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A NOTE ON RAKOTCH CONTRACTIONS

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Abstract. We establish fixed point and convergence theorems for certain mappings of contractive type which take a closed subset of a complete metric space X into X.

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One of the most important results in fixed point theory is the Banach fixed point theorem [1]. As far as we know, the first significant generalization of Banach's theorem was obtained in 1962 by Rakotch [4], who replaced Banach's strict contractions with contractive mappings, that is, with those mappings which satisfy condition (1) below. Since then, such mappings, as well as their numerous modifications, were studied and used by many authors [3]. Recently, a renewed interest in contractive mappings has arisen [2]. See, for example, [6, 7] where well-posedness and genericity results were established. Another important topic in fixed point theory is the search for fixed points of nonself-mappings. In the present paper, as in [5], we combine these

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two themes by proving fixed point and convergence theorems for contractive nonself-mappings.

In Theorem 1 we provide a new sufficient condition for the existence and approximation of the unique fixed point of a contractive mapping which maps a nonempty and closed subset of a complete metric space X into X. In Theorem 2 we present a new proof of the fixed point theorem established in [5, Theorem 1(A)]. This new proof is based on Theorem 1. In Theorem 3 we obtain a convergence result for (unrestricted) infinite products [8] of mappings which satisfy a weak form of condition (1). Its proof is analogous to the proof of Theorem 1(B) in [5].

Let K be a nonempty and closed subset of a complete metric space (X, ρ). For each $x \in X$ and $r > 0$, set

$$B(x, r) = \{y \in X : \ \rho(x, y) \leq r\}.$$

Theorem 1. *Assume that $T : K \to X$ satisfies*

$$\rho(Tx, Ty) \leq \phi(\rho(x, y))\rho(x, y) \text{ for all } x, y \in K, \tag{1}$$

where $\phi : [0, \infty) \to [0, 1]$ is a monotonically decreasing function such that $\phi(t) < 1$ for all $t > 0$.

Assume that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset K$ such that

$$\lim_{n \to \infty} \rho(x_n, Tx_n) = 0. \tag{2}$$

Then there exists a unique point $\bar{x} \in K$ such that $T\bar{x} = \bar{x}$.

Proof. The uniqueness of $\bar{x}$ is obvious. To establish its existence, let $\epsilon \in (0, 1)$ and choose a positive number γ such that

$$\gamma < (1 - \phi(\epsilon))\epsilon/8. \tag{3}$$

By (2), there is a natural number n_0 such that

$$\rho(x_n, Tx_n) < \gamma \text{ for all integers } n \geq n_0. \tag{4}$$

Suppose that the integers $m, n \geq n_0$. We claim that $\rho(x_m, x_n) \leq \epsilon$. Assume the contrary. Then

$$\rho(x_m, x_n) > \epsilon. \tag{5}$$

By (3), (1), (5), the monotonicity of ϕ, and (4),

$$\rho(x_m, x_n) \leq \rho(x_m, Tx_m) + \rho(Tx_m, Tx_n) + \rho(Tx_n, x_n)$$

$$\leq 2\gamma + \phi(\rho(x_m, x_n))\rho(x_m, x_n) \leq 2\gamma + \phi(\epsilon)\rho(x_m, x_n)$$

$$= \rho(x_m, x_n) - (1 - \phi(\epsilon))\rho(x_m, x_n) + 2\gamma$$

$$< \rho(x_m, x_n) - (1 - \phi(\epsilon))\rho(x_m, x_n) + (1 - \phi(\epsilon))\epsilon/4$$

$$< \rho(x_m, x_n) - (1 - \phi(\epsilon))\rho(x_m, x_n)(3/4)$$

$$= \rho(x_m, x_n)[(1/4) + (3/4)\phi(\epsilon)] < \rho(x_m, x_n),$$

a contradiction.

The contradiction we have reached proves that $\rho(x_m, x_n) \leq \epsilon$ for all integers $m, n \geq n_0$, as claimed.

Since ϵ is an arbitrary number in $(0, 1)$, we conclude that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$. By (1), for all integers $n \geq 1$,

$$\rho(T\bar{x}, \bar{x}) \leq \rho(T\bar{x}, Tx_n) + \rho(Tx_n, x_n) + \rho(x_n, \bar{x})$$

$$\leq 2\rho(x_n, \bar{x}) + \rho(Tx_n, x_n) \to 0 \text{ as } n \to \infty.$$

This concludes the proof of Theorem 1. □

Next we use Theorem 1 to present a new proof of [5, Theorem 1(A)].

