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SECOND ORDER RETARDED DIFFERENTIAL INCLUSIONS IN BANACH SPACES AND HENSTOCK-LEBESGUE INTEGRAL

B. SATCO

Ştefan cel Mare University of Suceava,
Faculty of Electrical Engineering and Computer Science,
Universităţii 13, Suceava, Romania

Email: bisatco@eed.usv.ro

Abstract. The aim of this paper is to obtain the existence of continuous solutions for a second order retarded differential inclusion under some hypothesis of Henstock-Lebesgue integrability. Also, some compactness properties in the sense of Sobolev-type spaces are given.

Keywords: retarded differential inclusion, Henstock integral.

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1. Introduction

In the present paper, we study, in a separable Banach space, the existence of solutions and the properties of solutions set for the second order retarded differential inclusion with three boundary conditions

$$\begin{cases} u"(t) \in F(t, u_t, u'(t)), \text{ a.e. } t \in [0,1] \\ u(0) = 0, u(\theta) = u(1). \end{cases}$$

Here $\theta \in]0,1[$, r is a positive number and, for each $t \in [0,1]$, $u_t : [-r, 0] \to X$ defined by $u_t(s) = u(t+s)$ is a measure of the taking into account the system's history. The multifunction governing the inclusion is

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$F : [0,1] \times \mathcal{HL}([-r,0],X) \times X \to \mathcal{P}_{kc}(X)$, where $\mathcal{HL}([-r,0],X)$ is the collection of X-valued Henstock-Lebesgue-integrable functions on $[-r,0]$ (in the sense of [3]). The set-valued function is supposed to be Henstock-integrably bounded.

We obtain, applying Kakutani-Ky Fan's fixed point theorem and a characterization of Henstock-integrable compact convex-valued multifunctions given in [6], that the family of $W^{2,1}_{HL,X}$-solutions is nonempty. By $W^{2,1}_{HL,X}$ we mean the Sobolev-type space containing all X-valued functions u that are continuous on $[0,1]$, a.e. differentiable with the derivative u' continuous and a.e. differentiable and the second derivative u'' Henstock-Lebesgue integrable. Finally, we study the properties of solutions set. We prove its compactness, as well as some properties similar to those already known in the classical setting.

Our main theorem is more general than the previously obtained results in the theory of retarded differential equations. The generalization concerns two aspects. Firstly, the vector-valued integral used throughout is more general than those involved in the previous papers and secondly, we consider the set-valued case. Let us remind that retarded equations were studied in the classical continuous case and in the Carathéodory setting (see [9], [11]), then in the case of Lebesgue integrable functions and lately in the less restrictive case of Henstock-Kurzweil integral (e.g. in [5], [18]). Moreover, problems involving such type of equations were investigated in the case of a general Banach space via Bochner integral, respectively Henstock-Lebesgue integral (see [17]).

2. Notations and preliminary facts

The Henstock-type integrals in Banach spaces are defined following the same line as for the Henstock-Kurzweil integral on the real line. Let us then begin by recalling the basic facts on Henstock-Kurzweil integrability, a concept that extends the classical Lebesgue integrability.

A gauge δ on $[0,1]$ (the real unit interval provided with the σ-algebra Σ of Lebesgue measurable sets and with the Lebesgue measure μ) is a positive function. A partition of $[0,1]$ is a finite family $(I_i,t_i)_{i=1}^n$ of nonoverlapping intervals covering $[0,1]$ with tags $t_i \in I_i$; a partition is said to be δ-fine if for each $i \in \{1,...,n\}$, $I_i \subset]t_i - \delta(t_i), t_i + \delta(t_i)[$. A function $f : [0,1] \to \mathbb{R}$ is said to be Henstock-Kurzweil (shortly HK-) integrable if there exists a real, denoted by (HK) $\int_0^1 f(t)dt$, satisfying that, for every $\varepsilon > 0$, there is a gauge

δ_ε such that $\left|\sum_{i=1}^n f(t_i)\mu(I_i) - (\mathrm{HK})\int_0^1 f(t)dt\right| < \varepsilon$, for every δ_ε-fine partition $\mathcal{P}=(I_i,t_i)_{i=1}^n$ of $[0,1]$. For properties of HK-integral, we refer the reader to [8].

