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ON THE MULTIPLICITY OF THE CIRCUMFERENCE IN PLANAR POLYNOMIAL VECTOR FIELDS

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Abstract. In this work we consider planar polynomial differential systems of the form:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

where $P(x, y)$ and $Q(x, y)$ are polynomials with real coefficients whose maximum degree is d. We only consider systems of this form with the circumference $x^2 + y^2 - 1 = 0$ as a periodic orbit. These systems take the form:

$$\dot{x} = -y\, c(x, y) + f(x, y)\, a(x, y), \quad \dot{y} = x\, c(x, y) + f(x, y)\, b(x, y),$$

where $f(x, y) = (x^2 + y^2 - 1)/2$ and a, b and c are real polynomials. Our interest in this work is to study the multiplicity of the circumference as periodic orbit of the aforementioned system. This work contains some theorems that characterize when the circumference is a limit cycle of multiplicity m and when it belongs to a period annulus. Moreover, if we assume that the system is of a particular form, we will give an upper bound for the possible multiplicities that the circumference may have as a limit cycle. Finally, we apply our results to some examples.

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1. Introduction and statement of the main results

In this work we consider planar polynomial differential systems of the form:

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials with real coefficients whose maximum degree is d.

We only consider systems of the form (1.1) with the circumference $x^2+y^2-1=0$ as periodic orbit. We denote by Γ the graphic of this periodic orbit. In the works [3, 7], it is shown that any polynomial differential system with the circumference as periodic orbit takes the form:

$$\dot{x} \,=\, -y\,c(x,y) \,+\, f(x,y)\,a(x,y), \quad \dot{y} \,=\, x\,c(x,y) \,+\, f(x,y)\,b(x,y), \tag{1.2}$$

where $f(x,y) = (x^2+y^2-1)/2$ and a, b and c are real polynomials. Moreover, since $f(x,y)=0$ needs to be a periodic orbit, the circumference cannot contain any critical point of the system. Therefore, we assume that there is no real point in the intersection of $f(x,y)=0$ and $c(x,y)=0$. In the case of $c \equiv 0$, Γ is filled with critical points and then $f=0$ is not a periodic orbit.

Our interest in this work is to study the multiplicity of the circumference as periodic orbit of system (1.2). The following section contains the definition of multiplicity of a periodic orbit.

In order to state our results we prefer to write system (1.2) in the equivalent Pfaffian form: $\omega = 0$ with

$$\omega \,=\, c\,df \,+\, f\,\omega_0,$$

where $c(x,y)$ is the aforementioned polynomial and ω_0 is the polynomial 1-form given by $\omega_0 \,=\, b\,dx \,-\, a\,dy$. We recall that $df \,=\, xdx + ydy$.

We present a generalization of Melnikov functions to study the multiplicity of the circumference as a periodic orbit of system (1.2). Melnikov functions are classically used for the study of the bifurcations of limit cycles from a period annulus. We give a description of these functions and its use in the following section.

The most important results we present are the following ones.

Theorem 1.1. *Γ is a limit cycle of multiplicity 1 of system (1.2) if, and only if,* $\displaystyle\oint_{f=0} \frac{\omega_0}{c} \neq 0.$

We are going to use the following result due to Françoise [11] which characterizes the decomposition of any polynomial 1-form ω in relation with the algebraic curve $f = 0$.

Proposition 1.2. [11] *Given any* polynomial 1*-form ω, there exist g, S polynomials in $\mathbb{R}[x, y]$ and ψ a polynomial in $\mathbb{R}[x]$, such that*

$$\omega \;=\; g\, df \;+\; dS \;+\; \psi(f)\,(y\, dx - x\, dy),$$

where $f = (x^2 + y^2 - 1)/2$.

In particular, we have that $\oint_{f=0} \omega_0 = 0$ if, and only if, there exist g_0, S_0 polynomials and a polynomial 1-form ω_1, such that $\omega_0 = g_0\, df + dS_0 + f\, \omega_1$, where ω_1 is the following 1-form $\omega_1 = \psi(f)\,(y\, dx - x\, dy)$.

We consider the particular case in which c is a constant different from zero and by scaling we take $c \equiv 1$. Using the aforementioned result, we can prove the following one.

Theorem 1.3. *Consider the particular case $c(x, y) \equiv 1$. Then, Γ is a limit cycle of multiplicity 2 of system (1.2) if, and only if, $\omega_0 = g_0 df + dS_0 + f\omega_1$ and* $\displaystyle\oint_{f=0} e^{-S_0}(\omega_1 - g_0 dS_0) \neq 0.$

The previous theorem shows that to characterize when the circumference is a limit cycle of multiplicity two, one needs to compute an integral involving elementary functions. Since we are mainly interested in the algebraic properties of the multiplicity, we are going to restrict ourselves to the particular case in which $c \equiv 1$ and S_0 is a constant, that is, we are going to determine when the circumference is a limit cycle of multiplicity m, with $m \geq 2$, of the following 1-form:

$$\omega \;=\; df + f\, g_0\, df \;+\; f^2\, \omega_1, \tag{1.3}$$

where, $f = (x^2 + y^2 - 1)/2$, g_0 is a polynomial in $\mathbb{R}[x, y]$ and ω_1 is the following 1-form $\omega_1 = \psi(f)\,(y\, dx - x\, dy)$, with ψ a polynomial in $\mathbb{R}[x]$.

