Fixed Point Theory, Volume 9, No. 1, 2008, 189-197
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A FUNCTIONAL-INTEGRAL EQUATION WITH LINEAR MODIFICATION OF THE ARGUMENT, VIA WEAKLY PICARD OPERATORS

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Abstract. In this paper we give existence results for the solutions of a functional-integral equation with linear modification of the argument, in Banach space. By weakly Picard operators' technique (see I.A. Rus [19], [23]-[25] and I.A. Rus, S. Muresan [27]), the data dependence is also studied.
Key Words and Phrases: Picard operators, weakly Picard operators, functional-integral equations, solutions set, data dependence.
2000 Mathematics Subject Classification: 34K15, 34G20, 45N05, 47H10.

1. Introduction

Functional-integral equations with modified argument arise in a wide variety of scientific and technical applications, including the modeling of problems from the natural and social sciences such as physics, chemistry, biology, economics, engineering. The theory of functional-integral equations has developed very much. Many monographs appeared: Bellman and Cooke [2](1963), Halanay [7](1966), Elsgoltz and Norkin [5](1971), Bernfeld and Lakshmikantham [3](1974), Hale [8](1977), Azbelev, Maksimov and Rahmatulina [1](1991), Hale and Verdyn Lunel [9](1993), Guo and Lakshmikantham [6](1996) such as a large number of papers. We quote here [11], [12], [16], [25], [27], [28].

This paper was presented at the International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from July 4 to July 8, 2007.

The aim of this paper is to study the following functional-integral equation with linear modification of the argument, in Banach space:

$$x(t) = g(t, x(t), x(\lambda t), x(0) + \int_0^t K(t, s, x(s), x(\lambda s))ds, \quad t \in [0, b], 0 < \lambda < 1. \tag{1.1}$$

We use weakly Picard operators' technique and the same method as in the paper [25].

In [4] and [6] were considered some particular cases of this equation.

2. Weakly Picard operators

Let (X, d) be a metric space and $A : X \longrightarrow X$ an operator. We shall use the following notations:
$P(X) := \{Y \subseteq X | Y \neq \emptyset\}$;
$F_A := \{x \in X | A(x) = x\}$ - the fixed point set of A;
$I(A) := \{Y \in P(X) | A(Y) \subseteq Y\}$;
$A^0 := 1_X, A^1 := A, \ldots, A^{n+1} := A \circ A^n, \ldots, n \in \mathbb{N}$.

Definition 2.1. *(Rus [20]) A is Picard operator if there exists $x^* \in X$ such that*

1) $F_A = \{x^*\}$;
2) *the successive approximation sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.*

Definition 2.2. *(Rus[19]) A is weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and its limit (which may depend on x_0) is a fixed point of A.*

If A is weakly Picard operator we consider $A^\infty : X \longrightarrow X$, $A^\infty(x) = \lim_{n\to\infty} A^n(x)$. We remark that $A^\infty(X) = F_A$.

Definition 2.3. *(Rus[24]) Let A be a weakly Picard operator and $c > 0$. A is c-weakly Picard operator if*

$$d(x, A^\infty(x)) \leq cd(x, A(x)), \quad \textit{for all} \quad x \in X.$$

We have

Theorem 2.1. *(Rus[23], [24]) Let (X,d) be a metric space and $A : X \longrightarrow X$ an operator. A is weakly Picard operator (c-weakly Picard operator) if and only if there exists a partition of $X, X = \underset{\mu\in\Lambda}{\cup} X_\mu$, such that:*

(a) $X_\mu \in I(A)$, *for all* $\mu \in \Lambda$*;*

(b) $A|_{X_\mu} : X_\mu \longrightarrow X_\mu$ *is a Picard (c-Picard) operator, for all* $\mu \in \Lambda$.

