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FIXED POINTS FOR MULTIVALUED OPERATORS ON A SET ENDOWED WITH VECTOR-VALUED METRICS AND APPLICATIONS

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Abstract. In this paper we present the fixed point theory for multivalued operators on a set endowed with vector valued metrics. Our results extend, to the multivalued case, the result given by A.I. Perov, A.I. Perov and A.V. Kibenko, J. Matkowski, T. Shibata, as well as, some recent work of C. Bacoţiu, D. O'Regan, N. Shahzad and R.P. Agarwal, D. O'Regan and R. Precup, R. Precup, etc.
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1. Introduction

It is well-known that, often, for the study of many processes, having a certain lack of precision, which arise from biology, economy or other sciences, we are interested to replace the following semilinear operator equations:

$$\begin{cases} x_1 = N_1(x_1, x_2) \\ x_2 = N_1(x_1, x_2) \end{cases}$$

(where $(X, |\cdot|)$ is a Banach space and $N_i : X \times X \to X$ for $i \in \{1, 2\}$) with the semilinear inclusion system:

$$\begin{cases} x_1 \in T_1(x_1, x_2) \\ x_2 \in T_1(x_1, x_2) \end{cases}$$

(where $T_i : X \times X \to P(X)$ for $i \in \{1, 2\}$ are multivalued operators, here $P(X)$ stands for the family of all nonempty subets of X). The system above can be represented as a fixed point problem of the form

$$x \in T(x) \quad (\text{where } T := (T_1, T_2) : X^2 \to P(X^2) \text{ and } x = (x_1, x_2)).$$

Hence, it is of great interest to give fixed point results for multivalued operators on a set endowed with vector-valued metrics or norms. However, the advantages of a vector-valued norm with respect to the usual scalar norms were already very nice pointed out by R. Precup in [13].

On the other hand, the concept of multivalued weakly Picard operator (briefly MWP operator) was introduced in close connection with the successive approximation method and the data dependence phenomenon for the fixed point set of multivalued operators on complete metric space, by I.A. Rus, A. Petruşel and A. Sântămărian (see [17]). Moreover, in [11] the theory of multivalued weakly Picard operators in L-spaces is developed. On the other hand, A.I. Perov [9] and A.I. Perov and A.V. Kibenko [10] proved a generalization of the Banach contraction principle for operators on a space endowed with vector-valued metrics. For generalizations of Perov's result, see J. Matkowski [4]-[5], T. Shibata [20], M. Turinici [21], C. Bacoţiu [1], etc. Some new fixed points theorems for singlevalued operators on a set with two vector-valued metric were established in D. O'Regan, N. Shahzad, R.P. Agarwal [8] and D. O'Regan and R. Precup [7]. Fixed point theorems for contractive multivalued operators in terms of vector-valued metrics were proved in M. Turinici [21], while the theory of multivalued contractions on a set endowed with two metrics was recently treated by A. Petruşel and I.A. Rus in [12]. For a comprehensive study of the above topics see also I.A. Rus, A. Petruşel, G. Petruşel [19]. For the theory of a (metrical) fixed point theorem see I.A. Rus [16].

The purpose of this paper is to present some new fixed point results for multivalued operators on a space endowed with one or two vector-valued metrics. Our theorems extend the works mentioned above and offer a nice tool for the study of semilinear inclusion systems which appear in nonlinear analysis.

2. Preliminaries

Let (X, d) be metric space. We will use the following notations:

$P(X)$ - *the set of all nonempty subsets of* X;

$\mathcal{P}(X) = P(X) \bigcup \{\emptyset\}$;

$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;

$P_b(X)$ - *the set of all nonempty bounded subsets of* X;

$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed, subsets of* X;

For two subsets $A, B \in P_b(X)$ we recall the following functionals.

$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, D(Z, Y) = inf\{d(x, y) : x \in Z \ , y \in Y\}$, $Z \subset X$ - *the gap functional.*

$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \delta(A, B) := sup\{d(a, b) | x \in A, b \in B\}$ - *the diameter functional*;

$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \rho(A, B) := sup\{D(a, B) | a \in A\}$ - *the excess functional*;

$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, H(A, B) := max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}$ - *the Pompeiu-Hausdorff functional.*

Definition 2.2. Let X be a nonempty set and consider the space $\mathbb{R}^m_+$ endowed with the usual component-wise partial order. The mapping $d : X \times X \to \mathbb{R}^m_+$ which satisfies all the usual axioms of the metric is called a generalized metric in the sense of Perov.