Theorem 1.4. *Consider the* 1*-form* (1.3)*. Then,* Γ *is a limit cycle of multiplicity* m *if, and only if,* $\omega = df + f\,g_0\,df + f^m\,\phi(f)\,(ydx - xdy)$*, with* ϕ *a polynomial in* $\mathbb{R}[x]$ *such that* $\phi(0) \neq 0$.

The following corollary specifies when the circumference Γ belongs to a period annulus for the 1-form described in (1.3).

Corollary 1.5. *Consider the* 1*-form* (1.3)*. Then,* Γ *belongs to a continuum of periodic orbits if, and only if,* $\omega = df + f\,g_0\,df$*, where* g_0 *is any real polynomial.*

We remark that the 1-form $\omega = df + f\,g_0\,df$ has the function f as first integral and $1 + f\,g_0$ as inverse integrating factor. The definitions of these notions of integrability can be found in [7, 13] and the references therein.

We have characterized when the circumference Γ is a limit cycle of multiplicity m and when it belongs to a period annulus. We assume that the 1-form (1.3) has degree d and the following result gives an upper bound for the possible multiplicities that the circumference may have as limit cycle of (1.3) in terms of d. Since this upper bound is sharp, we have the value of the cyclicity of Γ inside the family of systems of degree d whose associated 1-form reads for (1.3). The definition of cyclicity and its relationship with the multiplicity is described in the following section.

We denote by $\lfloor x \rfloor$ the floor of the real number x.

Theorem 1.6. *Consider the* 1*-form* (1.3) *and assume that it is of degree* d*. Then, the cyclicity of* Γ *as a limit cycle is* $\lfloor (d-1)/2 \rfloor$.

This work is organized as follows. The next section contains the definitions and previous results needed to state and prove our theorems. In addition, this section contains the description of the state of the art of the set forth problem. The first definition that we need is the multiplicity of a limit cycle of a planar differential system, which is given in terms of the Poincaré return map. One of the classical tools to determine the multiplicity of a limit cycle consists in changing the system to local coordinates and to characterize the Poincaré return map in these coordinates. We describe how this method gives rise to several formulae to tackle our problem. However the determination of these formulae is computationally very difficult. Another way to do this study is to consider the notion of analytic m-solution. This notion is introduced in [13] and it allows the computation of the multiplicity of a limit cycle, provided that one knows an analytic m-solution. In order to complete the understanding of the meaning of multiplicity, we give its relationship with the notion of cyclicity. We remark that the cyclicity is related with the possible bifurcations of a limit cycle, whereas the multiplicity is defined for a fixed system.

We introduce the concept of bifurcation of limit cycles from a continuum of periodic orbits. This bifurcation can be related to the stability problem as we will explain in Section 2. To start with, we describe how the definition of successive Melnikov functions allows to control the number and distribution of limit cycles bifurcating from a period annulus. A summary of the main classical results in this context and the corresponding references is given in Section 2. Françoise in [10] and Gasull and Torregrosa in [14] showed the connection between the bifurcation of limit cycles from the Hamiltonian system with $H = x^2 + y^2$ and the determination of the order (or equivalently the multiplicity) of a non-degenerate weak focus. The Melnikov functions corresponding to adequate perturbations of the aforementioned Hamiltonian system give rise to the computation of the first non-vanishing Liapunov constant of the weak focus, that is, its order and its stability can be given.

Our main aim is to give the determination of the multiplicity of a limit cycle through the use of Melnikov functions, that is, to generalize the previous result from a weak focus to a limit cycle. We will only consider the circumference as a limit cycle. We are able to give explicit formulae which determine when the circumference is a limit cycle of multiplicity 1 or 2, cf. Theorems 1.1 and 1.3, respectively. Given a natural number m, with $m \geq 2$, Theorem 1.4 gives explicit systems with the circumference as limit cycle of multiplicity m. The proof of these theorems is given in Section 3. Finally, Section 4 contains several examples to illustrate our results.

2. Definitions and previous results

In this section we first recall the definition of multiplicity for a periodic orbit of a planar analytic differential system. We describe two ways to study the multiplicity of a limit cycle of a differential system. The first way consists in changing the coordinates (x, y) to curvilinear or local coordinates, in which the determination of the Poincaré map can be done in a recursive form. The second way is to show the equivalence between having a limit cycle of multiplicity m

and the existence of an analytic m-solution related to it. The definition of analytic m-solution and the proof of the mentioned equivalence is given in [13].