The following auxiliary result was proved in [16]:

Lemma 1. *Let $(f_n)_n$ be an uniformly HK-integrable, pointwisely bounded sequence of real functions defined on $[0,1]$. Then $\widetilde{f}_n(\cdot) = (\mathrm{HK})\int_0^{\cdot} f_n(t)dt$ is an uniformly equi-continuous sequence.*

Through the paper, X is a real separable Banach space, X^* is its topological dual with unit ball B^* and $\mathcal{P}_{kc}(X)$ stands for the family of its compact convex subsets. On $\mathcal{P}_{kc}(X)$ the Hausdorff distance D is considered and, for every $A \in \mathcal{P}_{kc}(X)$, we put $|A| = D(A,\{0\})$.

The following notion extends the real HK-integral to the vector case.

Definition 2. *A function $f:[0,1]\to X$ is Henstock-Lebesgue (HL-) integrable if there exists $\widetilde{f}:[0,1]\to X$ such that, for every $\varepsilon>0$, there is a gauge $\delta_\varepsilon>0$ satisfying that $\sum_{i=1}^n \left\|f(t_i)\mu((x_{i-1},x_i)) - \left[\widetilde{f}(x_i)-\tilde{f}(x_{i-1})\right]\right\| < \varepsilon$, for each δ_ε-fine partition $\mathcal{P} = ((x_{i-1},x_i),t_i)_{i=1}^n$. The notation $(\mathrm{HL})\int_0^t f(s)ds = \widetilde{f}(t)$ is used.*

A Henstock-Lebesgue integrable function is Henstock-Lebesgue integrable on every subinterval. Any Bochner integrable function is Henstock-Lebesgue-integrable. As the following theorem states, the Henstock-Lebesgue integrable Banach-valued functions possess (like the Bochner integrable ones) an important property of differentiability, essential for solving our problem.

Theorem 3. ([3] or [8] for $X=\mathbb{R}$) *Let $f:[0,1]\to X$ be Henstock-Lebesgue-integrable. Then $\widetilde{f}$ is continuous, a.e. differentiable and $\left(\widetilde{f}\right)'(t) = f(t)$ a.e.*

The following integration by parts result can be proved, with some obvious modifications, in the same way as Theorem 12.8 in [8]:

Lemma 4. *Let $f:[0,1]\to X$ be Henstock-Lebesgue integrable and $g:[0,1]\to\mathbb{R}$ be absolutely continuous. Then fg is Henstock-Lebesgue integrable and*

$$(\mathrm{HL})\int_0^t f(s)g(s)ds = g(t)(\mathrm{HL})\int_0^t f(s)ds - \int_0^t \left(g'(s)(\mathrm{HL})\int_0^s f(\tau)d\tau\right)ds, \forall t\in[0,1] .$$

We denote the support functional of $A\in\mathcal{P}_{kc}(X)$ by $\sigma(\cdot,A)$. A function $f:[0,1]\to X$ is a selection of Γ if $f(t)\in\Gamma(t)$ a.e. For all concepts of measurability, we refer the reader to [4]. A multifunction Γ is said to be:

i) integrably bounded if the real function $|\Gamma(\cdot)|$ is Lebesgue integrable.

ii) scalarly HK-integrable if, for every $x^* \in X^*$, $\sigma(x^*, \Gamma(\cdot))$ is HK-integrable;

iii) A $\mathcal{P}_{kc}(X)$-valued function Γ is "Henstock-integrable in $\mathcal{P}_{kc}(X)$" (shortly, Henstock-integrable) [6] if there exists (H) $\int_0^1 \Gamma(t)dt \in \mathcal{P}_{kc}(X)$ satisfying that, for every $\varepsilon > 0$, there is a gauge δ_ε such that for any δ_ε-fine partition of $[0,1]$,

$$D\left((\mathrm{H})\int_0^1 \Gamma(t)dt, \sum_{i=1}^n \Gamma(t_i)\mu(I_i)\right) < \varepsilon.$$

iv) A $\mathcal{P}_{kc}(X)$-valued function Γ is said to be Henstock-integrably bounded if every measurable selection is Henstock-Lebesgue-integrable. In this case, the HL-integral of Γ is defined (in the Aumann way) by the collection of HL-integrals of its integrable selections.