Notice that the generalized metric in the sense of Perov is a particular case of K-metric. See P.P. Zabrejko [22] for a nice survey on this topic. For some fixed point theorems for singlevalued operators on K-metric spaces, see I.A. Rus, A. Petruşel, M.A. Şerban [18].

Let (X, d) be a generalized metric space in Perov' sense. Here, if $v, r \in \mathbb{R}^m$, $v := (v_1, v_2, \cdots, v_m)$ and $r := (r_1, r_2, \cdots, r_m)$, then by $v \leq r$ we mean $v_i \leq r_i$, for each $i \in \{1, 2, \cdots, m\}$, while $v < r$ stands for $v_i < r_i$, for each $i \in \{1, 2, \cdots, m\}$. Also, $|v| := (|v_1|, |v_2|, \cdots, |v_m|)$. If $u, v \in \mathbb{R}^m$, with $u := (u_1, u_2, \cdots, u_m)$ and $v := (v_1, v_2, \cdots, v_m)$, then $max(u, v) :=$

$(max(u_1, v_1), \cdots, max(u_m, v_m))$ and, if $c \in \mathbb{R}$ then $v \leq c$ means $v_i \leq c$, for each $i \in \{1, 2, \cdots, m\}$.

Notice that in a generalized metric space in Perov' sense the concepts of Cauchy sequence, convergent sequence, completeness, open and closet subsets are similar defined as those in a metric space. If $x_0 \in X$ and $r \in \mathbb{R}^m_+$ with $r_i > 0$ for each $i \in \{1, 2, \cdots, m\}$ we will denote by $B(x_0; r) := \{x \in X|\ d(x_0, x) < r\}$ the open ball centered in x_0 with radius $r := (r_1, r_2, \cdots, r_m)$ and by $\widetilde{B}(x_0; r) := \{x \in X|\ d(x_0, x) \leq r\}$ the closed ball centered in x_0 with radius r.

If $T : X \to P(X)$ is a multivalued operator, then we denote by $Fix(T)$ the fixed point set of T, i.e. $Fix(T) := \{x \in X|x \in T(x)\}$ and by $SFix(T)$ the strict fixed point set of T, i.e. $SFix(T) := \{x \in X|\{x\} = T(x)\}$. The symbol $Graph(T) := \{(x, y) \in X \times X : y \in T(x)\}$ denotes the graph of T.

Definition 2.3. Let $(X, \to)$ be a generalized metric space in the sense of Perov. Then $T : X \to P(X)$ is a multivalued weakly Picard operator (briefly MWP operator), if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point of T.

Notice that the sequence$(x_n)_{n\in\mathbb{N}}$ having the properties (i)-(ii) is called a sequence of successive approximations for T starting from (x, y).

For several examples of MWP operators, see [11], [17].

Throughout this paper we denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m\times m$ matrices with positive elements, by Θ the zero $m\times m$ matrix and by I the identity $m\times m$ matrix. If $A \in M_{m,m}(\mathbb{R}_+)$, then the symbol A^τ stands for the transpose matrix of A.

Recall that a matrix A issaid to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.

Notice that, for the proof of the main results, we need the following theorem, see [14].

Theorem 2.4. *Let $A \in M_{m,m}(\mathbb{R}_+)$. The following are equivalents:*

(i) $A^n \to 0$ as $n \to \infty$;

(ii)The eigen-values of A are in the open unit disc, i.e. $|\lambda| < 1$*, for every* $\lambda \in \mathbb{C}$ *with* $det(A - \lambda I) = 0$*;*

(iii) The matrix $I - A$ *is non-singular and*

$$(I - A)^{-1} = I + A + ... + A^n + ...;$$

(iv) The matrix $I - A$ *is non-singular and* $(I - A)^{-1}$ *has nonnenegative elements.*

(v) $A^n q \to 0$ *and* $qA^n \to 0$ *as* $n \to \infty$*, for each* $q \in \mathbb{R}^m$*.*

For examples and other considerations on matrices which converges to zero, see I.A. Rus [14], M. Turinici [21], etc.

3. Main results

Throughout this section (X, d) is a generalized metric space in Perov' sense. We start our considerations by presenting a local fixed point theorem for a multivalued operator on a generalized metric space in the sense of Perov.

Definition 3.1. Let $Y \subset X$ and $T : Y \to P(X)$ be a multivalued operator. Then, T is called a multivalued left A-contraction if $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and for each $x, y \in Y$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $d(u, v) \leq Ad(x, y)$.

Remark 3.2 In particular, if T is singlevalued we obtain the concept of A-contraction given by A.I. Perov in [9] and A.I. Perov and A.V. Kibenko in [10].