Given a planar analytic differential system of the form:

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{2.1}$$

defined in an open set $\mathcal{U} \subseteq \mathbb{R}^2$, we say that a periodic orbit $\Gamma \subset \mathcal{U}$ is a *limit cycle* if there exists a neighborhood of Γ without any other periodic orbit, that is, Γ is an isolated periodic orbit for system (2.1). We denote by $(x,y) = \gamma(t)$ the equations corresponding to this closed trajectory and, thus, $\Gamma := \{\gamma(t) \mid 0 \leq t < T\}$, where $T > 0$ is the minimal period of the limit cycle.

We are interested in the behavior of the orbits of system (2.1) in a neighborhood of a periodic orbit Γ. It may happen that Γ belongs to a continuous band of periodic orbits. In such a case, it is no longer a limit cycle because it is not isolated. The classical definitions and results related to periodic orbits can be encountered in the following books [1, 5, 16, 19, 20, 22, 23]. We say that a limit cycle Γ is stable (resp. unstable) when there exists a neighborhood of it such that any orbit with initial condition in this neighborhood has Γ as limit set when evolving the time to $+\infty$ (resp. $-\infty$). A limit cycle is called semi-stable when it is stable in a inner neighborhood and unstable in an outer neighborhood, or the other way round, unstable in a inner neighborhood and stable in an outer one. Any periodic orbit of a planar analytic differential system is either stable, unstable, semi-stable or it belongs to a continuous band of periodic orbits.

In order to determine the stability of a periodic orbit Γ, we consider the *Poincaré map* associated to it. To define the Poincaré map, we consider a point $p_0 \in \Gamma$ and a section Σ through the point p_0, that is, Σ is an arc of a curve containing the point p_0 and such that the considered differential system is not tangent at any point of this arc. Since Γ is a periodic orbit, for each point q of Σ, the solution of system (2.1) starting in q cuts Σ again in another point, denoted by $\Pi(q)$, for some positive time. We notice that since Γ is a periodic orbit and $p_0 \in \Gamma$, we have that $\Pi(p_0) = p_0$. In fact, any fixed point of this map corresponds to a periodic orbit of system (2.1). The function $\Pi : \Sigma \to \Sigma$ defined in this way is called the Poincaré map for Γ.

The qualitative properties of the Poincaré map do not depend on the chosen point p_0 or the chosen section Σ. Moreover, the Poincaré map is a diffeomorphism with the same regularity than system (2.1), so in our context is an analytic diffeomorphism. We parameterize the section Σ by a real parameter $\sigma \in (-\varepsilon, \varepsilon)$, with $\varepsilon > 0$ and $\sigma = 0$ corresponding to the point $p_0 \in \Gamma$.

The Poincaré map gives the definition of the *displacement map* associated to the periodic orbit Γ, $d : \Sigma \to \Sigma$ which is $d(\sigma) = \Pi(\sigma) - \sigma$. The displacement map shows the stability of Γ because, clearly from its definition, when $d(\sigma)$ is increasing for σ near 0 then Γ is unstable; when $d(\sigma)$ is decreasing for σ near 0 then Γ is stable; and if $\sigma = 0$ is an inflection point of $d(\sigma)$ then Γ is semi-stable. If $d(\sigma) \equiv 0$ for σ near 0, then Γ is contained in a continuous band of periodic orbits.

Moreover, the Poincaré map gives rise to the definition of *multiplicity* of a limit cycle as follows:

a) If $\Pi \equiv Id$, then Γ belongs to a continuous band of periodic orbits, and it is not a limit cycle,
b) if $\Pi(\sigma) = c_1\,\sigma + \mathcal{O}(\sigma^2)$ with $c_1 \neq 1$, then Γ is said to be a limit cycle of multiplicity 1 (or a hyperbolic limit cycle),
c) if $\Pi(\sigma) = \sigma + c_m\,\sigma^m + \mathcal{O}(\sigma^{m+1})$ and $c_m \neq 0$, then Γ is said to be a limit cycle of multiplicity m.

Note that $c_1 \neq 0$ because Π is diffeomorphism and $\Pi'(0) = c_1 > 0$ because Π is an increasing map.

The study of the multiplicity of a limit cycle also shows the stability of the periodic orbit Γ, because when c_1 is greater than one, then Γ is unstable, if c_1 is lower than one then Γ is stable. In the case that $c_1 = 1$ we need to determine the parity of m. If m is even then Γ is semi-stable and if m is odd then when c_m is positive Γ is unstable and when c_m is negative then Γ is stable.

2.1. **Curvilinear coordinates method.** We define the local or curvilinear coordinates associated to a periodic orbit, see [16, 22], to study the multiplicity of a periodic orbit of a planar analytic differential system (2.1).

Since Γ is an oval, we can describe it with its arc-length parameter s, $\Gamma := \{(\varphi(s), \psi(s)) \,:\, s \in [0, L]\}$, where $L > 0$ is the total length of the oval. Given a point (x, y) in a neighborhood of Γ, we consider its projection $(\varphi(s), \psi(s))$ over Γ and the value n which is the length of the normal from (x, y) to $(\varphi(s), \psi(s))$.

