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APPROXIMATION OF THE SOLUTIONS OF THE DARBOUX PROBLEM FOR THIRD ORDER HYPERBOLIC INCLUSIONS

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Abstract. In this paper we consider the Darboux problem for a third order hyperbolic inclusion of the form $u_{xyz} \in F(x, y, z, u)$. Using the notion of uniform convergence on compact domains as defined by Arrigo Cellina for a sequence of single-valued functions $f_k : \Lambda \to \mathbb{R}^n$ such that $f_k \to F$, where F is a multifunction, it is considered a sequence of approximating univalued equations of the form $u_{xyz} = f_k(x, y, z, u)$ and it is proved that they have a unique solution based on Schauder's Fixed Point Theorem. Using a characterization theorem for the solutions of the Darboux Problem for the specified inclusion, it is proved that the sequence of solutions to the univalued equations uniformly converges, on compact sets, to a solution of the Darboux Problem for the considered inclusion.

Key Words and Phrases: multifunction, hyperbolic inclusion, upper semi-continuity, initial values, absolutely continuous in Carathéodory's sense function, Aumann integral, uniform convergence of a sequence of single-valued functions on compact sets to a multifunction.
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1. Introduction

In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form

$$\frac{\partial^3 u(x, y, z)}{\partial x\, \partial y\, \partial z} \in F(x, y, z, u), \ (x, y, z) \in D = [0, a] \times [0, b] \times [0, c], \ u \in \Omega \subset \mathbb{R}^n \tag{1.1}$$

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with the initial values

$$\begin{cases} u\,(x,y,0)=\varphi\,(x,y)\,, & (x,y)\in D_1=[0,a]\times[0,b]\,,\\ u\,(0,y,z)=\psi\,(y,z)\,, & (y,z)\in D_2=[0,b]\times[0,c]\,,\\ u\,(x,0,z)=\chi\,(x,z)\,, & (x,z)\in D_3=[0,a]\times[0,c] \end{cases} \tag{1.2}$$

where φ,ψ,χ are absolutely continuous in Carathéodory's sense functions [2, §565 - §570], $\varphi\in C^*\,(D_1;\mathbb{R}^n)$, $\psi\in C^*\,(D_2;\mathbb{R}^n)$, $\chi\in C^*\,(D_3;\mathbb{R}^n)$ and they satisfy the conditions

$$\begin{cases} u\,(x,0,0)=\varphi\,(x,0)=\chi\,(x,0)=v^1\,(x)\,, & x\in[0,a]\,,\\ u\,(0,y,0)=\varphi\,(0,y)=\psi\,(y,0)=v^2\,(y)\,, & y\in[0,b]\,,\\ u\,(0,0,z)=\psi\,(0,z)=\chi\,(0,z)=v^3\,(z)\,, & z\in[0,c]\,,\\ u\,(0,0,0)=v^1\,(0)=v^2\,(0)=v^3\,(0)=v^0, & \end{cases} \tag{1.3}$$

where $F:D\times\Omega\rightarrow 2^{\mathbb{R}^n}$ is a multifunction with compact convex and non-empty values and $\Omega\subset\mathbb{R}^n$ is an open subset.

Under suitable assumptions, we proved in [27] an existence theorem for a local solution of the Darboux Problem (1.1) + (1.2) and that the set of its solutions is compact in Banach space $C\,(D_0;\mathbb{R}^n)$, $D_0=[0,x_0]\times[0,y_0]\times[0,z_0]\subseteq D$; moreover, as a function of the initial values this set defines an upper semi-continuous multifunction.

In [28] we proved a theorem of prolongation for the solutions of the considered problem and also an existence theorem for a saturated solution.

In [29] we proved a characterization theorem for the solutions of Darboux Problem (1.1) + (1.2) using the Aumann integral defined for multifunctions.

In this paper, using the notion of uniform convergence on compact sets as defined by Arrigo Cellina for a sequence of single-valued functions $f_k:\Lambda\rightarrow\mathbb{R}^n$ such that $f_k\rightarrow F$, where F is a multifunction, we consider a sequence of approximating univalued equations of the form $u_{xyz}=f_k\,(x,y,z,u)$ and we prove that they have a unique solution based on Schauder's Fixed Point Theorem. Using a characterization theorem for the solutions of the Darboux Problem for the specified inclusion, we prove that the sequence of solutions to the approximating univalued equations uniformly converges, on compact sets, to a solution of the Darboux Problem (1.1) + (1.2) for the considered inclusion. This study has been suggested by [26] and it provides an extension of the results in that article.

2. Preliminaries

The definitions and Theorems 2.1-2.7 plus Propositions 2.1-2.4 in this section are taken from [1], [2], [5]-[29].

Definition 2.1. Let X and Y be two non-empty sets. A *multifunction* $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup_{x \in X} \Phi(x)$ is the *range* of Φ. $\Phi(X) = \left\{ \bigcup_{x \in X} \Phi(x) \mid x \in X \right\}$.

Definition 2.2. Let us consider $\Phi : X \to 2^Y$.

a) If $A \subset X$, the *image* of A by Φ is $\Phi(A) = \bigcup_{x \in A} \Phi(x)$;

b) If $B \subset Y$, the *counterimage* of B by Φ is

$$\Phi^{-}(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) The *graph* of Φ, denoted graph Φ, is the set

$$\operatorname{graph} \Phi = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}.$$

Definition 2.3. Let us now take $\Phi : X \to 2^Y$. An element $x \in X$ with the property $x \in \Phi(x)$ is called a *fixed point* of the multifunction Φ.

Definition 2.4. A univalued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Definition 2.5. Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is *upper semi-continuous* if, for any closed $B \subset Y$, $\Phi^{-}(B)$ is closed in X.

Definition 2.6. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* if $\Phi^{-}(B) \in \mathcal{F}$ for every closed subset $B \subset Y, \mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^{-}(B)$ is measurable.