In the particular case of a single-valued function, the definition *iii*) gives the notion of vector Henstock integral (see [3]). Obviously, any Henstock-Lebesgue integrable function is Henstock-integrable too (the two notions coincide in the finite dimensional case, thanks to Saks-Henstock's Lemma, see [8]). We can consider (using the fact, proved in [14], that linear continuous functionals on the space of real HK-integrable functions are given by bounded variation functions) the space of all Henstock-integrable X-valued functions provided with the topology induced by the tensor product of the space of real functions of bounded variation and X^* (we call it the weak-Henstock-Kurzweil-Pettis topology and denote it by w-HKP, as in [16]). That is: $f_\alpha \to f$ if, for every $g : [0,1] \to \mathbb{R}$ of bounded variation and every $x^* \in X^*$, (HK) $\int_0^1 g(s) \langle x^*, f_\alpha(s)\rangle ds \to$ (HK) $\int_0^1 g(s) \langle x^*, f(s)\rangle ds$.

Let us recall following characterizations of Henstock-integrable multifunctions (for definition and properties of Pettis set-valued integral we refer to [7]):

Theorem 5. (Theorem 1 in [6]) *Let $\Gamma : [0,1] \to \mathcal{P}_{kc}(X)$ be scalarly HK-integrable. Then the following conditions are equivalent:*

i) Γ *is Henstock-integrable;*

ii) Γ *has at least one Henstock-integrable selection and for any Henstock-integrable selection f there exists a Pettis integrable multifunction $\Gamma_1 : [0,1] \to \mathcal{P}_{kc}(X)$ such that, for every $t \in [0,1]$, $\Gamma(t) = f(t) + \Gamma_1(t)$;*

iii) *each measurable selection of Γ is Henstock-integrable.*

and

Proposition 6. (Proposition 1 in [6]) *If* $\Gamma : [0,1] \to \mathcal{P}_{kc}(X)$ *is scalarly HK-integrable, then the following conditions are equivalent:*

i) Γ *is Henstock-integrable;*

ii) the collection $\{\sigma(x^*, \Gamma(\cdot)); x^* \in B^*\}$ *is HK-equi-integrable;*

iii) each countable subset of $\{\sigma(x^*, \Gamma(\cdot)); x^* \in B^*\}$ *is HK-equi-integrable.*

Applying Theorem 5 gives that every Henstock-integrably bounded multifunction is Henstock-integrable and the HL-integral coincides with the Henstock integral, therefore it is compact and convex. Moreover, using a weak compactness result on the family of integrable selections in the Pettis integrability settings (which can be found in [1]), one can prove that:

Proposition 7. *The family of all Henstock-Lebesgue-integrable selections of a Henstock-integrably bounded set-valued function is sequentially w-HKP compact.*

In the study of three boundary value second order differential inclusions, we will use a Hartman-type function (as those considered for the first time in the study of two boundary problems for ordinary differential equations in [10]). Consider $G : [0,1] \times [0,1] \to \mathbb{R}$ the Hartman function introduced in [2]:

$$\text{if } 0 \leq t < \theta, \qquad G(t,s) = \begin{cases} -s, \text{ if } 0 \leq s \leq t; \\ -t, \text{ if } t < s \leq \theta; \\ \frac{t(s-1)}{1-\theta}, \text{ if } \theta < s \leq 1, \end{cases}$$

and

$$\text{if } \theta \leq t \leq 1, \qquad G(t,s) = \begin{cases} -s, \text{ if } 0 \leq s < \theta; \\ \frac{\theta(s-t)+s(t-1)}{1-\theta}, \text{ if } \theta \leq s \leq t; \\ \frac{t(s-1)}{1-\theta}, \text{ if } t < s \leq 1. \end{cases}$$