Hence, any point (x, y) in a neighborhood of Γ can be described by means of these *curvilinear coordinates* (s, n). We notice that the change from cartesian (x, y) to curvilinear coordinates (s, n) is given by:

$$x = \varphi(s) - n\,\psi'(s) \quad y = \psi(s) + n\,\varphi'(s), \tag{2.2}$$

where $\psi'(s)$ and $\varphi'(s)$ denote the derivative of $\psi(s)$ and $\varphi(s)$ with respect to s. Since the pair $(\varphi(s), \psi(s))$ parameterize the oval Γ, we have that $\mid \varphi'(s) \mid + \mid \psi'(s) \mid \neq 0$ and, thus, the jacobian of the change described in (2.2) is different from zero in a neighborhood of $n = 0$.

We apply this change of coordinates to system (2.1). Since Γ is a periodic orbit of system (2.1), ds/dt is different from zero in a neighborhood of $n = 0$ and, therefore, we can write the new system in the form of the ordinary differential equation:

$$\frac{dn}{ds} = F(s, n), \tag{2.3}$$

where we take s as the new independent variable. Moreover, $F(s, 0) \equiv 0$, because in these coordinates the orbit Γ is described by $n = 0$. This change to curvilinear coordinates affects an annular neighborhood of the periodic orbit Γ and this neighborhood can be seen as a cylinder with $n \in (-\varepsilon, \varepsilon)$, for $\varepsilon > 0$ small enough to ensure the good change to curvilinear coordinates, and the L-periodic variable s. Hence, in these coordinates, we study ordinary differential equations defined over a cylindrical manifold. In this way, in a neighborhood of a periodic orbit, a system in the plane can always be considered as an ordinary differential equation over a cylinder. We remark that there are differential equations over a cylinder that cannot be transformed to a planar polynomial differential system.

We develop the function $F(s, n)$ in a series of n:

$$F(s, n) = \sum_{j \geq 1} F_j(s)\, n^j,$$

where the functions $F_j(s)$ are L-periodic in s because $\varphi(s)$ and $\psi(s)$ are L-periodic.

Let us consider the flow $\Psi(s; n_0)$ of equation (2.3), that is, $\Psi(s; n_0)$ is the function satisfying:

$$\frac{\partial \Psi(s; n_0)}{\partial s} = F\left(s, \Psi(s; n_0)\right) \quad \text{and} \quad \Psi(0; n_0) = n_0. \tag{2.4}$$

We develop the flow $\Psi(s; n_0)$ in terms of n_0 and we get:

$$\Psi(s; n_0) \; = \; \sum_{j \geq 1} \Psi_j(s)\, n_0^j. \tag{2.5}$$

We note that since $n = 0$ is a solution to (2.3), we have that $\Psi(s; 0) \equiv 0$.

We also note that, in these coordinates, the Poincaré map associated to $n = 0$ is given by $\Pi(n_0) \; = \; \Psi(L; n_0)$, or equivalently, $\Pi(n_0) \; = \; \sum_{j \geq 1} \Psi_j(L)\, n_0^j$.

We impose that $\Psi(s; n_0)$ satisfies the initial value problem (2.4) and we develop in terms of n_0. In this way we get a series of recurrent linear differential equations for the functions $\Psi_j(s)$, $j \geq 1$. For instance, equating the coefficient of n_0 in the development of (2.4), we deduce that:

$$\Psi_1'(s) \; = \; F_1(s) \Psi_1(s)$$

and $\Psi_1(0) = 1$. Hence, we have that:

$$\Psi_1(L) = \exp\left(\int_0^L F_1(s)\, ds \right).$$

In the same way, equating the coefficient of n_0^2 in the development of (2.4), we get that:

$$\Psi_2'(s) \; = \; F_1(s) \Psi_2(s) \; + \; F_2(s) \Psi_1^2(s),$$

and $\Psi_2(0) = 0$. Therefore, we have that

$$\Psi_2(s) \; = \; \Psi_1(s) \left(\int_0^s \Psi_1(\sigma)\, F_2(\sigma)\, d\sigma \right).$$

In a recursive way, we can give formulas for any $\Psi_j(s)$ provided that $\Psi_i(s)$ with $i = 1, 2, \ldots, j-1$ are known. In this recursive way we can compute the values of $\Psi_j(L)$ and we can determine the multiplicity of the periodic orbit Γ.

Since $\Pi(n_0) \; = \; \Psi(L; n_0)$, we deduce that if $\Psi_1(L) \neq 1$, we have that Γ is of multiplicity 1 and if $\Psi_1(L) = 1$, $\Psi_j(L) = 0$ for $j = 2, 3, \ldots, m-1$ and $\Psi_m(L) \neq 0$, then Γ is of multiplicity m. In case that $\Psi_1(L) = 1$ and $\Psi_j(L) = 0$ for all $j \geq 2$, we have that Γ belongs to a continuous band of periodic orbits.