Theorem 2.1. [21]. *Let X and Y be two metric spaces, Y compact and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:*

i) *the multifunction Φ is upper semi-continuous;*

ii) *the graph of Φ is a closed subset of $X \times Y$;*

iii) any would be the seguences $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$*, from* $x_n \to x$*,* $y_n \in \Phi(x_n)$ *and* $y_n \to y$ *it follows that* $y \in \Phi(x)$*.*

Definition 2.7. [2], [10], [11] The function $u : \triangle \to \mathbb{R}^n, \triangle \subset \mathbb{R}^2$, is *absolutely continuous in Carathéodory's sense* [2, §565 - §570] if and only if it is continuous on $\triangle$, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x,y)$ is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesque-integrable on $\triangle$.

Theorem 2.2. [2], [7], [24] *The function* $u : \triangle \to \mathbb{R}^n$*,* $\triangle = [0,a] \times [0,b] \subset \mathbb{R}^2$*, is absolutely continuous in Carathéodory's sense on* $\triangle$ *if and only if there exist* $f \in L^1(\triangle;\mathbb{R}^n)$*,* $g \in L^1([0,a];\mathbb{R}^n)$*,* $h \in L^1([0,b];\mathbb{R}^n)$ *such that*

$$u(x,y) = \int_0^x \int_0^y f(s,t)\, ds\, dt + \int_0^x g(s)\, ds + \int_0^y h(t)\, dt + u(0,0).$$

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(\triangle;\mathbb{R}^n)$, [11], [12]. In [7], this space is denoted by $AC(\triangle;\mathbb{R}^n)$.

Theorem 2.3. [7] *The space* $C^*(\triangle;\mathbb{R}^n)$ *endowed with the norm*

$$\|u(\cdot,\cdot)\| = \int_0^a \int_0^b \|u_{xy}(s,t)\|\, ds\, dt + \int_0^a \|u_x(s,0)\|\, ds + \int_0^b \|u_y(0,t)\|\, dt + \|u(0,0)\|,$$

where $\triangle = [0,a] \times [0,b] \subset \mathbb{R}^2$*, and* $\|\cdot\|$ *is the Euclidean norm, is a Banach space.*

Definition 2.8. [2], [12] The function $u : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^3$, is absolutely *continuous in Carathéodory's sense* [2, §565 - §570] if and only if $u(x,y,z)$ is continuous on D, absolutely continuous in each variable (for any pair of the other two variables) and similarly for $u_x(x,y,z)$, $u_y(x,y,z)$, $u_z(x,y,z)$, $u_{xy}(x,y,z)$, $u_{yz}(x,y,z)$, $u_{xz}(x,y,z)$, and u_{xyz} is Lebesque-integrable on D.

Theorem 2.4. [7] *The function* $u : D \to \mathbb{R}^n$*,* $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$*, is absolutely continuous in Carathéodory's sense on* D *if and only if there exist* $f \in L^1(D;\mathbb{R}^n)$*,* $g_1 \in L^1(D_1;\mathbb{R}^n)$*,* $g_2 \in L^1(D_2;\mathbb{R}^n)$*,* $g_3 \in L^1(D_3;\mathbb{R}^n)$*,*

$h_1 \in L^1([0,a];\mathbb{R}^n)$, $h_2 \in L^1([0,b];\mathbb{R}^n)$, $h_3 \in L^1([0,c];\mathbb{R}^n)$, *such that*

$$\begin{aligned} u(x,y,z) = & \int_0^x \int_0^y \int_0^z f(r,s,t)\,dr\,ds\,dt + \int_0^x \int_0^y g_1(r,s)\,dr\,ds + \\ & + \int_0^y \int_0^z g_2(s,t)\,ds\,dt + \int_0^x \int_0^z g_3(r,t)\,dr\,dt + \\ & + \int_0^x h_1(r)\,dr + \int_0^y h_2(s)\,ds + \int_0^z h_3(t)\,dt + u(0,0,0). \end{aligned}$$

We denote the class of absolutely continuous functions in Carathéodory's sense on D by $C^*(D;\mathbb{R}^n)$ [12].

Theorem 2.5. [7] *The space $C^*(D;\mathbb{R}^n)$ endowed with the norm*

$$\begin{aligned} \|u(\cdot,\cdot,\cdot)\| = & \int_0^a \int_0^b \int_0^c \|u_{xyz}(r,s,t)\|\,dr\,ds\,dt + \int_0^a \int_0^b \|u_{xy}(r,s,0)\|\,dr\,ds + \\ & + \int_0^b \int_0^c \|u_{yz}(0,s,t)\|\,ds\,dt + \int_0^a \int_0^c \|u_{xz}(r,0,t)\|\,dr\,dt + \\ & + \int_0^a \|u_x(r,0,0)\|\,dr + \int_0^b \|u_y(0,s,0)\|\,ds + \\ & + \int_0^c \|u_z(0,0,t)\|\,dt + \|u(0,0,0)\|, \end{aligned}$$

where $\|\cdot\|$ is the Euclidean norm, is a Banach space.

We denote by $d(x,y)$ the Euclidean distance from x to y, $x,y \in \mathbb{R}^n$, $\mathbb{R}^n$ is the Euclidean space. $B[x,r]$ is the closed ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. If $A \subset \mathbb{R}^n$, $d(x,A) = \inf\{d(x,y) \mid y \in A\}$ and $B[A,r] = \{x \mid d(x,A) \leq r\}$. If $A, B \subset \mathbb{R}^n$, $d^*(A,B) = \sup\{d(x,B) \mid x \in A\}$. $\operatorname{Conv} A$ is the convex covering of $A \subset \mathbb{R}^n$ and

$$|A| = \sup\{\|\zeta\| \mid \zeta \in A\}.$$

$\mathcal{C}(\mathbb{R}^n)$ is the set of compact and non-empty subsets of $\mathbb{R}^n$.