Theorem 3.3. *Let* (X, d) *be a complete generalized metric space,* $x_0 \in X$ *and* $r := (r_i)_{i=1}^m \in \mathbb{R}^m_+$ *with* $r_i > 0$ *for each* $i \in \{1, 2, \cdots, m\}$*. Let* $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ *be a multivalued left* A*-contraction. Suppose that:*

(i) if $v, r \in \mathbb{R}^m_+$ *are such that* $v \cdot (I - A)^{-1} \leq (I - A)^{-1} \cdot r$*, then* $v \leq r$*;*

(ii) there exists $x_1 \in T(x_0)$ *such that* $d(x_0, x_1)(I - A)^{-1} \leq r$*.*

Then, T *has at least one fixed point.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$ such that $d(x_0, x_1)(I - A)^{-1} \leq r \leq (I - A)^{-1} \cdot r$. Then, by (i), $x_1 \in \widetilde{B}(x_0; r)$. Now, by the contraction condition, there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) \leq Ad(x_0, x_1)$. Thus $d(x_1, x_2)(I - A)^{-1} \leq Ad(x_0, x_1)(I - A)^{-1} \leq Ar$. Notice that $x_2 \in \widetilde{B}(x_0; r)$. Indeed, since $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$ we get that $d(x_0, x_2)(I - A)^{-1} \leq d(x_0, x_1)(I - A)^{-1} + d(x_1, x_2)(I - A)^{-1} \leq Ir + Ar \leq (I - A)^{-1} r$, which immediately implies (by (i)) that $d(x_0, x_2) \leq r$.

By induction, we construct the sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{B}(x_0;r)$ having the properties:

(a) $x_{n+1}\in T(x_n), n\in\mathbb{N}$;

(b) $d(x_0,x_n)(I-A)^{-1}\le(I-A)^{-1}r$, for each $n\in\mathbb{N}^*$, that means (by (i)) $d(x_0,x_n)\le r$;

(c) $d(x_n,x_{n+1})(I-A)^{-1}\le A^n r$, for each $n\in\mathbb{N}$.

By (c) we get that

$$d(x_n,x_{n+p})(I-A)^{-1}\le A^n(I-A)^{-1}r, \text{ for all } n\in\mathbb{N} \text{ and } p\in\mathbb{N}^*.$$

Thus, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in the complete metric space $(\widetilde{B}(x_0;s),d)$. Denote by x^* its limit in $\widetilde{B}(x_0;r)$.

We prove now that $x^*\in T(x^*)$. If $n\in\mathbb{N}^*$, for each $x_n\in T(x_{n-1})$ there exists $u_n\in T(x^*)$ such that $d(x_n,u_n)\le Ad(x_{n-1},x^*)$.

On the other hand $d(x^*,u_n)\le d(x^*,x_n)+d(x_n,u_n)\le d(x^*,x_n)+Ad(x_{n-1},x^*)\to 0$ as $n\to\infty$. Hence $\lim\limits_{n\to\infty}u_n=x^*$.

Since $u_n\in T(x^*)$ for $n\in\mathbb{N}^*$ and using the fact that $T(x^*)$ is closed, it follows that $x^*\in T(x^*)$. The proof is complete. □

As a consequence of the previous theorem, if $T:X\to P_{cl}(X)$ is a multivalued left A-contraction on the complete generalized metric space (X,d), then we have the following result (see also Theorem 8.1 in [11]):

Corollary 3.4. *Let (X,d) be a complete generalized metric space and $T:X\to P_{cl}(X)$ be a multivalued left A-contraction. Then, T is a MWP operator.*

A dual concept is given in the following definition.

Definition 3.5. Let $Y\subset X$ and $T:Y\to P(X)$ be a multivalued operator. Then, T is called a multivalued right A-contraction if $A\in\mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and for each $x,y\in Y$ and each $u\in T(x)$ there exists $v\in T(y)$ such that $d(u,v)^\tau\le d(x,y)^\tau A$.

Remark 3.6. *Notice that, since $(d(x,y)^\tau A)^\tau=A^\tau d(x,y)$, the right A-contraction condition on the multivalued operator T is equivalent to the left A^τ-contraction condition given in Definition 3.1. It is also obvious that the matrix A converges to zero if and only if the matrix A^τ converges to zero (since A and A^τ have the same eigenvalues) and $[(I-A)^{-1}]^\tau=(I-A^\tau)^{-1}$.*

From Remark 3.6 and Theorem 3.3 we get the following dual result:

Theorem 3.7. *Let (X,d) be a complete generalized metric space, $x_0 \in X$ and $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ with $r_i > 0$ for each $i \in \{1,2,\cdots,m\}$. Let $T : \widetilde{B}(x_0;r) \to P_{cl}(X)$ be a multivalued right A-contraction. Suppose that:*

(i) if $v, r \in \mathbb{R}_+^m$ are such that $(I-A)^{-1} \cdot v \le r \cdot (I-A)^{-1}$, then $v \le r$;

(ii) there exists $x_1 \in T(x_0)$ such that $(I-A)^{-1} d(x_0,x_1) \le r$.