This way of studying the multiplicity of a limit cycle Γ is computationally very difficult because, we first need to know these curvilinear coordinates. Secondly we need to change to these new coordinates and moreover, the recurrent differential equations giving the expressions of $\Psi_j(L)$ are more complicated as j increases. We also remark that the computation of $\Psi_j(L)$ involves j iterated integrals.

Remark 2.1. This study of the multiplicity can be particularized for a monodromic critical point, that is a singular point of center or focus type, provided that this point has no real characteristic directions. Then, by blowing-up, we can think of it as a periodic orbit. For further information about monodromic points, characteristic directions and blow-ups, see the book [23]. We can also use curvilinear coordinates and the multiplicity corresponds to the order of the point as a focus of the system.

Lloyd in [17] studies some properties of the flow of the ordinary differential equation (2.3) and proves the following formulae. We use the notation we have defined to state his result. Lloyd considers system (2.3) and proves:

$$\Pi'(n_0) = \exp\left(\int_0^L \frac{\partial F}{\partial n}\,(s, \Psi(s; n_0))\; ds\right).$$

Let us define the following two functions:

$$\begin{aligned} E(\phi, n_0) &= \exp\left(\int_0^\phi \frac{\partial F}{\partial n}\,(s, \Psi(s; n_0))\; ds\right) \\ D(\phi, n_0) &= E(\phi, n_0)\,\frac{\partial^2 F}{\partial n^2}\,(\phi, \Psi(\phi; n_0)). \end{aligned}$$

Lloyd showed that:

$$\begin{aligned} \Pi''(n_0) &= E(L, n_0) \int_0^L D(\phi, n_0)\, d\phi, \\ \Pi'''(n_0) &= E(L, n_0) \left[\frac{3}{2}\left(\int_0^L D(\phi, n_0)\, d\phi\right)^2 + \right. \\ &\qquad \left. + \int_0^L (E(\phi, n_0))^2\, \frac{\partial^3 F}{\partial n^3}\,(\phi, \Psi(\phi; n_0))\; d\phi\right]. \end{aligned}$$

These formulae coincide with the values $\Psi_1(L), \Psi_2(L)$ and $\Psi_3(L)$ in the case $n_0 = 0$.

2.2. **Analytic m-solution method.** An alternative way to study the multiplicity of a limit cycle is given in [13] where the concept of analytic m-solution is introduced. This notion is related with the concept of multiplicity of an invariant curve and it arises as a generalization of the result given in [15] where only the case of multiplicity $m = 1$ is treated. We state the corresponding definitions, which always concern a planar differential system (2.1) with a periodic orbit Γ contained in an open set $\mathcal{U} \subseteq \mathbb{R}^2$. All the considered functions are assumed to be analytic in $\mathcal{U}$.

Definition 2.2. We say that a curve $f(x, y) = 0$, where f is analytic function defined in an open set $\mathcal{U}$, is an *invariant curve* of system (2.1) if the following equality occurs:

$$P(x,y)\frac{\partial f}{\partial x} + Q(x,y)\frac{\partial f}{\partial y} = k_0(x,y)f(x,y), \tag{2.6}$$

where $k_0(x, y)$ is an analytic function in $\mathcal{U}$ called the *cofactor* of $f(x, y)$.

The following result shows that there always exists an invariant curve which implicitly describes the considered periodic orbit Γ. This result is proved in [1, 22], see also [13].

Lemma 2.3. [1, 13] *If system* (2.1) *has a limit cycle* Γ*, then there exists an analytic invariant curve* $f(x, y) = 0$ *for system* (2.1) *defined in a neighborhood* $\mathcal{U}$ *of* Γ *and such that* $\Gamma \subseteq \{(x, y) \in \mathcal{U} : f(x, y) = 0\}$*. Moreover, the curve* $f(x, y) = 0$ *can always be chosen such that the vector* $\nabla f(x, y)$ *is different from zero in all the points of* Γ*.*

Once the notion of invariant curve has been introduced, we give the definition of generalized exponential factor.

Definition 2.4. [13] A function $F_d = \exp\left(g_d/f^d\right)$ with $d \in \mathbb{N}$, $d \geq 1$ is called a *generalized exponential factor* of order d associated to the invariant curve $f(x, y) = 0$ for system (2.1) if it satisfies

$$P(x,y)\,\frac{\partial F_d}{\partial x} + Q(x,y)\,\frac{\partial F_d}{\partial y} = k_d(x,y)\,F_d(x,y),$$

where $k_d(x, y)$ is an analytic function in $\mathcal{U}$ called the cofactor of $F_d(x, y)$ and $g_d(x, y)$ is an analytic function in $\mathcal{U}$ with $g_d(p) \neq 0$, $\forall p \in \Gamma$.