Theorem 2.2. *(Rus[24]) Let (X,d) be a metric space and $A_i : X \longrightarrow X,\ i = 1,2$. We suppose that*

(i) *the operator A_i is c_i-weakly Picard operator, $i = 1,2$;*

(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x), A_2(x)) \leq \eta, \quad \text{for all} \quad x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \ \max(c_1, c_2),$$

where H stands for Pompeiu-Hausdorff functional defined by

$$H(F_{A_1}, F_{A_2}) : \max\{ \sup_{a\in F_{A_1}} \inf_{b\in F_{A_2}} d(a,b), \sup_{b\in F_{A_2}} \inf_{a\in F_{A_1}} d(a,b)\} \cup \{\infty\}.$$

Theorem 2.3. *(Rus[23]) Let $(X,d,\leq)$ be an ordered metric space and $A : X \longrightarrow X$ such that*

(i) *A is monotone increasing;*

(ii) *A is weakly Picard operator.*

Then the operator A^∞ is monotone increasing.

Theorem 2.4. *(Rus[23]) Let $(X,d,\leq)$ be an ordered metric space and $A, B, C : X \longrightarrow X$ be such that:*

(i) $A \leq B \leq C$*;*

(ii) *the operators A, B, C are weakly Picard operators;*

(iii) *the operator B is monotone increasing*

Then $x \leq y \leq z$ implies $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

3. The solutions set of the equation (1.1)

Let $(X, \|\cdot\|, \leq)$ be an ordered Banach space and $C([0,b], X)$ endowed with the following Bielecki norm:

$$\|x\|_B := \max_{t\in[0,b]} (\|x(t)\| e^{-\tau t}), \quad \text{where} \quad \tau > 0.$$

So, $(C([0,b],X),\|\cdot\|_B)$ is a Banach space denoted in what follows by $C([0,b],X)$.

Consider the equation(1.1) and suppose that the following conditions are satisfied:

(c_1) $g\in C([0,b]\times X\times X\times X,X)$, $K\in C([0,b]\times[0,b]\times X\times X,X)$;

(c_2) there exists $L_k>0$ such that

$$\|K(t,s,u_1,u_2)-K(t,s,v_1,v_2)\|\leq L_k(\|u_1-v_1\|+\|u_2-v_2\|),$$

for all $t,s\in[0,b]$ and all $u_i,v_i\in X,\ i=1,2$;

(c_3) there exists $L_g<\frac{1}{2}$ such that

$$\|g(t,u_1,u_2,\alpha)-g(t,v_1,v_2,\alpha)\|\leq L_g(\|u_1-v_1\|+\|u_2-v_2\|),$$

for all $t\in[0,b]$ and all $u_i,v_i,\alpha\in X,\ i=1,2$.

Consider the operator $A:C([0,b],X)\longrightarrow C([0,b],X)$ defined by $A(x)(t):=g(t,x(t),x(\lambda t),x(0))+\int_0^t K(t,s,x(s),x(\lambda s))ds,\ t\in[0,b],\ \lambda\in]0,1[$.

In our considerations the following equation

$$g(0,\alpha,\alpha,\alpha)=\alpha,\quad \alpha\in X \tag{3.1}$$

plays an important role. We denote by S_g the solutions set of the equation (3.1).

Then we have

Remark 3.1. *If x is a solution of the equation (1.1) (i.e. $x\in F_A$), then $x(0)\in S_g$.*

Remark 3.2. *Let $X_\alpha:=\{x\in C([0,b],X)\mid x(0)=\alpha\}$ be.*
It is clear that

$$C([0,b],X)=\bigcup_{\alpha\in X}X_\alpha$$

is a partition of $C([0,b],X)$.

Remark 3.3. *$X_\alpha\in I(A)$ if and only if $\alpha\in S_g$.*

We denote by $A_\alpha:=A\mid_{X_\alpha}:X_\alpha\longrightarrow X_\alpha$.

We have

Theorem 3.1. *We suppose that the conditions* $(c_1), (c_2)$ *and* (c_3) *are satisfied. Then the operator*

$$A\mid_{\underset{\alpha\in S_g}{\cup} X_\alpha} : \underset{\alpha\in S_g}{\cup} X_\alpha \longrightarrow \underset{\alpha\in S_g}{\cup} X_\alpha$$

is a weakly Picard operator and $CardF_A = CardS_g$.

