in the following ways:

(i) Our implicit iterative scheme is different from the implicit iterative scheme therein and contains four sequences of iterative parameters;

(ii) The technique of proof of Theorem 4.2, uses the existence and uniqueness of fixed points of $P_{\mathrm{Fix}(S)\cap VI(C,T)}f$ in D and so is very much different.

(iii) Our Theorem 4.2 reduces to Theorem 3.1 in [4] when $\alpha_n + \beta_n = 1$ and $\gamma_n = 0$ for all $n \geq 0$.

5. Applications

Let C be a subset of a real Hilbert space H. A mapping $V : C \to C$ is called strictly pseudocontractive if there exists $k \in [0, 1)$ such that

$$\|Vx - Vy\|^2 \leq \|x - y\|^2 + k\|(I - V)x - (I - V)y\|^2, \quad \text{for all } x, y \in C.$$

If $k = 0$, then V is nonexpansive. Put $T = I - V$, where $V : C \to C$ is a strictly pseudocontractive mapping with constant k. Then T is $\frac{1-k}{2}$-inverse strongly monotone. Actually, we have, for all $x, y \in C$,

$$\|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2 + k\|Tx - Ty\|^2, \quad \text{for all } x, y \in C.$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I-T)x-(I-T)y\|^2 = \|x-y\|^2 + \|Tx-Ty\|^2 - 2\langle x-y, Tx-Ty\rangle, \text{ for all } x, y \in C.$$

Hence we have

$$\langle x - y, Tx - Ty\rangle \geq \frac{1-k}{2}\|Tx - Ty\|^2, \quad \text{for all } x, y \in C.$$

As applications of Theorem 3.1, we prove strong convergence theorems for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem 5.1. Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction, S a nonexpansive self-mapping on C and V a strictly pseudocontractive self-mapping on C with constant α such that $\mathrm{Fix}(S) \cap \mathrm{Fix}(V) \neq \emptyset$. Let A, B be strongly positive bounded linear operators on H. Let $0 < \gamma < \frac{\beta}{k}$. Suppose that $\{x_n\}$ is a sequence in C generated by

$x_0 \in C$ as:

$$\begin{cases} y_n = (1-\lambda_n)x_n + \lambda_n V x_n, \\ x_{n+1} = P_C\{(I-\alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))]\}, \text{ for all } n \geq 0 \end{cases}$$

where $\{\alpha_n\} \subset [0,1)$, $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 1-\alpha$. Assume that the conditions (C1)-(C3) of Theorem 3.1 hold. Then $\{x_n\}$ converges strongly to the unique $q \in \mathrm{Fix}(S) \cap \mathrm{Fix}(V)$ which solves the following variational inequality:

$$\langle [A - I + \eta(B - \gamma f)]q, q-p\rangle \leq 0, \quad \text{for all } p \in \mathrm{Fix}(S) \cap \mathrm{Fix}(V).$$

Proof. Put $T = I - V$. Then T is $\frac{1-\alpha}{2}$-inverse-strongly monotone. We have $\mathrm{Fix}(V) = \mathrm{VI}(C,T)$ and $P_C(x_n - \lambda_n T x_n) = (1-\lambda_n)x_n + \lambda_n V x_n$. So by Theorem 3.1, we obtain the result. □

Theorem 5.2. Let C be a closed convex subset of a real Hilbert space H. Let $f : H \to H$ be a contraction, $S : H \to H$ a nonexpansive mapping and $T : H \to H$ an α-inverse-strongly monotone mapping such that $\mathrm{Fix}(S) \cap T^{-1}0 \neq \emptyset$. Let A, B be stronglyly positive bounded linear operators on H. Let $0 < \gamma < \frac{\beta}{k}$. Suppose that $x_0 \in H$ and $\{x_n\}$ is generated by:

$$\begin{cases} y_n = x_n - \lambda_n T x_n, \\ x_{n+1} = (I-\alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))], \quad \text{for all } n \geq 0 \end{cases}$$

where $\{\alpha_n\} \subset [0,1)$, $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\alpha$. Assume that the conditions (C1)-(C3) of Theorem 3.1 hold. Then $\{x_n\}$ converges strongly to the unique $q \in \mathrm{Fix}(S) \cap T^{-1}0$ which solves the following variational inequality:

$$\langle [A - I + \eta(B - \gamma f)]q, q-p\rangle \leq 0, \quad \text{for all } p \in \mathrm{Fix}(S) \cap T^{-1}0.$$