Then, T has at least one fixed point.

We are now interested for the problem that, if $T : \widetilde{B}(x_0;r) \to P_{cl}(X)$ is a multivalued A-contraction, wheather the closed ball $\widetilde{B}(x_0;r)$ is invariant with respect to T.

For example, for the case of a multivalued right A-contraction, we have:

Theorem 3.8. *Let (X,d) be a complete generalized metric space, $x_0 \in X$ and $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ with $r_i > 0$ for each $i \in \{1,2,\cdots,m\}$. Let $T : \widetilde{B}(x_0;r) \to P_{cl}(X)$ be a multivalued right A-contraction. Suppose also that $d(x_0,u)(I-A)^{-1} \le r$, for each $u \in T(x_0)$.*

Then, the following assertions hold:

a) $\widetilde{B}(x_0;r)$ is invariant with respect to T;

b) T is a MWP operator on $\widetilde{B}(x_0;r)$.

Proof. a) In order to prove that $\widetilde{B}(x_0;r)$ is invariant with respect to T, let us consider $x \in \widetilde{B}(x_0;r)$. Then, we have to show that $T(x) \subseteq \widetilde{B}(x_0;r)$. For this purpose, let $y \in T(x)$ be arbitrarily chosen. Then, by the contraction condition, there exists $u \in T(x_0)$ such that $d(y,u) \le d(x_0,x)A$. Then, by the triangle inequality, we get that:
$d(x_0,y)(I-A)^{-1} \le d(x_0,u)(I-A)^{-1} + d(u,y)(I-A)^{-1} \le r + d(x_0,x)A(I-A)^{-1} \le r + rA(I-A)^{-1} = r[I + A(I-A)^{-1}] = r[I + A(I + A + A^2 + \cdots)] = r(I-A)^{-1}$. Thus, we get that $d(x_0,y) \le r$. Hence, the proof of a) is complete.
b) Since $T : \widetilde{B}(x_0;r) \to P_{cl}(\widetilde{B}(x_0;r))$, Corollary 3.4 (see also Theorem 8.1 in [11]) applies and the conclusion follows. □

A dual result is:

Theorem 3.9. *Let (X,d) be a complete generalized metric space, $x_0 \in X$ and $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ with $r_i > 0$ for each $i \in \{1,2,\cdots,m\}$. Let $T : \widetilde{B}(x_0;r) \to P_{cl}(X)$ be a multivalued left A-contraction. Suppose also that $(I-A)^{-1} d(x_0,u) \le r$, for each $u \in T(x_0)$.*

Then, the following assertions hold:

a) $\widetilde{B}(x_0;r)$ is invariant with respect to T;

b) T is a MWP operator on $\widetilde{B}(x_0;r)$.

Notice that, if T is singlevalued, then we obtain a local result for nonself singlevalued A-contractions.

Corollary 3.10. *Let (X,d) be a complete generalized metric space, $x_0 \in X$ and $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ with $r_i > 0$ for each $i \in \{1,2,\cdots,m\}$. Let $f : \widetilde{B}(x_0;r) \to X$ be a singlevalued left A-contraction. Suppose also that $(I-A)^{-1}d(x_0,f(x_0)) \le r$.*

Then, f has a unique fixed point in $\widetilde{B}(x_0;r)$.

Notice that a generalized Popmpeiu-Hausdorff functional can be introduced in the setting of a generalized metric space in the sense of Perov. Namely, if (X,d) is a generalized metric space in the sense of Perov with $d := (d_1,\cdots,d_m)$ and if H_i denotes the Pompeiu-Hausdorff metric on $P_{b,cl}(X)$ generated by d_i, (where $i \in \{1,2,\cdots,m\}$) then be denote by $H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbb{R}_+^m$, $H := (H_1,\cdots,H_m)$ the vector-valued Pompeiu-Hausdorff metric on $P_{b,cl}(X)$.