Theorem 2. *Let $T : K \to X$ satisfy*

$$\rho(Tx, Ty) \leq \phi(\rho(x, y))\rho(x, y) \text{ for all } x, y \in K,$$

where $\phi : [0, \infty) \to [0, 1]$ is a monotonically decreasing function such that $\phi(t) < 1$ for all $t > 0$.

Assume that $K_0 \subset K$ is a nonempty and bounded set with the following property:

For each natural number n, there exists $y_n \in K_0$ such that $T^i y_n$ is defined for all $i = 1, \ldots, n$.

Then the mapping T has a unique fixed point $\bar{x}$ in K.

Proof. By Theorem 1, it is sufficient to show that for each $\epsilon \in (0, 1)$, there is $x \in K$ such that $\rho(x, Tx) < \epsilon$. Indeed, let $\epsilon \in (0, 1)$. There is $M > 0$ such that

$$\rho(y_0, y_i) \leq M, \ i = 1, 2, \ldots. \tag{6}$$

By (1) and (6), for each integer $i \geq 1$,

$$\rho(y_i, Ty_i) \leq \rho(y_i, y_0) + \rho(y_0, Ty_0) + \rho(Ty_0, Ty_i) \leq 2M + \rho(y_0, Ty_0). \tag{7}$$

Choose a natural number $q \geq 4$ such that

$$(q-1)\epsilon(1-\phi(\epsilon)) > 4M + 2\rho(y_0, Ty_0). \tag{8}$$

Set $T^0 z = z$, $z \in K$.

We will show that $\rho(T^{q-1}y_q, T^q y_q) < \epsilon$. Assume the contrary. Then by (1),

$$\rho(T^i y_q, T^{i+1} y_q) \geq \epsilon,\ i = 0, \dots, q-1. \tag{9}$$

In view of (1), (9) and the monotonicity of ϕ, we have for $i = 0, \dots, q-2$,

$$\rho(T^{i+1}y_q, T^{i+2}y_q) \leq \phi(\rho(T^i y_q, T^{i+1}y_q))\rho(T^i y_q, T^{i+1}y_q) \leq \phi(\epsilon)\rho(T^i y_q, T^{i+1}y_q)$$

and

$$\rho(T^i y_q, T^{i+1}y_q) - \rho(T^{i+1}y_q, T^{i+2}y_q) \geq (1-\phi(\epsilon))\rho(T^i y_q, T^{i+1}y_q) \geq (1-\phi(\epsilon))\epsilon. \tag{10}$$

By (7) and (10),

$$\begin{aligned} 2M + \rho(y_0, Ty_0) &\geq \rho(y_q, Ty_q) - \rho(T^{q-1}y_q, T^q y_q) \\ &\geq \sum_{i=0}^{q-2}[\rho(T^i y_q, T^{i+1}y_q) - \rho(T^{i+1}y_q, T^{i+2}y_q)] \\ &\geq (q-1)(1-\phi(\epsilon))\epsilon \end{aligned}$$

and

$$2M + \rho(y_0, Ty_0) \geq (q-1)(1-\phi(\epsilon))\epsilon.$$

This contradicts (8). The contradiction we have reached shows that

$$\rho(T^{q-1}y_q, T^q y_q) < \epsilon,$$

as required. Theorem 2 is proved. □

Now we establish a convergence result for (unrestricted) infinite products of mappings which satisfy a weak form of condition (1).

Theorem 3. *Let $\phi : [0, \infty) \to [0, 1]$ be a monotonically decreasing function such that $\phi(t) < 1$ for all $t > 0$.*

Let

$$\bar{x} \in K,\ T_i : K \to X,\ i = 0, 1, \dots,\ T_i\bar{x} = \bar{x},\ i = 0, 1, \dots, \tag{11}$$

and assume that

$$\rho(T_i x, \bar{x}) \leq \phi(\rho(x, \bar{x}))\rho(x, \bar{x}) \text{ for each } x \in K,\ i = 0, 1, \dots. \tag{12}$$

Then for each $M, \epsilon > 0$, there exist $\delta > 0$ and a natural number k such that for each integer $n \geq k$, each mapping $r : \{0, 1, \dots, n-1\} \to \{0, 1, \dots\}$, and each sequence $\{x_i\}_{i=0}^n \subset K$ satisfying

$$\rho(x_0, \bar{x}) \leq M \text{ and } \rho(x_{i+1}, T_{r(i)}x_i) \leq \delta, \; i = 0, \dots, n-1,$$

we have

$$\rho(x_i, \bar{x}) \leq \epsilon, \; i = k, \dots, n. \tag{13}$$

Proof. Choose $\delta_0 > 0$ such that

$$\delta_0 < M(1 - \phi(M/2))/4. \tag{14}$$

Assume that

$$y \in K \cap B(\bar{x}, M), \; i \in \{0, 1, \dots\}, \; z \in X \text{ and } \rho(z, T_i y) \leq \delta_0. \tag{15}$$