For $S \subset \mathbb{R}^n$ we have

Definition 2.5'. [6] The multifunction $\Gamma : S \to \mathcal{C}(\mathbb{R}^n)$ is *upper semi-continuous* on S if for each $y \in S$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Gamma(B[y,\delta]) \subset B[\Gamma(y),\varepsilon].$$

Similarly with [1], [6], [25], we define the *Aumann integral* for multifunctions of three variables.

Definition 2.9. [29] Let $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$. For each $(x,y,z) \in D$, let $H(x,y,z)$ be a non-empty subset of $\mathbb{R}^n$. Let $\mathcal{H}$ be the set of functions $h : D \to \mathbb{R}^n$ integrable on D and $h(x,y,z) \in H(x,y,z)$ for each $(x,y,z) \in D$. Then, by the integral of the multifunction $H : D \to 2^{\mathbb{R}^n}$ we mean the set

$$\iiint\limits_D H(x,y,z)\,dx\,dy\,dz = \left\{ \iiint\limits_D h(x,y,z)\,dx\,dy\,dz \mid h \in \mathcal{H} \right\}.$$

In what follows we list some properties of the integral defined above, similarly with [6], [25] (in the first three propositions).

Proposition 2.1. [29] *If $H : D \to 2^{\mathbb{R}^n}$ is an upper semi-continuous multifunction and there exists a positive real number C such that*

$$|H(x,y,z)| = \sup\{\|\zeta\| \mid \zeta \in H(x,y,z)\} \leq C$$

for each $(x,y,z) \in D$, then

$$\iiint\limits_D H(x,y,z)\,dx\,dy\,dz = \iiint\limits_D \operatorname{conv} H(x,y,z)\,dx\,dy\,dz.$$

Proposition 2.2. [29] *If $H_k : D \to 2^{\mathbb{R}^n}$, $k \in \mathbb{N}$, are upper semi-continuous multifunctions and there exists a positive real number C such that $|H_k(x,y,z)| \leq C$ for each $(x,y,z) \in D$ and $k \in N$, then*

$$\iiint\limits_D \underline{\lim} H_k(x,y,z)\,dx\,dy\,dz \subset \underline{\lim} \iiint\limits_D H_k(x,y,z)\,dx\,dy\,dz.$$

Taking into account Definition 2 in [6], we have $(x,y,z) \in \underline{\lim} H_k(x,y,z)$ iff each neighbourhood of (x,y,z) intersects all the sets $H_k(x,y,z)$ with k large enough.

Proposition 2.3. [29] *If A is a compact subset of $\mathbb{R}^n$, independent of (x,y,z), then*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} A\,dx\,dy\,dz = (x_2 - x_1)(y_2 - y_1)(z_2 - z_1) \operatorname{conv} A,$$

where $(x_1,y_1,z_1), (x_2,y_2,z_2) \in D$.

Moreover, we need the following proposition.

Proposition 2.4. *If K is a convex set in a Banach space X, then the set $K_\varepsilon = \bigcup_{x \in K} B[x, \varepsilon]$ is convex.*

Definition 2.10. (Def. 6 in [6]). Let $\Gamma_n : \Omega \subset E^{q+1} \to \mathcal{C}(E^q)$ be a sequence of multifunctions, Ω is an open set, E^q is the Euclidean space. $\Gamma_n \to \Gamma$ *uniformly on compact sets* if for $\varepsilon > 0$ and all compact subset $K \subset \Omega$, there exists N such that for $n \geq N$, $d^*(G_n, G) < \varepsilon$, where G_n and G are the graphs of restrictions of Γ_n and Γ to K.

$$G_n = \operatorname{graph} \Gamma_n|_K, \quad G = \operatorname{graph} \Gamma|_K, \quad K \subset \Omega.$$

Theorem 2.6. (Th. 2 in [6]) *Let $\Gamma : \Omega \to \mathcal{C}(E^q)$ be an upper semi-continuous multifunction, $\Omega \subset E^{q+1}$ and $\Gamma(x,t)$ is convex for $(x,t) \in \Omega$. Then there exists a sequence of single-valued continuous functions $f_n : \Omega \to E^q$, such that $f_n \to \Gamma$ uniformly on compact sets.*

We have, from Definition 2.10, $d^*(F_n, G) < \varepsilon$ where $F_n = \operatorname{graph} f_n|_K$, $G = \operatorname{graph} \Gamma|_K$, $K \subset \Omega$ being a compact set.

This Theorem can be extended to

Theorem 2.7. *Let $F : \Lambda \subset \mathbb{R}^{n+3} \to 2^{\mathbb{R}^n}$ be an upper semi-continuous multifunction defined on the open set Λ and whose values are non-empty, convex and compact sets in $\mathbb{R}^n$. Then there exists a sequence of single-valued continuous functions $f_k : \Lambda \to \mathbb{R}^n$, $k \in \mathbb{N}$, such that $f_k \to F$ uniformly on compact sets.*

The proof is completely analogous with that of Theorem 2.6 (Theorem 2 in [6]).

3. Results

In [27], the notion of a *local solution* for the Darboux Problem $(1.1) + (1.2)$ is a defined and it is proved an existence theorem for a local solution of this problem, together with some properties of the set of its solutions, namely that this set is a compact subset in Banach space $C(D; \mathbb{R}^n)$ and, as a function of initial values, it defines an upper semi-continuous multifunction on $D_0 = [0, x_0] \times [0, y_0] \times [0, z_0] \subseteq D$.