In [2] it is proved that $G(\cdot, s)$ is differentiable, for every $s \in [0,1]$:

$$\text{if } 0 \leq t < \theta, \qquad \frac{\partial G}{\partial t}(t,s) = \begin{cases} 0, \text{ if } 0 \leq s < t; \\ -1, \text{ if } t < s \leq \theta; \\ \frac{s-1}{1-\theta}, \text{ if } \theta < s \leq 1, \end{cases}$$

and

$$\text{if } \theta \leq t \leq 1, \qquad \frac{\partial G}{\partial t}(t,s) = \begin{cases} 0, \text{ if } 0 \leq s \leq \theta; \\ \frac{s-\theta}{1-\theta}, \text{ if } \theta < s < t; \\ \frac{s-1}{1-\theta}, \text{ if } t < s \leq 1. \end{cases}$$

Obviously,

Lemma 8. *For every $t \in [0,1]$, $G(t,\cdot)$ and $\frac{\partial G}{\partial t}(t,\cdot)$ are differentiable on $[0,1]\setminus\{t,\theta\}$ and their derivatives are absolutely continuous.*

3. Main result

Consider the Sobolev-type space $W^{2,1}_{\mathrm{HL},X}([0,1])$ of all X-valued functions u that are continuous on $[0,1]$, a.e. differentiable with the derivative u' continuous and a.e. differentiable and the second derivative u" Henstock-Lebesgue integrable.

Let r be some positive number and, following [9], we define, for each $t \in [0,1]$, $u_t : [-r,0] \to X$ defined by $u_t(s) = u(t+s)$. It is an expression of the taking into account the history of process modeling the evolution of the system. Let ϕ be a fixed X-valued Henstock-Lebesgue-integrable function on $[-r,0]$ and make the convention that whenever for a continuous function on $[0,1]$ the function u_s will intervene, u will be considered prolonged to $[-r,0]$ by ϕ.

Using the Hartman-type function G we can obtain $W^{2,1}_{\mathrm{HL},X}([0,1])$-functions:

Lemma 9. *Let $f : [0,1] \to X$ be a Henstock-Lebesgue integrable function. Then:*
1) *for every $t \in [0,1]$, $G(t,\cdot)f(\cdot)$ and $\frac{\partial G}{\partial t}(t,\cdot)f(\cdot)$ are Henstock-Lebesgue integrable and the function $u_f : [0,1] \to X$, $u_f(t) = (\mathrm{HL})\int_0^1 G(t,s)f(s)ds, \forall t \in [0,1]$ satisfies the following conditions: $u_f(0) = 0$, $u_f(\theta) = u_f(1)$, and*
2) *u_f is continuous;*
3) *u_f is a.e. differentiable and its derivative is $(u_f)'(t) = (\mathrm{HL})\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds$;*
4) *$(u_f)'$ is a.e. differentiable and its derivative satisfies $(u_f)" = f$.*

Proof. Lemma 8 and Lemma 4 yield the first assertion. By definition, $u_f(\theta) = u_f(1)$ and $u_f(0) = (\mathrm{HL})\int_0^1 G(0,s)f(s)ds = 0$. In order to prove the assertions 2) – 4), consider only the case $t \in [0,\theta[$. Then $u_f(t) = (\mathrm{HL})\int_0^t -sf(s)ds - t(\mathrm{HL})\int_t^\theta f(s)ds + t(\mathrm{HL})\int_\theta^1 \frac{s-1}{1-\theta}f(s)ds$. By Theorem 3, it is a.e. differentiable and $(u_f)'(t) = -(\mathrm{HL})\int_t^\theta f(s)ds + (\mathrm{HL})\int_\theta^1 \frac{s-1}{1-\theta}f(s)ds = (\mathrm{HL})\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds$ and also $(u_f)'$ is a.e. differentiable and $(u_f)"(t) = f(t)$. □