By definition, a multivalued operator $T : Y \subseteq X \to P_{b,cl}(X)$ is said to be a multivalued left A-contraction in the sense of Nadler if $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and and

$$H(T(x),T(y)) \le Ad(x,y), \text{ for all } x,y \in Y.$$

Notice that for $m = 1$ we get the well-known concept of contraction mapping introduced by S.B. Nadler Jr. [6]. We point out also that, by the properties of the functional H, if T is a multivalued left A-contraction, then T is a multivalued left A-contraction in the sense of Nadler.

Remark 3.11. *If (X,d) is a complete generalized metric space and $T : X \to P_{b,cl}(X)$is a multivalued left A-contraction in the sense of Nadler, then it is an open problem to establish a fixed point theory for T.*

We will present now some applications of the above results.

Theorem 3.12. *Let $(X,|\cdot|)$ be a Banach space and $T_1,T_2 : X \to P_{cl}(X)$ be two multivalued operators. Suppose there exist $a_{ij} \in \mathbb{R}_+$, $i,j \in \{1,2\}$ such that:*

(1) for each $u := (u_1,u_2), v := (v_1,v_2) \in X \times X$ and each $y_1 \in T_1(u_1,u_2)$ there exists $z_1 \in T_1(v_1,v_2)$ such that:

$$|y_1 - z_1| \le a_{11}|u_1 - v_1| + a_{12}|u_2 - v_2|$$

(2) for each $u := (u_1, u_2), v := (v_1, v_2) \in X \times X$ *and each* $y_2 \in T_2(u_1, u_2)$ *there exists* $z_2 \in T_2(v_1, v_2)$ *such that*

$$|y_2 - z_2| \leq a_{21}|u_1 - v_1| + a_{22}|u_2 - v_2|.$$

In addition assume that the matrix $A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *converges to* 0.

Then, the semilinear inclusion system:

$$\begin{cases} u_1 \in T_1(u_1, u_2) \\ u_2 \in T_1(u_1, u_2) \end{cases}$$

has at least one solution in $X \times X$*.*

Proof. Consider the multivalued operator $T : X^2 \to P_{cl}(X^2)$ given by $T := (T_1, T_2)$. Then, the conditions (1)+(2) can be represented in the follwing form: for each $u, v \in X^2$ and each $y \in T(u)$ there exists $z \in T(v)$ such that

$$\|y - z\| \leq A \cdot \|u - v\|.$$

Hence, Corollary 3.4 applies (with $d(u, v) := \|u - v\| := \begin{pmatrix} |u_1 - v_1| \\ |u_2 - v_2| \end{pmatrix}$) and T has at least one fixed point $u \in T(u)$. □

Notice that, similar results can be obtained for arbitrary $n \in \mathbb{N}$. See also S. Czerwik [2] and M. Turinici [21] for other theorems of this type.

We will prove now an open operator principle in generalized Banach spaces in Perov' sense. For this purpose, we recall some useful concepts and results.

Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ be a matrix convergent to zero. Then:

$$(\alpha)\ r \leq (I - A)^{-1} r, \text{ for each } r \in \mathbb{R}^n_+;$$

$$(\beta) \text{ the set } P_A := \{\rho \in \mathbb{R}^n \ : \rho > 0,\ (I - A)\rho > 0\} \text{ is nonempty}$$

$$\text{and coincide with the set } Q_A := \{(I - A)^{-1} r \ : r \in \mathbb{R}^n, r > 0\}.$$

$$(\gamma) \text{ a subset } U \subset X \text{ is open if and only if for each } x \in U \text{ there exists } \rho \in P_A$$

$$\text{such that } \widetilde{B}(x; \rho) \subset U.$$

Let E be a Banach space and $Y \subset E$. Given an operator $f : Y \to E$, the operator $g : Y \to E$ defined by $g(x) := x - f(x)$ is called the field associated with f. An operator $f : Y \to E$ is said to be open if for any open subset U of Y the set $f(U)$ is open in E too.

As a consequence of Corollary 3.10 we have the following domain invariance theorem for contraction type fields in generalized Banach spaces.

Theorem 3.13. *Let E be a generalized Banach space in Perov' sense and let U be an open subset of E. Let $f : U \to E$ be a left A-contraction. Let $g : U \to E$ $g(x) := x - f(x)$ be the associated field. Then $g : U \to E$ is an open operator.*

Proof. We will prove that for each open subset V of U, the set $g(V)$ is open in E too. For this purpose, we will show that for each $y \in g(V)$ there exists $r \in \mathbb{R}^n$, $r > 0$ such that $\widetilde{B}(y, r) \subset g(V)$. Since $y \in g(V)$, it is enough to prove that $\widetilde{B}(g(x), r) \subset g(V)$, for some $x \in V$. Now, since V is open, for each $x \in V$ there exists (by (γ)) $p \in P_A$ such that $\widetilde{B}(x, p) \subset V$. Thus, $g(\widetilde{B}(x, p)) \subset g(V)$. The proof is complete if we prove that there exists $r > 0$, $r \in \mathbb{R}^n$ such that $\widetilde{B}(g(x), r) \subset g(\widetilde{B}(x, p))$. We will show that the relation take place with $r := (I - A)p$. Notice that $r > 0$, by (β).