Let the following hypotheses be satisfied:

(H_1) $F : D \times \Omega \to 2^{\mathbb{R}^n}$ is multifunction with compact convex non-empty values in $\mathbb{R}^n, D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$, and $\Omega \subset \mathbb{R}^n$ is an open subset;

(H_2) For any $(x,y,z) \in D$, the mapping $u \to F(x,y,z,u)$ is upper semi-continuous on Ω;

(H_3) For any $u \in \Omega$, the mapping $(x,y,z) \to F(x,y,z,u)$ is Lebesque-measurable on D;

(H_4) There exists a function $k : D \to \mathbb{R}_+, k \in \mathcal{L}^1(D;\mathbb{R}^n)$ such that

$$\|\zeta\| \leq k(x,y,z), \quad (\forall)\,\zeta \in F(x,y,z,u), \quad (\forall)\,(x,y,z) \in D, \quad (\forall)\,u \in \Omega;$$

(H_5) The functions $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3,\mathbb{R}^n)$ are absolutely continuous in Carathéodory's sense functions and satisfy condition (1.3).

Remark 1. The function $\alpha : D \to \mathbb{R}^n$ defined by

$$\begin{aligned}\alpha(x,y,z) = {} & \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \\ & - \varphi(0,y) - \psi(0,z) + \psi(0,0) = \\ & = \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0,\end{aligned} \tag{3.1}$$

is an absolutely continuous in Carathéodory's sense function on D, $\alpha \in C^*(D;\mathbb{R}^n)$ [2, §565-§570].

Remark 2. Denote by $M \subset \Omega$ the convex compact set in which the function $\alpha : D \to \mathbb{R}^n$, defined by (3.1), takes its values for all $(x,y,z) \in D_0$.

Remark 3. Let $(x_0,y_0,z_0) \in]0,a]\times]0,b]\times]0,c]$ be a point such that

$$\int_0^{x_0}\int_0^{y_0}\int_0^{z_0} k(r,s,t)\,dr\,ds\,dt < d(M,C_\Omega),$$

where $d(M,C_\Omega)$ is the distance from M to $C_\Omega = \mathbb{R}^n - \Omega$, an inequality immediately resulting from the integrability of function k.

Definition 3.1. [27] The *Darboux Problem* for the hyperbolic inclusion (1.1) means to determine a *solution* of this inclusion which satisfies the initial conditions (1.2).

Definition 3.2. [27] A *local solution* of Darboux Problem (1.1) + (1.2) is defined as a function $U : D_0 \to \Omega, U \in C^*(D_0;\mathbb{R}^n)$, absolutely continuous in Carathéodory's sense [2,§565-§570], which satisfies (1.1) for a.e. $(x,y,z) \in D_0$, and also initial conditions (1.2) for all $(x,y) \in [0,x_0]\times[0,y_0]$, all $(y,z) \in [0,y_0]\times[0,z_0]$, all $(x,z)\in[0,x_0]\times[0,z_0]$.

In [27] we proved the following

Theorem 3.1. [27] *Let the hypotheses $(H_1)-(H_5)$ be satisfied. Then:*

- (i) *there exists at least a local solution U of Darboux Problem* (1.1)+(1.2)*;*
- (ii) *the set S_α of the local solutions U is compact in Banach space $C(D_0;\mathbb{R}^n)$;*
- (iii) *the multifunction $\alpha \to S_\alpha$ is upper semi-continuous on $C^*(D_0;\mathbb{R}^n)$, taking values in $C(D_0;\mathbb{R}^n)$.*

The solution U is a fixed point of a suitable multifunction which satisfies the Kakutani-Ky Fan Fixed Point Theorem and it is of the form

$$U(x,y,z)=\alpha(x,y,z)+\int_0^x\int_0^y\int_0^z \beta(r,s,t)\,dr\,ds\,dt,\quad (x,y,z)\in D_0, \quad (3.2)$$

where

$$\beta(x,y,z)\in\Gamma(x,y,z)\subset F(x,y,z,U(x,y,z)) \text{ for a.e. } (x,y,z)\in D_0, \quad (3.3)$$

β is a measurable selection of the multifunction $\Gamma: D_0\to C(\mathbb{R}^n)$, [3], [4], [27].

Definition 3.3. [28] A local solution for the Darboux Problem (1.1) + (1.2) $U: D_0\to\Omega$ is *prolongable* (or *non-saturated*) if there exists a solution $\widetilde{U}: \widetilde{D}\to\mathbb{R}^n$ for the Darboux Problem (1.1) + (1.2) such that

$$\begin{cases} D_0\subseteq\widetilde{D}, \quad D_0\neq\widetilde{D} \\ \widetilde{U}(x,y,z)=U(x,y,z), \quad (x,y,z)\in D_0, \end{cases}$$

where $\widetilde{D}\subseteq D$ is a union of D_0 with a finite number of adjacent parallelepipeds.

In [28] we proved the following theorems:

Theorem 3.2. [28] *Let the hypotheses $(H_1)-(H_5)$ be satisfied together with the hypotheses:*

- (H_6) *The set Ω is bounded, that is there exists a constant $C\in\mathbb{R}_+$ such that $\|u\|\le C$, $(\forall)\,u\in\Omega$.*
- (H_7) *The multifunction F maps bounded sets onto bounded sets, hence a constant $K\in\mathbb{R}_+$ exists such that*

$$\sup\{\|\zeta\| \mid \zeta\in F(x,y,z,u)\}\le K,$$

for any $(x, y, z, u) \in D \times \Omega$.

Then the local solution U *is prolongable.*

Theorem 3.3. [28] *We assume the hypotheses* $(H_1) - (H_7)$ *to be satisfied. If* $U : D_0 \to \Omega$ *is a local solution of Darboux Problem* $(1.1) + (1.2)$ *that is non-saturated, hence prolongable, then there exists a saturated solution* $U^* : D^* \to \Omega$ *of the Darboux Problem* $(1.1) + (1.2)$ *such that*

$$\begin{cases} D_0 \subseteq D^*, \quad D_0 \neq D^*, \quad D^* \subseteq D, \\ U^*(x, y, z) = U(x, y, z), \quad (x, y, z) \in D_0, \end{cases}$$

hence U^* *is a prolongation of* U *onto* D^* *that has been built by joining* D_0 *with a union of parallelepipeds adjacent to* D_0.