One can easily deduce

Proposition 10. *Let $f : [0,1] \to X$ be a Henstock-Lebesgue integrable function. Then the second order differential equation*

$$\begin{cases} u"(t) = f(t), \ \ a.e. \ t \in [0,1] \\ u(0) = 0, u(\theta) = u(1) \end{cases}$$

has an unique $W^{2,1}_{\mathrm{HL},X}([0,1])$-solution, namely $u_f(t) = (\mathrm{HL}) \int_0^1 G(t,s)f(s)ds$.

We proceed now to give the main result of the paper, on the existence of $W^{2,1}_{\mathrm{HL},X}([0,1])$-solutions for the second order retarded differential inclusion considered at the beginning of the paper. In order to obtain the existence of solutions, we make use of Kakutani-Ky Fan's fixed point theorem (as in [2]).

Theorem 11. *Let $\Gamma : [0,1] \to \mathcal{P}_{kc}(X)$ be measurable, Henstock-integrably bounded and $F : [0,1] \times \mathcal{HL}([-r,0],X) \times X \to \mathcal{P}_{kc}(X)$ satisfy the following conditions:*
1) *$F(t,x,y) \subset \Gamma(t)$, for all $t \in [0,1]$, $x \in \mathcal{HL}([-r,0],X)$ and $y \in X$;*
2) *$F(\cdot,x,y)$ is measurable, for every $x \in \mathcal{HL}([-r,0],X)$ and $y \in X$;*
3) *$F(t,\cdot,\cdot)$ is upper semi-continuous on $\mathcal{HL}([-r,0],X) \times X$, for each $t \in [0,1]$, where $\mathcal{HL}([-r,0],X)$ is considered endowed with the topology of pointwise convergence. Then the $W^{2,1}_{\mathrm{HL},X}([0,1])$-solutions set of retarded differential inclusion*

$$\begin{cases} u"(t) \in F(t,u_t,u'(t)), \text{a.e.} t \in [0,1] \\ u(0) = 0, u(\theta) = u(1) \end{cases}$$

is nonempty and $C([0,1],X)$-compact. Moreover, if a sequence $(u_n)_n$ of solutions converges uniformly to u, then the sequence $(u_n')_n$ pointwisely converges to u' and $(u_n")_n$ converges to $u"$ with respect to the w-HKP topology.

Proof. Step I. Let us prove, at this point, that if $\Gamma : [0,1] \to \mathcal{P}_{kc}(X)$ is a Henstock-integrably bounded set-valued function, then the $W^{2,1}_{\mathrm{HL},X}([0,1])$-solutions set of the second order differential inclusion

$$\begin{cases} u"(t) \in \Gamma(t), \ \ \text{a.e.} \ t \in [0,1] \\ u(0) = 0, u(\theta) = u(1) \end{cases}$$

is nonempty, convex and compact in $C([0,1],X)$ provided with the topology of the uniform convergence. We will also show that if a sequence $(u_n)_n$ of solutions converges uniformly to u, then the sequence $(u_n')_n$ pointwisely converges to u' and $(u_n")_n$ converges to $u"$ with respect to the w-HKP topology.

By Lemma 9, any $W^{2,1}_{\mathrm{HL},X}([0,1])$-solution u of this inclusion is characterized by the existence of a Henstock-Lebesgue-integrable selection f of Γ such that

$$u(t) = u_f(t) = (\mathrm{HL})\int_0^1 G(t,s)f(s)ds, \forall t \in [0,1].$$

The set of solutions is non-empty and it is convex, since Γ is convex-valued. We will use Ascoli's theorem to prove the compactness of solution set.
By Lemma 4, the $\mathcal{P}_{kc}(X)$-valued function $s \mapsto s\Gamma(s)$ is Henstock-integrably bounded. Fix $\varepsilon > 0$. By Proposition 6 and Lemma 1, one can find $\delta_\varepsilon > 0$ such that $\max\left(\left|(\mathrm{HK})\int_{t_1}^{t_2}\sigma(x^*,\Gamma(s))ds\right|, \left|(\mathrm{HK})\int_{t_1}^{t_2}\sigma(-x^*,\Gamma(s))ds\right|\right) < \varepsilon$, for all $x^* \in B^*$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_\varepsilon$. From the fact that for every HL-integrable selection of Γ,