For this aim, let us consider an arbitrary $z \in \widetilde{B}(g(x), r)$. Then $\|g(x)-z\| \leq r$. Let us show that there exists $u \in \widetilde{B}(x, p)$ such that $g(u) = z$, which also means that u is a fixed point for thet operator $h : \widetilde{B}(x, p) \to E$, $h(t) := f(t)+z$. Obviously, since f is a left A-contraction, h is a left A-contraction too. On the other hand, we have:
$\|h(x) - x\| = \|g(x) - z\| \leq r := (I - A)p$. By multiplying on the left with $(I - A)^{-1}$ we get that

$$(I - A)^{-1}\|h(x) - x\| \leq p.$$

By Corollary 3.10 we get that there exists a unique $u \in \widetilde{B}(x, p)$ such that $h(u) = u$. The proof is now complete. □

A homotopy result for multivalued operators on a set endowed with a vector-valued metric is the following.

Theorem 3.14. *Let (X, d) be a generalized complete metric space in Perov' sense, U be an open subset of X and V be a closed subset of X, with $U \subset V$.*

Let $G : V \times [0,1] \to P(X)$ be a multivalued operator with closed (with respect to d) graph, such that the following conditions are satisfied:

(a) $x \notin G(x,t)$, for each $x \in V \setminus U$ and each $t \in [0,1]$;

(b) there exists a matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ convergent to zero such that for each $t \in [0,1]$, each $x, y \in Y$ and each $u \in G(x,t)$ there exists $v \in G(y,t)$ such that $d(u,v) \leq Ad(x,y)$;

(c) there exists a continuous increasing function $\phi : [0,1] \to \mathbb{R}^m$ such that for all $t, s \in [0,1]$, each $x \in V$ and each $u \in G(x,t)$ there exists $v \in G(x,s)$ such that $d(u,v) \leq |\phi(t) - \phi(s)|$;

(d) if $v, r \in \mathbb{R}^m_+$ are such that $v \cdot (I-A)^{-1} \leq (I-A)^{-1} \cdot r$, then $v \leq r$;

Then $G(\cdot,0)$ has a fixed point if and only if $G(\cdot,1)$ has a fixed point.

Proof. Suppose $G(\cdot,0)$ has a fixed point z. From (a) we have that $z \in U$. Define

$$Q := \{(t,x) \in [0,1] \times U |\ x \in G(x,t)\}.$$

Clearly $Q \neq \emptyset$, since $(0,z) \in Q$. Consider on Q a partial order defined as follows:

$$(t,x) \leq (s,y) \text{ if and only if } t \leq s \text{ and } d(x,y) \leq 2[\phi(s) - \phi(t)] \cdot (I-A)^{-1}.$$

Let M be a totally ordered subset of Q and consider $t^* := \sup\{t|(t,x) \in M\}$. Consider a sequence $(t_n, x_n)_{n\in\mathbb{N}^*} \subset M$ such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ for each $n \in \mathbb{N}^*$ and $t_n \to t^*$, as $n \to +\infty$. Then

$$d(x_m, x_n) \leq 2[\phi(t_m) - \phi(t_n)] \cdot (I-A)^{-1}, \text{ for each } m, n \in \mathbb{N}^*, m > n.$$

When $m, n \to +\infty$ we obtain $d(x_m, x_n) \to 0$ and, thus, $(x_n)_{n\in\mathbb{N}^*}$ is d-Cauchy. Thus $(x_n)_{n\in\mathbb{N}^*}$ is convergent in (X,d). Denote by $x^* \in X$ its limit. Since $x_n \in G(x_n, t_n), n \in \mathbb{N}^*$ and since G is d-closed, we have that $x^* \in G(x^*, t^*)$. Thus, from (a), we have $x^* \in U$. Hence $(t^*, x^*) \in Q$. Since M is totally ordered we get that $(t,x) \leq (t^*, x^*)$, for each $(t,x) \in M$. Thus (t^*, x^*) is an upper bound of M. By Zorn's Lemma, Q admits a maximal element $(t_0, x_0) \in Q$. We claim that $t_0 = 1$. This will finish the proof.