Proof. We have $T^{-1}0 = \mathrm{VI}(C,T)$. So putting $P_H = I$, we obtain the result by Theorem 3.1. □

Theorem 5.3. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction, S a nonexpansive self-mapping on C and V a strictly pseudocontractive self-mapping on C with constant α such that $\mathrm{Fix}(S) \cap \mathrm{Fix}(V) \neq \emptyset$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in C

generated by $x_0 \in C$ as

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n((1-\lambda_n)x_n + \lambda_n V x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n S((1-\lambda_n)y_n + \lambda_n V y_n), \text{ for all } n \geq 0 \end{cases}$$

where $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 1-\alpha$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ satisfying the conditions (i)-(iv) of Theorem 4.1. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique $q \in \mathrm{Fix}(S) \cap \mathrm{Fix}(V)$ which solves the following variational inequality:

$$\langle f(q) - q, q - p \rangle \leq 0, \quad \text{for all } p \in \mathrm{Fix}(S) \cap \mathrm{Fix}(V).$$

Proof. Put $T = I - V$. Then T is $\frac{1-\alpha}{2}$-inverse-strongly monotone. We conclude that $\mathrm{Fix}(V) = VI(C,T)$, $P_C(x_n - \lambda_n T x_n) = (1-\lambda_n)x_n + \lambda_n V x_n$ and

$$P_C(y_n - \lambda_n T y_n) = (1-\lambda_n)y_n + \lambda_n V y_n.$$

So by Theorem 4.1, we obtain the result. □

Theorem 5.4. Let H be a real Hilbert space. Let $f : H \to H$ be a contraction, $S : H \to H$ a nonexpansive mapping and $T : H \to H$ an α-inverse-strongly monotone mapping such that $\mathrm{Fix}(S) \cap T^{-1}0 \neq \emptyset$. Suppose that $x_0 \in H$ and $\{x_n\}$ is given by:

$$\begin{cases} y_n = x_n - \gamma_n \lambda_n T x_n, \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n S(y_n - \lambda_n T y_n), \quad \text{for all } n \geq 0 \end{cases}$$

where $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < 2\alpha$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the conditions (i)-(iv) of Theorem 4.1. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique $q \in \mathrm{Fix}(S) \cap T^{-1}0$ which solves the following variational inequality

$$\langle f(q) - q, q - p \rangle \leq 0, \quad \text{for all } p \in \mathrm{Fix}(S) \cap T^{-1}0.$$

Proof. We have $T^{-1}0 = VI(C,T)$. So putting $P_H = I$, we have

$$y_n = (1-\gamma_n)x_n + \gamma_n P_H(x_n - \lambda_n T x_n) = x_n - \gamma_n \lambda_n T x_n.$$

By Theorem 4.1, we obtain the result. □

Remark 5.1. Our Theorems 5.1-5.4 generalize and improve the corresponding results in [4, 3] for very much different iterative schemes involving four parametric sequences.

Acknowledgement. In this research, the first author ws partially supported by the National Science Foundation of China (10771141), Ph.D. Program Foundation of Ministry of Education of China (20070270004), Science and Technology Commission of Shanghai Municipality grant (075105118) and Shanghai Leading Academic Discipline Project (S30405). The second and third authors were supported by a KFUPM Funded Research Project IN070362. The fourth author was partially supported by the grant NSC 97-2115-M-110-001.

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Received: March 18, 2008; Accepted: October 21, 2008.