$$\left|(\mathrm{HK})\int_{t_1}^{t_2}\langle x^*, f(s)\rangle ds\right| \leq \max\left(\left|(\mathrm{HK})\int_{t_1}^{t_2}\sigma(x^*,\Gamma(s))ds\right|, \left|(\mathrm{HK})\int_{t_1}^{t_2}\sigma(-x^*,\Gamma(s))ds\right|\right),$$

we deduce that $\|(HL)\int_{t_1}^{t_2} f(s)ds\| < \varepsilon$, $\forall |t_1 - t_2| < \delta_\varepsilon$. It is not difficult to see that one can choose δ_ε such that $\|(HL)\int_{t_1}^{t_2} sf(s)ds\| < \varepsilon$ when $|t_1 - t_2| < \delta_\varepsilon$. Considering the three possible cases ($t_1 < t_2 < \theta$, $t_1 < \theta \leq t_2$ and $\theta \leq t_1 < t_2$) we obtain $\|u_f(t_1) - u_f(t_2)\| \leq \varepsilon$, thus the equi-continuity is proved.
Fix now $t \in [0,1]$. Again by Lemma 4, $G(t,\cdot)\Gamma(\cdot)$ is Henstock-integrably bounded. For every solution $u = u_f$ of our inclusion (where f is a Henstock-Lebesgue-integrable selection of Γ),

$$u(t) = u_f(t) = (\mathrm{HL})\int_0^1 G(t,s)f(s)ds \in (\mathrm{HL})\int_0^1 G(t,s)\Gamma(s)ds$$

that is compact and convex.
It remains us to prove only the closeness of the solution set in $C([0,1],X)$. Consider a sequence $(u_n)_n$ of solutions, uniformly convergent to $u \in C([0,1],X)$, and prove that u is a solution too.
We can find a sequence $(f_n)_n$ of Henstock-Lebesgue-integrable selections of Γ such that $u_n(t) = u_{f_n}(t) = (\mathrm{HL})\int_0^1 G(t,s)f_n(s)ds$, for every $t \in [0,1]$. By Proposition 7, we are able to extract a subsequence $(f_{k_n})_n$ which w-HKP converges to a HL-integrable selection f of Γ such that the sequence $s \mapsto sf_{k_n}(s)$ be w-HKP convergent to the function $s \mapsto sf(s)$. By considering again the

two possible cases ($t \in [0, \theta[$ and $t \in [\theta, 1]$), we obtain that $(u_{k_n})_n$ pointwisely weakly converges to $u_f(t) = \text{(HL)} \int_0^1 G(t,s)f(s)ds$, therefore $u_f = u$; the solutions set is closed in $C([0,1],X)$ and thus the compactness is proved.
Similarly, we can prove that $(u_{f_n})'(t) = \text{(HL)} \int_0^1 \frac{\partial G}{\partial t}(t,s)f_n(s)ds$ converges to $(u_f)'(t) = \text{(HL)} \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds$. Since a.e. $(u_{f_n})" = f_n$ and $(u_f)" = f$, the w-HKP convergence of $(u_{f_n})"$ to $(u_f)"$ follows from the sequential w-HKP compactness of the set of all Henstock-Lebesgue-integrable selections of Γ.