Suppose $t_0 < 1$. Choose $r \in \mathbb{R}^*_+$ and $t \in]t_0, 1]$ such that $B(x_0, r) \subset U$ and $r := 2[\phi(t) - \phi(t_0)] \cdot (I-A)^{-1}$. Since $x_0 \in G(x,t_0)$, by (c), there exists $x_1 \in G(x_0, t)$ such that $d(x_0, x_1) \leq |\phi(t) - \phi(t_0)|$. Thus, $d(x_0, x_1)(I-A)^{-1} \leq |\phi(t) - \phi(t_0)| \cdot (I-A)^{-1} < r$.

Since $\overline{B}(x_0, r) \subset V$, the multivalued operator $G(\cdot, t) : \overline{B}(x_0, r) \to P_{cl}(X)$ satisfies, for all $t \in [0, 1]$, the assumptions of Theorem 3.3. Hence, for all $t \in [0, 1]$, there exists $x \in \overline{B}(x_0, r)$ such that $x \in G(x, t)$. Thus $(t, x) \in Q$. Since $d(x_0, x) \leq r = 2[\phi(t) - \phi(t_0)](I - A)^{-1}$, we immediately get that $(t_0, x_0) < (t, x)$. This is a contradiction with the maximality of (t_0, x_0).

Conversely, if $G(\cdot, 1)$ has a fixed point, then putting $t := 1 - t$ and using first part of the proof we get the conclusion. □

Remark 3.15. *1) Usually in the above result, we take $Q = \overline{U}$. Notice that in this case, condition (a) becomes:*

(a') $x \notin G(x, t)$, for each $x \in \partial U$ and each $t \in [0, 1]$.

2) If, in the above theorem, we replace (c) by the following condition:

(c') there exists a continuous increasing function $\phi : [0, 1] \to \mathbb{R}^m$ such that for all $t, s \in [0, 1]$, each $x \in V$ and each $u \in G(x, t)$ and $v \in G(x, s)$ we have:

$$d(u, v) \leq |\phi(t) - \phi(s)|;$$

then the same conclusion holds in the absence of assumption (d). Here we apply Theorem 3.9 instead of Theorem 3.3.

Some other important concepts will be presented now.

Definition 3.16. a) Let (X, d) be a generalized metric space in Perov' sense and $T : X \to P(X)$ be an MWP operator. Then, we define the multivalued operator $T^\infty : Graph(T) \to P(Fix(T))$ by the formula $T^\infty(x, y) := \{ z \in Fix(T) \mid$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z \}$.
b) Let (X, d) be a generalized metric space in Perov' sense and $T : X \to P(X)$ be a MWP operator. Then, T is called a C-multivalued weakly Picard operator (briefly C-MWP operator) if and only if $C \in \mathcal{M}_{m,m}(\mathbb{R}_+) \setminus \{\Theta\}$ and there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x, y)) \leq C\ d(x, y), \text{ for all } (x, y) \in Graph(T).$$

By the proof of Theorem 3.3 and by Remark 3.4 and the above definition, we have:

Theorem 3.17. *Let (X, d) be a complete generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued left A-contraction. Then, T is a $(I - A)^{-1}$-MWP operator.*

Proof. By the proof of Theorem 3.3., we have that for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from (x, y) such that:

$$d(x_n, x_{n+p}) \leq A^n(I - A)^{-1}d(x_0, x_1), \text{ for all } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*.$$

The sequence converges and its limit is a fixed point $t^\infty(x, y)$ of T. By letting $p \to +\infty$ and putting $n := 0$ we get that:

$$d(x, t^\infty(x, y)) \leq (I - A)^{-1}d(x, y). \square$$

As in metric spaces, the following abstract data dependence result holds. Notice that, if $d := (d_1, \cdots, d_m)$, then we denote by H, the vector-valued Pompeiu-Hausdorff (generalized) metric with components H_{d_i}, for $i \in \{1, 2, \cdots, m\}$.

Theorem 3.18. *Let (X, d) be a generalized metric space and $T_1, T_2 : X \to P_{cl}(X)$ be two multivalued operators. We suppose that:*

i) T_i is a C_i-MWP operator, for $i \in \{1, 2\}$;

ii) there exists $\eta \in \mathbb{R}^m_+$ with $\eta_i > 0$ for each $i \in \{1, 2, \cdots, m\}$, such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.

Then

$$H(Fix(T_1), Fix(T_2)) \leq \max \{ C_1\eta, C_2\eta\}.$$

We are going to prove now a strict fixed point theorem for a multivalued Perov type operator.