We can determine the multiplicity m of the considered periodic orbit $\Gamma = \{\gamma(t) : 0 \leq t < T\}$, contained in the invariant curve $f(x, y) = 0$ and whose existence is ensured by Lemma 2.3, using the notion of *analytic m-solution.* Given a positive integer m, we say that $f(x, y) = 0$ is an analytic m-solution of system (2.1) if there exist $m-1$ generalized exponential factors of consecutive orders $d = 1, 2, 3, ..., m-1$, whose cofactors satisfy that:

$$\int_0^T k_j(\gamma(t))\, dt = 0 \text{ for } j = 0, 1, 2, ..., m-2 \quad \text{and} \quad \int_0^T k_{m-1}(\gamma(t))\, dt \neq 0.$$

The following theorem is proved in [13].

Theorem 2.5. [13] *If Γ is a limit cycle of system* (2.1) *and $f(x, y) = 0$ is an analytic invariant curve defined in a neighborhood $\mathcal{U}$ of Γ and such that $\Gamma \subseteq \{(x, y) \in \mathcal{U} : f(x, y) = 0\}$ with $\nabla f(x, y)$ different from zero in all the points of Γ, then Γ has multiplicity m if, and only if, $f(x, y) = 0$ is an analytic m-solution of* (2.1).

The particular case of multiplicity $m = 1$ is already treated in the following theorem, proved in [15].

Theorem 2.6. [15] *Let us consider a system* (1.1) *and $\Gamma := \{\gamma(t) : 0 < t < T\}$ a periodic orbit of minimal period $T > 0$. Assume that $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is an invariant curve with $\gamma \subseteq \{(x, y) : f(x, y) = 0\}$ and let $k_0(x, y)$ be the $\mathcal{C}^1$ function given in* (2.6). *We assume that if $p \in \mathcal{U}$ is such that $f(p) = 0$ and $\nabla f(p) = 0$, then p is a singular point of system* (1.1). *Then,*

$$\int_0^T k_0(\gamma(t))dt \; = \; \int_0^T \mathrm{div}(\gamma(t))dt,$$

where $\mathrm{div}(x, y)$ *is the divergence of system* (1.1).

2.3. **Relationship with the cyclicity.** We have seen the definition of multiplicity of a periodic orbit, but we are also interested in another concept related with the bifurcations of a limit cycle of multiplicity m. This concept is the cyclicity whose definition is the following.

We consider system (2.1) and any unfolding of it of the form:

$$\dot{x} \; = \; P_\lambda(x, y), \quad \dot{y} \; = \; Q_\lambda(x, y), \tag{2.7}$$

where $P_\lambda(x, y)$ and $Q_\lambda(x, y)$ are analytic functions in the same neighborhood $(x, y) \in \mathcal{U}$ and analytic in the set of parameters $\lambda \in \mathbb{R}^k$. Moreover, when $\lambda = \mathbf{0}$, $P_{\mathbf{0}}(x, y)$ and $Q_{\mathbf{0}}(x, y)$ coincide with the functions P and Q defining the unperturbed system (2.1).

We recall, see for instance [1], that a fixed system (2.7) and system (2.1) are said to be *δ-close* if $\|\lambda\| < \delta$. We say that two ovals Γ_1, Γ_2 are ε-close if $d(\Gamma_1, \Gamma_2) < \varepsilon$ where $d(\cdot,\cdot)$ denotes the Hausdorff distance between compact sets.

Definition 2.7. We say that the *cyclicity* of a periodic orbit Γ of system (2.1) is m if the following two conditions hold.

(i) There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for any system of the form (2.7) which is δ-close to system (2.1), with $\delta < \delta_0$, has at most m limit cycles which are ε-close to Γ, with $\varepsilon < \varepsilon_0$.

(ii) Given $\varepsilon > 0$, $\delta > 0$, with $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$, there exists a system of the form (2.7) which is δ-close to system (2.1) with exactly m limit cycles ε-close to Γ.

As we have defined the notion of cyclicity, when we perturb system (2.1), we are considering any analytic unfolding of it, that is any system of the form (2.7) with any $\lambda \in \mathbb{R}^k$ and any $k \in \mathbb{N}$. Thus, as it is stated and proved in [1], a limit cycle Γ of multiplicity m has cyclicity exactly m. The proof of this fact uses the Weierstrass Preparation Theorem, see for instance [5]. In the particular case that $m = 1$, we say that Γ is a limit cycle of multiplicity 1, and we have that any perturbation of the system of the form (2.7) has one, and only one, limit cycle bifurcating from Γ. This property gives rise to the denomination of hyperbolic for any limit cycle Γ of multiplicity 1.

However, bifurcation problems do not usually consider any analytic unfolding but the restriction to a particular family. This particular family usually has a finite number of parameters and, more concretely, is usually a polynomial family. That is, the cyclicity is usually studied inside a certain fixed family of systems of the form (2.7). In this sense, the multiplicity establishes an upper bound for the cyclicity inside the restricted family of study, which does not need to be attained.

One of our interests is to study the maximum multiplicity of the circumference as limit cycle inside of the family of planar polynomial differential systems of degree d. This maximum multiplicity is what we denote the cyclicity of the circumference inside the aforementioned family. We know that this number is finite due to the following result of Françoise and Pugh [9].