Step II. Denote by $\mathcal{K}$ the $W^{2,1}_{\text{HL},X}([0,1])$-solutions set of

$$\begin{cases} u"(t) \in \Gamma(t), \text{ a.e. } t \in [0,1] \\ u(0) = 0, \quad u(\theta) = u(1) \end{cases}$$

Following the first step of the proof, $\mathcal{K}$ is convex and compact in $C([0,1],X)$. On $\mathcal{K}$, consider the set-valued function defined by

$$\Xi(u) = \left\{ v \in \mathcal{K} : \ v"(t) \in F(t, u_t, u'(t)), \text{ a.e. } \right\}.$$

Show that Ξ satisfies the hypothesis of Kakutani-Ky Fan's fixed point theorem.
Ξ is nonempty-valued. Indeed, passing through a sequence of simple functions as in the first part of the proof of Theorem 3.3 in [2], we can find a measurable selection of $F(\cdot, u_\cdot, u'(\cdot))$ which is a selection of Γ, so it is Henstock-Lebesgue-integrable. The claim follows from Lemma 9.
It is obvious that Ξ has convex values. Let us prove that is is closed- (thus, compact-) valued. Take a sequence $(v_n)_n \subset \Xi(u)$ uniformly convergent to $v \in C([0,1],X)$ and show that $v \in \Xi(u)$. The set-valued function Γ has a sequence $(f_n)_n$ of Henstock-Lebesgue-integrable selections satisfying that $f_n(t) \in F(t, u_t, u'(t))$ a.e. and $v_n(t) = \text{(HL)} \int_0^1 G(t,s)f_n(s)ds$. By Lemma 9, $(f_n)_n$ w-HKP converges to $v"$ and, following Proposition 7, $v"(t) \in F(t, u_t, u'(t))$ a.e.
It remains us to show that Ξ is upper semi-continuous or, equivalently in this case, to show that its graph is closed.
Consider $(u_n, v_n)_n \subset \text{Graph}(\Xi)$ convergent in $C([0,1],X) \times C([0,1],X)$ to (u,v) and prove that $(u,v) \in \text{Graph}(\Xi)$. For all $n \in \mathbb{N}$, $v_n"(t) \in F(t, u_{n_t}, u_n'(t))$ a.e. Again by Lemma 9, $(u_n')_n$ (resp. $(v_n')_n$) pointwisely converges to u' (resp. v') and $(u_n")_n$ (resp. $(v_n")_n$) w-HKP converges to $u"$ (resp. $v"$).

As $F(t,\cdot,\cdot)$ is upper semi-continuous, for every neighborhood V of the origin there are $W^2_{t,V}$ neighborhood of 0 in X and $W^1_{t,V}$ neighborhood of the origin of $\mathcal{HL}([-r,0],X)$ such that, for all z_1, z_2 verifying $z_1 - u_t \in W^1_{t,V}$ and $z_2 - u'(t) \in W^2_{t,V}$, one has $F(t,z_1,z_2) \subset F(t,u_t,u'(t))+V$. Since $(u_n)_n$ $C([0,1],X)$-converges to u and $(u'_n)_n$ pointwisely converges to u', there exists $n_{t,V} \in \mathbb{N}$ such that $u_t - u_{n_t} \in W^1_{t,V}$ and $u'(t) - u'_n(t) \in W^2_{t,V}$, for each $n \geq n_{t,V}$ (here we had to take into account the convention made at the beginning of the section). Consequently, $F(t,u_{n_t},u'_n(t)) \subset F(t,u_t,u'(t)) + V$, $\forall n \geq n_{t,V}$.
Using a classical result on measurable selections of a compact convex-valued multifunction (see e.g. [16] for the case of the weak topology), we obtain a sequence of convex combinations of $\{v_m", m \geq n\}$ a.e. convergent to $v"$, and then $v"(t) \in F(t,u_t,u'(t))$ a.e. Thus, Graph(Ξ) is closed.
We can now apply Kakutani-Ky Fan's fixed point theorem and deduce that Ξ has a fixed point, and so, our differential inclusion possess a continuous solution. $\square$

Remark 12. *Theorem* 11 *is new in the field of retarded differential equations and it extends the results given in the ordinary case in* [2] (*via Bochner and Pettis integrals*) *and* [15] (*using Henstock-Lebesgue integral*).

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