Proof. By using (c_2) and (c_3) we obtain

$$\|A_\alpha(x)(t) - A_\alpha(z)(t)\| \leq \|g(t, x(t), x(\lambda t), \alpha) - g(t, z(t), z(\lambda t), \alpha)\|$$
$$+ \int_0^t \|K(t, s, x(s), x(\lambda s)) - K(t, s, z(s), z(\lambda s))\| ds \leq$$
$$L_g(\|x(t) - z(t)\|e^{-\tau t}e^{\tau t} + \|x(\lambda t) - z(\lambda t)\|e^{-\tau\lambda t}e^{\tau\lambda t} +$$
$$+L_K \int_0^t (\|x(s) - z(s)\|e^{-\tau s}e^{\tau s} + \|x(\lambda s) - z(\lambda s)\|e^{-\tau\lambda s}e^{\tau\lambda s})ds$$
$$\leq L_g\|x - z\|_B(e^{\tau t} + e^{\tau\lambda t}) + L_k\|x - z\|_B(\int_0^t e^{\tau s}ds + \int_0^t e^{\tau\lambda s}ds)$$
$$\leq 2L_g\|x - z\|_B e^{\tau t} + L_K\|x - z\|_B\left(\tfrac{e^{\tau t}-1}{\tau} + \tfrac{e^{\tau\lambda t}-1}{\lambda\tau}\right)$$
$$\leq e^{\tau t}\left(2L_g + L_k\tfrac{1+\frac{1}{\lambda}}{\tau}\right)\|x - z\|_B, \quad \text{for all} \quad t \in [0, b].$$

So,

$$\|A_\alpha(x)(t) - A_\alpha(z)(t)\|e^{-\tau t} \leq \left[2L_g + \frac{L_k(1 + \frac{1}{\lambda})}{\tau}\right]\|x - z\|_B, \quad \text{for all} \quad t \in [0, b].$$

Therefore,

$$\|A_\alpha(x)(t) - A_\alpha(z)(t)\|_B \leq \left[2L_g + \frac{L_k(1 + \frac{1}{\lambda})}{\tau}\right]\|x - z\|_B.$$

It follows that A_α is a Lipschitz operator with a Lipschitz constant

$$L_A = 2L_g + \frac{L_k}{\tau}(1 + \frac{1}{\lambda}).$$

But $2L_g < 1$ and by choosing τ large enough we have that A is a contraction. The proof follows from Contraction principle and from the result of Theorem 2.1.

By using Remark 3.1 we can define the operator $\varphi : F_A \longrightarrow S_g,\ x \longrightarrow x(0)$ and φ is a bijective operator. So, $CardF_A = CardS_g$. □

Theorem 3.2. *Consider the equation (1.1) with the conditions $(c_1),(c_2),(c_3)$. We suppose that*
(c_4) *$g(t,\cdot,\cdot,\cdot)$ and $K(t,s,\cdot,\cdot)$ are increasing, for all $t,s\in[0,b]$.*

If x and y are two solutions of the equation (1.1) then

$$x(0)\le y(0) \text{ implies } x\le y.$$

Proof. From the Theorem 3.1, the operator A is weakly Picard operator. By using (c_4) and the Theorem 2.3 we obtain that A^∞ is increasing. For $\alpha\in X$ we define $\tilde\alpha:[0,b]\longrightarrow X,\ \tilde\alpha(t):=\alpha$, for all $t\in[0,b]$. We have $x=A^\infty(\tilde x(0))$ and $y=A^\infty(\tilde y(0)$. So, $x(0)\le y(0)$ implies $x\le y$. □

Theorem 3.3. *Let $g_i,K_i,\ i=1,2,3$ be with the corresponding conditions $(c_1),(c_2),(c_3)$. We suppose that*

(i) *$g_2(t,\cdot,\cdot,\cdot)$ and $K_2(t,s,\cdot,\cdot)$ are increasing;*
(ii) *$g_1\le g_2\le g_3$ and $K_1\le K_2\le K_3$.*
(iii) *Let S_{g_i} be the solution set of the equation*

$$g_i(0,\alpha,\alpha,\alpha)=\alpha,\ \ i=1,2,3$$

and we suppose that $S_{g_1}=S_{g_2}=S_{g_3}$.