Theorem 3.4. [28] *Let the hypotheses* $(H_1) - (H_7)$ *be satisfied. If the saturated solution* U^* *is bounded on* D^*, *then* $D^* = D$.

Theorem 3.5. [28] *Let the hypotheses* $(H_1) - (H_7)$ *be satisfied together with the hypothesis:*

(H_8) *The multifunction* $F : D \times \Omega \to 2^{\mathbb{R}^n}$ *is sublinear, hence two constants* $k_1 > 0$ *and* $k_2 \in \mathbb{R}$ *exist with the property*

$$\sup \{\|\zeta\| \mid \zeta \in F(x, y, z, u)\} \leq k_1 \|u\| + k_2, \quad \text{for a.e. } (x, y, z) \in D, \quad u \in \Omega. \tag{3.4}$$

Then the saturated solution $U^* : D \to \Omega$ is *bounded on* D.

The saturated solution U^* has the form, by Theorem 3.1 [27],

$$U^*(x, y, z) = \alpha(x, y, z) + \int_0^x \int_0^y \int_0^z \beta^*(r, s, t)\, dr\, ds\, dt, \quad (x, y, z) \in D, \tag{3.5}$$

where $\alpha(x, y, z)$ is given by (3.1) and β^* is a measurable selection of the multivalued mapping Γ^* [3], [4], [27], defined on D with compact non-empty values in $\mathbb{R}^n$, i.e. $\Gamma^* : D \to \mathcal{C}(\mathbb{R}^n)$, such that

$$\beta^*(x, y, z) \in \Gamma^*(x, y, z) \subseteq F(x, y, z, U^*(x, y, z)) \text{ for a.e. } (x, y, z) \in D. \tag{3.6}$$

Definition 3.4. [28] A function $U : D \to \mathbb{R}^n$ is called a *solution* of the Darboux Problem (1.1) + (1.2) if it is absolutely continuous in Carathéodory's sense on $D, U \in C^*(D; \mathbb{R}^n)$, [2, §565 - §570], and satisfies (1.1) for a.e.

$(x,y,z)\in D$, and also initial conditions (1.2) for all $(x,y)\in D_1$, all $(y,z)\in D_2$, all $(x,z)\in D_3$.

Similarly with [6], [25], we proved in [29] the following theorem of characterization for the solutions to Darboux Problem (1.1) + (1.2).

Theorem 3.6. [29] *Let the hypotheses* (H_1'), (H_3), (H_4), (H_5) *of Theorem 3.1 be satisfied, where:*

(H_1') $F : D\times\Omega \to 2^{\mathbb{R}^n}$ *is an upper semi-continuous multifunction with compact convex non-empty values in* $\mathbb{R}^n$, $D=[0,a]\times[0,b]\times[0,c]\subset\mathbb{R}^3$ *and* $\Omega\subset\mathbb{R}^n$ *is an open bounded set.*

The hypothesis (H_1') *includes the hypothesis* (H_6). *Then, the continuous function* $U : D\to\mathbb{R}^n$ *is a solution of Darboux Problem* (1.1) + (1.2) *if and only if for each* $(x_1,y_1,z_1),(x_2,y_2,z_2)\in D$ *the membership relation*

$$\begin{aligned}&[U(x_2,y_2,z_2)-U(x_1,y_2,z_2)-U(x_2,y_1,z_2)+U(x_1,y_1,z_2)]-\\&-[U(x_2,y_2,z_1)-U(x_1,y_2,z_1)-U(x_2,y_1,z_1)+U(x_1,y_1,z_1)]\in\\&\in\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}F(x,y,z,U(x,y,z))\,dx\,dy\,dz,\end{aligned}\tag{3.7}$$

holds, and U *satisfies the conditions* (1.2).

The difference in (3.7) *is an extension of hyperbolic difference for functions in two variables.*

The main result of this paper is the following:

Theorem 3.7. *Assume the hypotheses* (H_1'), $(H_2)-(H_8)$ *to be satisfied, where* $F : D\times\overline{\Omega}\to 2^{\mathbb{R}^n}$. *Let* $f_k : D\times\overline{\Omega}\to\mathbb{R}^n$ *be a sequence of single-valued continuous functions such that* $f_k\to F$ *uniformly on compact sets and let* $u_k : D\to\mathbb{R}^n$, $k\in\mathbb{N}$, *be the solutions of the Darboux Problems* $(3.8_k)+(1.2)$, *where*

$$\frac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z}=f_k(x,y,z,u),\quad (x,y,z)\in D,\ u\in\overline{\Omega},\ k\in\mathbb{N}.\tag{3.8$_k$}$$

Then, there exists a solution $U : D\to\mathbb{R}^n$ *of the Darboux Problem* (1.1) + (1.2) *and a sequence of positive integers* $\{n_p\}_{p\in\mathbb{N}}$ *such that* $U_{n_p}(x,y,z)\to U(x,y,z)$ *uniformly on* D.