Theorem 2.8. [9] *A periodic orbit has finite cyclicity inside any analytic family of vector fields depending on a finite number of parameters.*

2.4. **Melnikov method.** In this section we describe a method to tackle a completely different problem to the one of determining the multiplicity of a periodic orbit, which is our concern. The method of Melnikov functions serves to determine the periodic orbits which persist from the perturbation of a period annulus. In the following section we describe how this method can be used to solve a related problem to ours: to give the multiplicity of a weak focus. This use of Melnikov method was introduced in [10, 14]. Since a weak focus can be seen as a limit cycle by blowing up, we base on the previous result to give the way of studying the multiplicity of a limit cycle by means of the Melnikov method.

Let us describe the context in which the Melnikov method is introduced. Let H be a real polynomial and let us consider the associated Hamiltonian system

$$\dot{x} = -H_y, \ \ \dot{y} = H_x. \tag{2.8}$$

We consider a perturbation of the previous system of the form:

$$\dot{x} = -H_y + \varepsilon\, P_1, \ \ \dot{y} = H_x + \varepsilon\, Q_1, \tag{2.9}$$

where P_1, Q_1 are real polynomials in (x, y). We assume that the unperturbed system (2.8) has a continuum of periodic orbits and we aim to study which ones of them persist in system (2.9). We parameterize the period annulus of system (2.8) by $h \in \mathcal{I} \subset \mathbb{R}$, where $\mathcal{I}$ is an open real interval and $H = h$ denotes one of the periodic orbits. The parameter h only has sense in the open interval $\mathcal{I}$ which is usually bounded.

We write system (2.9) as a 1-form so as to make the calculations easier:

$$H_x dx + H_y dy + \varepsilon(Q_1 dx - P_1 dy) = 0.$$

We denote by $\omega = Q_1 dx - P_1 dy$ and system (2.9) takes the form:

$$dH + \varepsilon\omega = 0.$$

Let $\Pi(h; \varepsilon)$ be the Poincaré map associated to system (2.9) over a transversal section Σ parameterized by h. Let $\gamma_\varepsilon(h)$ be the arc of orbit of system (2.9) with initial point $h \in \Sigma$ and ending point $\Pi(h; \varepsilon)$. In the particular case that for a value $h_0 \in \Sigma$ the corresponding $\gamma_\varepsilon(h)$ is a periodic orbit, we have that the initial and ending points of $\gamma_\varepsilon(h)$ are the same. That is, we have a function

$\bar{h}(\varepsilon)$ such that $\bar{h}(0) = h_0$ and $\Pi(\bar{h}(\varepsilon);\varepsilon) = \bar{h}(\varepsilon)$. In such a case, we say that the periodic orbit $H = h_0$ *persists*.

We can develop $\Pi(h;\varepsilon)$ in a neighborhood of $\varepsilon = 0$. We recall that when $\varepsilon = 0$ we have that $\Pi(h;0) \equiv h$ because we have a period annulus. Therefore,

$$\Pi(h;\varepsilon) = h + \Pi_1(h)\,\varepsilon + \mathcal{O}(\varepsilon^2),$$

The function $\Pi_1(h)$ is called the *first Melnikov function* and it can be shown, see for instance [10], that it can be computed through the following Abelian integral:

$$\Pi_1(h) = \int_{H=h} \omega. \tag{2.10}$$

Proposition 2.9. [9, 10] *Suppose that the orbit $H = h_0$ persists, then $\Pi_1(h_0) = 0$.*

The proof of this proposition can be found in [10].

We observe that at first order in ε, the maximum number of isolated zeros of the function $\Pi_1(h)$ is an upper bound for the number of the limit cycles that bifurcate from the considered period annulus.

This observation leads to the statement of the 16^{th} weak Hilbert problem. As stated in the paper of Arnold [2], this problem reads for:

> Given $H(x,y)$ a real polynomial of degree n and any continuous family of closed connected components of its level curves $H = h$ and given ω any polynomial 1-form of degree d; to determine the maximum number of isolated zeros of the function $\Pi_1(h)$ in terms of n and d.

The existence of an upper bound for the number of isolated zeros of $\Pi_1(h)$ in terms of n and d is proved by Varchenko [21]. This proof is an existential one and not a quantitative one.

In the previous Proposition 2.9 a necessary condition for an oval $H = h_0$ to persist after the perturbation is given. A sufficient condition is the next one:

Proposition 2.10. [9, 10] *If h_0 is such that $\Pi_1(h_0) = 0$ and $\Pi_1'(h_0) \neq 0$ (i.e., h_0 is a simple zero of the function $\Pi_1(h)$), then there exists a unique periodic orbit of system (2.7) which bifurcates from $H = h_0$.*

The proof of this result can be found in [11] and the references therein.

If instead of having a simple isolated zero of $\Pi_1(h)$, we have a multiple zero (although isolated), its multiplicity gives an upper bound for the number of periodic orbits which bifurcate from it as the following proposition shows.