If x_i is a solution of the corresponding equation (1.1), for $g_i,K_i,\ i=1,2,3$, then

$$x_1(0)\le x_2(0)\le x_3(0) \text{ implies } x_1\le x_2\le x_3.$$

Proof. Let $A_i,\ i=1,2,3$ be the corresponding operator for $g_i,K_i,\ i=1,2,3$. We remark that

$$x_i=A_i^\infty(\tilde x_i(0)),\ i=1,2,3.$$

The proof follows from the Theorem 2.4. □

Theorem 3.4. *Let $g_i,K_i,i=1,2$ be with the corresponding conditions $(c_1),(c_2),(c_3)$. We suppose that*

(i) *there exist $\eta_i>0,\ i=1,2$ such that*

$$|g_1(t,u_1,u_2,u_3)-g_2(t,u_1,u_2,u_3)|\le\eta_1,$$

for all $t\in[0,b]$ and all $u_i\in X,\ i=1,2,3$, and

$$|K_1(t,s,v_1,v_2)-K_2(t,s,v_1,v_2)|\le\eta_2,$$

for all $t,s\in[0,b]$ and all $v_i\in X,\ i=1,2$;

(ii) $S_{g_1} = S_{g_2}$.

Then

$$H_B(F_{A_1}, F_{A_2}) \leq \frac{\eta_1 + \eta_2 b}{1 - 2L_g - \frac{L_K}{\tau}(1 + \frac{1}{\lambda})},$$

where $L_g = \max(L_{g_1}, L_{g_2})$, $L_K = \max(L_{K_1}, L_{K_2})$, H_B *is the Pompeiu-Hausdorff functional corresponding to* $\|\cdot\|_B$ *and* τ *is suitable chosen.*

Proof. We have

$$\|A_1(x)(t) - A_2(x)(t)\| \leq \|g_1(t, x(t), x(\lambda t), x(0)) - g_2(t, x(t), x(\lambda t), x(0))\|$$

$$+ \int_0^b \|K_1(t, s, x(s), x(\lambda s)) - K_2(t, s, x(s), x(\lambda s))\| ds \leq \eta_1 + \eta_2 b.$$

It follows that $\|A_1(x) - A_2(x)\|_B \leq \eta_1 + \eta_2 b$.

The operator $A_i, i = 1, 2$ is c-weakly Picard operator with the constant

$$c = \frac{1}{1 - 2L_g - \frac{L_k}{\tau}(1 + \frac{1}{\lambda})},$$

where $L_g = \max(L_{g_1}, L_{g_2}), L_K = \max(L_{K_1}, L_{K_2})$ and τ is suitable chosen such that $2L_g + \frac{L_K}{\tau}(1 + \frac{1}{\lambda}) < 1$. So, the proof follows from the Theorem 2.2. □

Example 3.1. *Consider the following equation:*

$$x(t) = t + a_1 x(t) + a_2 x(\frac{t}{2}) + x(0) - a_3 + \int_0^t (t + s + \sin(x(s)) + \cos\left(x\left(\frac{s}{2}\right)\right)\Big) ds,\ t \in [0, 3], \tag{3.2}$$

where $a_i \in \mathbb{R}, i = \overline{1, 3}$.

We have

Theorem 3.5. *We suppose that* $a_1 + a_2 \neq 0$, $|a_1| < \frac{1}{4}$ *and* $|a_2| < \frac{1}{4}$. *Then the equation (3.2) has a unique solution.*

Proof. Here $g(t, u, v, w) = t + a_1 u + a_2 v + w - a_3$ and the condition (c_3) is satisfied with $L_g = \max(|a_1|, |a_2|)$. The equation $g(0, \alpha, \alpha, \alpha) = \alpha$ has a unique solution

$$\alpha = \frac{a_3}{a_1 + a_2}.$$

So, the proof follows from the Theorem 3.1. □

References

[1] N.V. Azbelev, V.P. Maksimov, L.F. Rahmatulina, *Introduction to Functional-Differential Equations Theory*, MIR, Moscow, 1991 (in Russian).