Proof. Suppose $D\times\overline{\Omega}\subset\Lambda\subset\mathbb{R}^{n+3}$. Taking into account the sublinearity of F and the uniform convergence of $\{f_k\}_{k\in\mathbb{N}}$ to F on compact sets, it follows that f_k, $k \in \mathbb{N}$, is sublinear. Given the compact set $K_1 = D \times \overline{\Omega}$, by Definition 2.10 for every $\varepsilon > 0$ there exists a natural number $k(\varepsilon)$ such that $d^*(\operatorname{graph} f_k, \operatorname{graph} F) < \varepsilon$, $(\forall)\, k > k(\varepsilon)$, hence $\sup d\{[(x,y,z,u,f_k(x,y,z,u)), \operatorname{graph} F]\} < \varepsilon$, $(\forall)\, k > k(\varepsilon)$, for $(x,y,z,u,f_k(x,y,z,u)) \in \operatorname{graph} f_k$, and moreover

$$\sup_{\substack{(x,y,z,u,f_k)\\ \in \operatorname{graph} f_k}} \left\{ \inf_{\substack{(\bar{x},\bar{y},\bar{z},\bar{u},\zeta)\\ \in \operatorname{graph} F}} \{d[(x,y,z,u,f_k(x,y,z,u)), (\bar{x},\bar{y},\bar{z},\bar{u},\zeta)]\} \right\} < \varepsilon, \quad (\forall)\, k > k(\varepsilon). \tag{3.9}$$

From (3.9), taking into account [5]

$$\begin{aligned} &d[(x,y,z,u,f_k(x,y,z,u)), (\bar{x},\bar{y},\bar{z},\bar{u},\zeta)] = \\ &= \max\{d[(x,y,z,u), (\bar{x},\bar{y},\bar{z},\bar{u})], d[f_k(x,y,z,u), \zeta]\} \end{aligned}$$

we get

$$d[(x,y,z,u), (\bar{x},\bar{y},\bar{z},\bar{u})] < \varepsilon \tag{3.10}$$

and

$$d[f_k(x,y,z,u), \zeta] = \|f_k(x,y,z,u) - \zeta\| < \varepsilon. \tag{3.11}$$

The relation (3.10)yields $d(u,\bar{u}) = \|u - \bar{u}\| < \varepsilon$. Thus

$$\|\bar{u}\| \le \|u - \bar{u}\| + \|u\| < \varepsilon + \|u\|. \tag{3.12}$$

Since by (H_8) F is sublinear, the inequality (3.4) holds for $\bar{u}$. From (3.11), (3.12) and (3.4) we deduce

$$\begin{aligned} \|f_k(x,y,z,u)\| &\le \|f_k(x,y,z,u) - \zeta\| + \|\zeta\| < \varepsilon + k_1\|\bar{u}\| + k_2 < \\ &< \varepsilon + k_1(\varepsilon + \|u\|) + k_2 = k_1\|u\| + k_1\varepsilon + k_2 + \varepsilon = \\ &= k_1\|u\| + k_3, \quad (x,y,z,u) \in D \times \overline{\Omega}, \end{aligned} \tag{3.13}$$

where $k_3 = k_1\varepsilon + k_2 + \varepsilon$, $k_3 \in \mathbb{R}$.

By (3.13) we conclude that f_k, $k \in \mathbb{N}$, is sublinear.

Because f_k, $k \in \mathbb{N}$, is continuous and sublinear, by Schauder's Fixed Point Theorem, the Darboux Problem $(3.8_k) + (1.2)$ has at least a solution $u_k : D \to \mathbb{R}^n$, $k \in \mathbb{N}$.

Indeed, $(3.8_k) + (1.2)$ is equivalent to the integral equation

$$u(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z f_k(r,s,t,u(r,s,t))\, dr\, ds\, dt, \quad k \in \mathbb{N}.$$

Define the operator $T_k : C(D;\mathbb{R}^n) \to C(D;\mathbb{R}^n)$, $k \in \mathbb{N}$, by

$$T_k u(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z f_k(r,s,t,u(r,s,t))\,dr\,ds\,dt,$$

$(x,y,z) \in D$, $k \in \mathbb{N}$. The operator T_k, $k \in \mathbb{N}$, is continuous. The set

$$B[\alpha;r] = \{u \mid u \in C(D;\mathbb{R}^n),\ \|u-\alpha\| \le r\}, \quad r > 0,$$

is convex and closed and it is mapped by the operator T_k, $k \in \mathbb{N}$, into itself. Indeed, let $u \in B[\alpha;r]$. Then, taking into account (3.13) one gets

$$\begin{aligned}\|T_k u - \alpha\| &\le \int_0^x \int_0^y \int_0^z \|f_k(r,s,t,u(r,s,t))\|\,dr\,ds\,dt \le \\ &\le \int_0^{x_0} \int_0^{y_0} \int_0^{z_0} \|f_k(r,s,t,u(r,s,t))\|\,dr\,ds\,dt \le \\ &\le \int_0^x \int_0^y \int_0^z [k_1\|u(r,s,t)\| + k_3]\,dr\,ds\,dt,\end{aligned}$$
$$(x,y,z) \in D_0 = [0,x_0] \times [0,y_0] \times [0,z_0], (x_0,y_0,z_0) \in D.$$

We have

$$\|u(r,s,t)\| \le \|u(r,s,t) - \alpha(r,s,t)\| + \|\alpha(r,s,t)\| \le r + \sup\|\alpha(r,s,t)\| = C_1,$$

$(r,s,t) \in D_0$ and thereby

$$\|T_k u - \alpha\| \le \int_0^{x_0} \int_0^{y_0} \int_0^{z_0} (k_1 C_1 + k_3)\,dr\,ds\,dt = (k_1 C_1 + k_3)\,x_0 y_0 z_0. \quad (3.14)$$

Choose $(x_0,y_0,z_0) \in D$ such that the condition

$$(k_1 C_1 + k_3)\,x_0 y_0 z_0 \le r \quad (3.15)$$

holds.