By (15) and (12),

$$\rho(\bar{x}, z) \leq \rho(\bar{x}, T_i y) + \rho(T_i, z) \leq \phi(\rho(\bar{x}, y))\rho(\bar{x}, y) + \delta_0. \tag{16}$$

There are two cases:

$$\rho(y, \bar{x}) \leq M/2 \tag{17}$$

and

$$\rho(y, \bar{x}) > M/2. \tag{18}$$

Assume that (17) holds. Then by (16), (17) and (14),

$$\rho(\bar{x}, z) \leq \rho(\bar{x}, y) + \delta_0 \leq M/2 + \delta_0 < M. \tag{19}$$

If (18) holds, then by (16), (15), (14) and the monotonicity of ϕ,

$$\rho(\bar{x}, z) \leq \delta_0 + \phi(M/2)\rho(\bar{x}, y) \leq \delta_0 + \phi(M/2)M$$
$$< (M/4)(1 - \phi(M/2)) + \phi(M/2)M \leq M.$$

Thus $\rho(\bar{x}, z) \leq M$ in both cases.

We have shown that

$$\text{if } y \in K \cap B(\bar{x}, M), \; i \in \{0, 1, \dots\}, \; z \in X, \; \rho(z, T_i y) \leq \delta_0, \text{ then } \rho(\bar{x}, z) \leq M. \tag{20}$$

Since M is any positive number, we conclude that there is $\delta_1 > 0$ such that

$$\text{if } y \in K \cap B(\bar{x}, \epsilon), \; i \in \{0, 1, \dots\}, \; z \in X, \; \rho(z, T_i y) \leq \delta_1, \text{ then } \rho(\bar{x}, z) \leq \epsilon. \tag{21}$$

Now choose a positive number δ such that

$$\delta < \min\{\delta_0, \delta_1, \ \epsilon(1-\phi(\epsilon))4^{-1}\} \tag{22}$$

and a natural number k such that

$$k > 4(M+1)((1-\phi(\epsilon))\epsilon)^{-1} + 4. \tag{23}$$

Let $n \geq k$ be a natural number. Assume that $r : \{0, \dots, n-1\} \to \{0, 1, \dots\}$ and that

$$\{x_i\}_{i=0}^n \subset K$$

satisfies

$$\rho(x_0, \bar{x}) \leq M \text{ and } \rho(x_{i+1}, T_{r(i)}x_i) \leq \delta, \ i = 0, \dots, n-1. \tag{24}$$

We claim that (13) holds. By (20), (24) and the inequality $\delta < \delta_0$,

$$\{x_i\}_{i=0}^n \subset B(\bar{x}, M). \tag{25}$$

Assume to the contrary that (13) does not hold. Then there is an integer j such that

$$j \in \{k, \dots, n\} \text{ and } \rho(x_j, \bar{x}) > \epsilon. \tag{26}$$

By (26) and (12),

$$\rho(x_i, \bar{x}) > \epsilon, \ i = 0, \dots, j. \tag{27}$$

Let $i \in \{0, \dots, j-1\}$. By (24), (12) and the monotonicity of ϕ,

$$\rho(x_{i+1}, \bar{x}) \leq \rho(x_{i+1}, T_{r(i)}x_i) + \rho(T_{r(i)}x_i, \bar{x}) \leq \delta + \phi(\rho(x_i, \bar{x}))\rho(x_i, \bar{x})$$

$$\leq \delta + \phi(\epsilon)\rho(x_i, \bar{x}).$$

When combined with (22) and (27), this implies that

$$\rho(x_{i+1}, \bar{x}) - \rho(x_i, \bar{x}) \leq \delta - (1-\phi(\epsilon))\rho(x_i, \bar{x}) \leq \delta - (1-\phi(\epsilon))\epsilon < -(1-\phi(\epsilon))\epsilon/2. \tag{28}$$

Finally, by (24), (28) and (26),

$$-M \leq -\rho(x_0, \bar{x}) \leq \rho(x_j, \bar{x}) - \rho(x_0, \bar{x})$$

$$= \sum_{i=0}^{j-1} [\rho(x_{i+1}, \bar{x}) - \rho(x_i, \bar{x})] \leq -j(1-\phi(\epsilon))\epsilon/2 \leq -k(1-\phi(\epsilon))\epsilon/2.$$

This contradicts (23). The contradiction we have reached proves (13) and Theorem 3 itself. □

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