[2] R.E. Bellman, K.L. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.

[3] S.R. Bernfeld, V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.

[4] C. Corduneanu, *Integral Equations and Applications,* Cambridge University Press, 1991.

[5] L.F. Elsgoltz, S.B. Norkin, *Introduction to the Theory of Differential Equations with Deviating Arguments,* MIR, Moscow, 1971 (in Russian).

[6] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstracts Spaces,* Kluwer, Dordrecht, 1996.

[7] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags,* Academic Press, New York, 1965.

[8] J.K. Hale, *Theory of Functional-Differential Equations,* Springer Verlag, 1977.

[9] J.K. Hale, M. Sjoerd, Lunel Verduyn, *Introduction to Functional-Differential Equations,* Springer Verlag, New York, 1993.

[10] M.W. Hirsch, C.C. Pugh, *Stable maniforlds and hyperbolic sets*, Proc. Symp. in Pure Math., **14**(1970), 133-163.

[11] A. Iserles, *On the generalized pantograph functional-differential equation*, European J. Appl. Math., 1992, 1-38.

[12] T. Kato, J.B. McLeod, *The functional-differential equation* $y'(x) = ay(\lambda x)+by(x)$, Bull. Amer. Math. Soc., **77**(1971), 891-937.

[13] V. Lakshmikanthan (ed.), *Trends in the Theory and Practice of Nonlinear Differential Equations*, Marcel Dekker, New York, 1984.

[14] V. Muresan, *Differential Equations with Affine Modification of the Argument*, Transilvania Press, Cluj-Napoca, 1997 (in Romanian).

[15] V. Muresan, *Functional-Integral Equations*, Ed. Mediamira, Cluj-Napoca, 2003.

[16] J.R. Ockendon, *Differential equations and industry*, The Math. Scientist., **5**(1980), 1-12.

[17] R. Precup, *Integral Equations*, Babes-Bolyai University, Cluj-Napoca, 1993 (in Romanian).

[18] I.A. Rus, *Metrical Fixed Point Theorems*, Univ. of Cluj-Napoca, 1979.

[19] I.A. Rus, *Weakly Picard mappings,* Comment. Math. Univ. Carolinae, **34**(1993), 769-773.

[20] I.A. Rus, *Picard operators and applications*, Babes-Bolyai Univ. of Cluj-Napoca, Preprint 3(1996).

[21] I.A. Rus, *Differential Equations, Integral Equations and Dynamical Systems*, Transilvania Press, Cluj-Napoca, 1996 (in Romanian).

[22] I.A. Rus, *A fiber generalized contraction theorem and applications*, Mathematica, **41(64)**(1999), 85-90.

[23] I.A. Rus, *Weakly Picard operators and applications,* Seminar on Fixed Point Theory Cluj-Napoca, **2**(2001), 41-58.

[24] I.A. Rus, *Generalized Contractions and Applications,* Cluj Univ. Press, Cluj-Napoca, 2001.

[25] I.A. Rus, *A class of nonlinear functional-integral equations, via weakly Picard operators*, Anal. Univ. Craiova, ser. Mat-Inf., **28**(2001), 10-15.

[26] I.A. Rus, *Picard operators and applications*, Scientia Mathematicae Japonicae, **58**(2003), 191-219.

[27] I.A. Rus, S. Muresan, *Data dependence of the fixed points set of weakly Picard operators*, Studia Univ. Babes-Bolyai Mathematica, **43**(1998), 79-83.

[28] A. Sincelean, *On a class of functional-integral equations*, Seminar on Fixed Point Theory Cluj-Napoca, **1**(2000), 87-92.

[29] Z.B. Tsalynk, *Volterra functional inequalities*, Yzv. Ucheb. Zaved, **3**(1969), 86-95 (in Russian).

Received: November 15, 2007; Accepted: January 18, 2007.