By (3.14), (3.15) we obtain $\|T_k u - \alpha\| \le r$, i.e. $T_k u \in B[\alpha;r]$, or $T_k B[\alpha;r] \subset B[\alpha;r]$, $k \in \mathbb{N}$. The set $T_k B[\alpha;r]$ is relatively compact by the Arzelà-Ascoli Theorem. By Schauder's Fixed Point Theorem, the operator T_k, $k \in \mathbb{N}$, has at least a fixed point u_k, $k \in \mathbb{N}$, which is a solution of the Darboux Problem $(3.8_k) + (1.2)$ on D_0. This solution can be extended to the whole D [28]. Then

$$u_k(x,y,z) = \alpha(x,y,z) + \int_0^{x_0} \int_0^{y_0} \int_0^{z_0} f_k(r,s,t,u_k(r,s,t))\,dr\,ds\,dt, \quad (3.16)$$

$(x,y,z)\in D$, $k\in\mathbb{N}$, is equivalent to (3.8_k) and (1.2_k).

$$\frac{\partial^3 u_k\left(x,y,z\right)}{\partial x\,\partial y\,\partial z}=f_k\left(x,y,z,u_k\left(x,y,z\right)\right),(x,y,z)\in D,u_k\in\overline{\Omega}\subset\mathbb{R}^n,\qquad (3.8_k)$$

$$\begin{cases} u_k\left(x,y,0\right)=\varphi\left(x,y\right), & (x,y)\in D_1=[0,a]\times[0,b]\,,\\ u_k\left(0,y,z\right)=\psi\left(y,z\right), & (y,z)\in D_2=[0,b]\times[0,c]\,,\\ u_k\left(x,0,z\right)=\chi\left(x,z\right), & (x,z)\in D_3=[0,a]\times[0,c]\end{cases}\qquad (1.2_k)$$

where u_k, $k\in\mathbb{N}$, denotes the extended solution on D.

The family $\{u_k\left(x,y,z\right)\}_{k\in\mathbb{N}}$ of solutions is equicontinuous and equibounded in the Banach space $C\left(D;\mathbb{R}^n\right)$.

For $\bar{h},\bar{k},\bar{l}\in\mathbb{R}$, $\left(x+\bar{h},y+\bar{k},z+\bar{l}\right)\in D$, from the absolute continuity of the integral, we get

$$\left\|u_k\left(x+\bar{h},y+\bar{k},z+\bar{l}\right)-z_k\left(x,y,z\right)\right\|<\varepsilon,\quad\text{for }\left|\bar{h}\right|,\left|\bar{k}\right|,\left|\bar{l}\right|<\delta\left(\varepsilon\right).$$

Thus, the family $\{u_k\left(x,y,z\right)\}_{k\in\mathbb{N}}$ is equicontinuous. Taking into account the sublinearity of f_k, $k\in\mathbb{N}$, by a Gronwall's type inequality [9], [28] we get from (3.16)

$$\begin{aligned}\left\|u_k\left(x,y,z\right)\right\|&\leq B\left[1+k_1\int_0^x\int_0^y\int_0^z\exp\left(\int_r^x\int_s^y\int_t^z k_1 d\xi\,d\eta\,d\zeta\right)dr\,ds\,dt\right]=\\&=B\left[1+k_1\int_0^x\int_0^y\int_0^z\exp\left(k_1\left(x-r\right)\left(y-s\right)\left(z-t\right)\right)dr\,ds\,dt\right]\leq\\&\leq B\left[1+k_1\int_0^x\int_0^y\int_0^z\exp\left(k_1xyz\right)dr\,ds\,dt\right]\leq\\&\leq B\left[1+k_1\exp\left(k_1abc\right)xyz\right]\leq\\&\leq B\left[1+k_1abc\exp\left(k_1abc\right)\right],\quad\text{for a.e. }(x,y,z)\in D,\end{aligned}$$

where

$$B=\sup\left\|\alpha\left(x,y,z\right)\right\|+\left|k_2\right|abc,\quad (x,y,z)\in D.$$

Thus, the family $\{u_k\left(x,y,z\right)\}_{k\in\mathbb{N}}$ is equibounded.

By the Arzelà-Ascoli Theorem, the sequence $\{u_k\left(x,y,z\right)\}_{k\in\mathbb{N}}$ contains an uniformly convergent subsequence to a continuous function $U\in C\left(D;\mathbb{R}^n\right)$, $u_{k_p}\left(x,y,z\right)\to U\left(x,y,z\right)$, $(x,y,z)\in D$.

We shall prove that the function above obtained is a solution of the Darboux Problem (1.1) + (1.2). To this end we shall show that the conditions of the characterization Theorem 3.6 [29] of a solution is fulfilled. The conditions

(1.2) from (1.2_k) are obviously satisfied. We have to prove the relation (3.7) for the compact $[x_1, x_2] \times [y_1, y_2] \times [z_1, z_2] \subseteq D$. Thus, we prove the inequality

$$\begin{gathered} d\Big([U(x_2,y_2,z_2) - U(x_1,y_2,z_2) - U(x_2,y_1,z_2) + U(x_1,y_1,z_2)] - \\ - [U(x_2,y_2,z_1) - U(x_1,y_2,z_1) - U(x_2,y_1,z_1) + U(x_1,y_1,z_1)], \\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F(x,y,z,U(x,y,z))\,dx\,dy\,dz\Big) < \varepsilon. \end{gathered} \tag{3.17}$$

Integrating (3.8_k) on $[x_1, x_2] \times [y_1, y_2] \times [z_1, z_2] \subseteq D$ one gets

$$\begin{gathered} [u_k(x_2,y_2,z_2) - u_k(x_1,y_2,z_2) - u_k(x_2,y_1,z_2) + u_k(x_1,y_1,z_2)] - \\ - [u_k(x_2,y_2,z_1) - u_k(x_1,y_2,z_1) - u_k(x_2,y_1,z_1) + u_k(x_1,y_1,z_1)] = \\ = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f_k(r,s,t,u_k(r,s,t))\,dr\,ds\,dt. \end{gathered} \tag{3.18}$$