Theorem 3.19. *Let (X, d) be a complete generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued left A-contraction. Suppose that $SFix(T) \neq \emptyset$. Then, $Fix(T) = SFix(T) = \{x^*\}$.*

Proof. Suppose that $x^* \in SFix(T)$. Then $T(x^*) = \{x^*\}$. We will prove that $Fix(T) \subset SFix(T)$. Indeed, if $y \in Fix(T)$, then, by the contraction condition we have that $d(y, x^*) \leq Ad(y, x^*)$. Hence, $(I - A)d(y, x^*) \leq 0$ and, by multiplying with $(I - A)^{-1}$, we get $d(y, x^*) \leq 0$. Thus $d(y, x^*) = 0$. $\square$

The concept of well-posedness for the fixed point problem is defined as follows.

Definition 3.20. Let (X, d) be a generalized metric space and $T : X \to P_{cl}(X)$. The fixed point problem for T is well posed if and only if:

a) $Fix(T) = \{x^*\}$;

b) if $(x_n, y_n) \in Graph(T)$, for $n \in \mathbb{N}$ is such that $d(x_n, y_n) \to 0$ as $n \to +\infty$, then $d(x_n, x^*) \to 0$ as $n \to +\infty$.

The following well-posedness result holds.

Theorem 3.21. *Let (X, d) be a complete generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued left A-contraction. Suppose that $SFix(T) \neq \emptyset$. Then, the fixed point problem for T is well-posed.*

Proof. By Theorem 3.18 we have that $Fix(T) = SFix(T) = \{x^*\}$. Let $(x_n, y_n) \in Graph(T)$, for $n \in \mathbb{N}$ such that $d(x_n, y_n) \to 0$ as $n \to +\infty$. Then, by the contraction condition, we have that $d(y_n, x^*) \leq Ad(x_n, x^*)$ for $n \in \mathbb{N}$. Then $d(x_n, x^*) \leq d(x_n, y_n) + d(y_n, x^*) \leq d(x_n, y_n) + Ad(x_n, x^*)$ for $n \in \mathbb{N}$. Thus $(I - A)d(x_n, x^*) \leq d(x_n, y_n) \to 0$ as $n \to +\infty$. The proof is complete. □

Next theorem is a fixed point result for a multivalued operator on a space endowed with two generalized metric spaces.

Theorem 3.22. *Let (X, d) be a complete generalized metric space and ρ be another generalized metric on X. Let $T : X \to P(X)$ be a multivalued operator.*

Suppose that:

(i) there exists $C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that $d(x, y) \leq C\rho(x, y)$ for each $x, y \in X$;

(ii) $T : (X, d) \to (P(X), H_d)$ has closed graph;

(iii) T is a multivalued left A-contraction with respect to ρ;

Then T has a fixed point. Moreover, T is a MWP operator with respect to d.

Proof. Let $x_0 \in X$ such that $x_1 \in T(x_0)$. Then for $x_2 \in T(x_1)$ we have $\rho(x_1, x_2) \leq A\rho(x_0, x_1)$. Thus we can define the sequence $\{x_n\} \in X$ such that $x_{n+1} \in T(x_n)$ and $\rho(x_n, x_{n+1}) \leq A^n\rho(x_{n-1}, x_n)$ for every $n \in \mathbb{N}$.

Then we have, for any $n \in \mathbb{N}$,

$$\rho(x_n, x_{n+1}) \leq A\rho(x_{n-1}, x_n) \leq ... \leq A^n\rho(x_0, x_1).$$

Hence, for any $m, n \in \mathbb{N}$ with $m \geq n$, and using Theorem 2.4 it follows that

$$\begin{aligned}\rho(x_n, x_{n+m}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + ... + \rho(x_{n+m-1}, x_{n+m}) \\ &\leq A^n\rho(x_0, x_1) + A^{n+1}\rho(x_0, x_1) + ... + A^{n+m-1}\rho(x_0, x_1) \\ &\leq A^n(I - A)^{-1}\rho(x_0, x_1).\end{aligned}$$

From (iii) we have that $A^n \to 0$ as $n \to \infty$. Thus, the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, ρ). From (i) it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy in (X, d). Thus, $(x_n)_{n\in\mathbb{N}}$ converges with respect to d to some point $x^* \in X$. We have to prove now that $x^* \in T(x^*)$. Since $x_n \in T(x_{n-1})$ for all $n \in \mathbb{N}^*$, by (ii), we get that $x^* \in T(x^*)$. The proof is now complete. □

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