By (3.17) and (3.18) we have

$$\begin{gathered} d\Big([U(x_2,y_2,z_2) - U(x_1,y_2,z_2) - U(x_2,y_1,z_2) + U(x_1,y_1,z_2)] - \\ - [U(x_2,y_2,z_1) - U(x_1,y_2,z_1) - U(x_2,y_1,z_1) + U(x_1,y_1,z_1)], \\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F(r,s,t,U(r,s,t))\,dr\,ds\,dt\Big) \le \\ \le d\Big([U(x_2,y_2,z_2) - U(x_1,y_2,z_2) - U(x_2,y_1,z_2) + U(x_1,y_1,z_2)] - \\ - [U(x_2,y_2,z_1) - U(x_1,y_2,z_1) - U(x_2,y_1,z_1) + U(x_1,y_1,z_1)], \\ [u_k(x_2,y_2,z_2) - u_k(x_1,y_2,z_2) - u_k(x_2,y_1,z_2) + u_k(x_1,y_1,z_2)] - \\ - [u_k(x_2,y_2,z_1) - u_k(x_1,y_2,z_1) - u_k(x_2,y_1,z_1) + u_k(x_1,y_1,z_1)] + \\ + d\Big(\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f_k(r,s,t,u_k(r,s,t))\,dr\,ds\,dt, \\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F(r,s,t,U(r,s,t))\,dr\,ds\,dt\Big). \end{gathered} \tag{3.19}$$

Denoting for simplicity $k = k_p$, from $u_k(x,y,z) \to U(x,y,z)$ uniformly on the compact set D, we deduce

$$\begin{gathered} d\Big([U(x_2,y_2,z_2) - U(x_1,y_2,z_2) - U(x_2,y_1,z_2) + U(x_1,y_1,z_2)] - \\ - [U(x_2,y_2,z_1) - U(x_1,y_2,z_1) - U(x_2,y_1,z_1) + U(x_1,y_1,z_1)], \\ [u_k(x_2,y_2,z_2) - u_k(x_1,y_2,z_2) - u_k(x_2,y_1,z_2) + u_k(x_1,y_1,z_2)] - \\ - [u_k(x_2,y_2,z_1) - u_k(x_1,y_2,z_1) - u_k(x_2,y_1,z_1) + u_k(x_1,y_1,z_1)] < \frac{\varepsilon}{2}, \\ (\forall)\, k \in N_1(\varepsilon). \end{gathered} \tag{3.20}$$

Since $d(\zeta, A) = d(\theta, \zeta - A)$, where θ is the null vector in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$, we deduce

$$\begin{gathered} d\Big(\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f_k(r,s,t,u_k(r,s,t))\,dr\,ds\,dt, \\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F(r,s,t,U(r,s,t))\,dr\,ds\,dt\Big) = \\ = d\Big(\theta, \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} [f_k(r,s,t,u_k(r,s,t)) - F(r,s,t,U(r,s,t))]\,dr\,ds\,dt. \end{gathered} \tag{3.21}$$

For $(r,s,t) \in [x_1,x_2] \times [y_1,y_2] \times [z_1,z_2]$, by the Lemma [6], there exists a natural number $N_\varepsilon = N_2(\varepsilon,(r,s,t))$ such that [5], [29]

$$f_k(r,s,t,u_k(r,s,t)) \in B[F(r,s,t,U(r,s,t)),\varepsilon],$$

for $(\forall)\, k \geq N_2(\varepsilon,(r,s,t))$, and therefore

$$d(f_k(r,s,t,u_k(r,s,t)), F(r,s,t,U(r,s,t))) < \varepsilon,$$

or

$$d(\theta, f_k(r,s,t,u_k(r,s,t)) - F(r,s,t,U(r,s,t))) < \varepsilon.$$

Thus, by Definition 2 [6]

$$\theta \in \underline{\lim}\, \{f_k(r,s,t,u_k(r,s,t)) - F(r,s,t,U(r,s,t))\}.$$

Then

$$\theta \in \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \underline{\lim}\, \{f_k(r,s,t,u_k(r,s,t)) - F(r,s,t,U(r,s,t))\}\,dr\,ds\,dt. \tag{3.22}$$

Using the Proposition 3.2, the membership relation (3.22) yields

$$\theta \in \underline{\lim} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \{ f_k (r, s, t, u_k (r, s, t)) - F (r, s, t, U (r, s, t)) \} \, dr \, ds \, dt. \tag{3.23}$$

By (3.23) and Definition 2 [6], each neighbourhoud of θ intersects the sets of the form

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \{ f_k (r, s, t, u_k (r, s, t)) - F (r, s, t, U (r, s, t)) \} \, dr \, ds \, dt,$$

for a large enough k, $(\forall)\, k \geq N_2 (\varepsilon, (r, s, t))$.

By (3.23) one gets

$$d\left(\theta, \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \{ f_k (r, s, t, u_k (r, s, t)) - F (r, s, t, U (r, s, t)) \} \, dr \, ds, dt \right) < \frac{\varepsilon}{2}, \tag{3.24}$$

for $k \geq N_2 (\varepsilon)$, or by (3.21) it results

$$\begin{gathered} d \left(\theta, \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f_k (r, s, t, u_k (r, s, t)) \, ds \, dr \, dt, \right. \\ \left. \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F (r, s, t, U (r, s, t)) \, dr \, ds, dt \right) < \frac{\varepsilon}{2}, \end{gathered} \tag{3.25}$$

for $k \geq N_2 (\varepsilon, (r, s, t))$.

Taking into account (3.20) and (3.25) the relation (3.19) yields (3.17) for $(\forall)\, k \geq \max \{ N_1 (\varepsilon), N_2 (\varepsilon, (r, s, t)) \}$.

But $F (r, s, t, U (r, s, t))$ is closed and then, by (3.17) for $\varepsilon \to 0$, one obtains (3.7).

Recalling that (3.7) together (1.2) are sufficient conditions for U be a solution of the Darboux Problem (1.1) + (1.2), the proof is complete.

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