Proposition 2.11. [9, 10] *Let h_0 such that*

$$\Pi_1(h_0) = 0, \Pi_1'(h_0) = 0, \ldots, \Pi_1^{m-1}(h_0) = 0, \;\; \Pi_1^m(h_0) \neq 0,$$

then there exist at most m periodic orbits (*not necessarily real ones*) *of system* (2.7) *which bifurcate from $H = h_0$.*

The proof of this proposition is analogous to the one of Proposition 2.10 but using Weierstrass Preparation Theorem instead of Implicit Function Theorem, see for instance [5].

Since $\Pi_1(h)$ is an analytic function of h, if it has a non-isolated zero, then $\Pi_1(h) \equiv 0$. In such a case we denote by $\omega_1 = Q_1 dx - P_1 dy$ and we have that the Pfaffian form $dH + \varepsilon \omega_1 = 0$ gives rise to a uniparametric family of the periodic orbits at first order in ε. For the sake of completeness, we consider an analytic perturbation of $dH = 0$:

$$dH \, + \sum_{i \geq 1} \varepsilon^i \, \omega_i \; = \; 0, \tag{2.11}$$

where ω_i are polynomial 1-forms. In case $\Pi_1(h) \equiv 0$, we need to compute the second order term in ε for $\Pi(h; \varepsilon) \, = \, h \, + \, \Pi_2(h) \, \varepsilon^2 \, + \, \mathcal{O}(\varepsilon^3)$.

We remark that the hypothesis $\int_{H=h} \omega_1 \equiv 0$ implies a certain expression for ω_1 in terms of H. For instance, any 1-form ω_1 is of the form $\omega_1 \, = \, g \, dH \, + \, dS$ with g and S real polynomials then, we always have that $\int_{H=h} \omega_1 \equiv 0$. This assertion is true because the first term in the integrand is zero since $H = h$ is constant and, thus $dH_{|H=h} \equiv 0$. Moreover, the second term is an integral of a differential of a polynomial over a closed curve and, thus, its value is zero.

Françoise in [10] gives the following definition:

Definition 2.12. [10] A real polynomial $H(x, y)$ satisfies the $\star$-*condition* if the next implication holds: given any polynomial 1-form ω such that $\int_{H=h} \omega \equiv 0$, then $\omega \, = \, g \, dH + dS$, where g, S are polynomials.

In fact, generic polynomials H satisfy the $\star$-condition as the following Theorem 2.13 shows. We remark that if H satisfies the $\star$-condition, we have completely restricted the form of any ω with $\int_{H=h} \omega \equiv 0$. Moreover, in general, if one knows the degree of ω, the degrees of the polynomials g and S are bounded in terms of the degrees of ω and H.

We recall that polynomials of Morse type are generic and that a polynomial is of Morse type if all its singular points (finite or infinite) have different tangents, i.e. the Hessian matrix in any critical point is a non degenerated bilinear form. The following result is proved in [18]:

Theorem 2.13. [18] *All the Morse polynomials satisfy the $\star$-condition.*

For example, the polynomial $H = (x^2 + y^2 - 1)/2$ is of Morse type, so it satisfies the $\star$-condition.

If H is a real polynomial that satisfies the $\star$-condition then, a constructive way to find the polynomials g and S is the following one. Suppose that $\int_{H=h} \omega \equiv 0$ and we know that there exist polynomials g and S such that $\omega = g dH + dS$. We calculate the differential of ω which needs to satisfy:

$$d\omega = dg \wedge dH + g \wedge d^{(2)} H + d^{(2)} S,$$

where $\wedge$ is the alternate product and we recall that the operator $d^{(2)} \equiv 0$. Thus, $d\omega = dg \wedge dH$, which defines a partial differential equation for g. This partial differential equation is always solvable by means of the characteristics' method. Once we know the polynomial g, we compute the polynomial S using that $\omega - g\, dH$ needs to be an exact 1-form.

Following the reasonings given in [11], we call the *second Melnikov function* associated to (2.11) to the following one:

$$\Pi_2(h) = \int_{H=h} (\omega_2 - g_1 \omega_1).$$

In the case that $\int_{H=h} (\omega_2 - g_1 \omega_1) \equiv 0$, we have the following equality $\omega_2 - g_1 \omega_1 = g_2 dH + dS_2$, where g_2 and S_2 are polynomials. By an analogous reasoning, we have that, in this case, $\Pi(h; \varepsilon) = h + \varepsilon^3 \, \Pi_3(h) + \mathcal{O}(\varepsilon^4)$, with

$$\Pi_3(h) = \int_{H=h} (\omega_3 - g_1 \omega_2 - g_2 \omega_1),$$

which is defined to be the *third Melnikov function.*

A recursive analysis, which can be found [11], allows to find an explicit formula for $\Pi_n(h)$, called the nth Melnikov function, provided that the $n-1$ previous Melnikov functions are all identically null.

2.5. **Order of a weak focus through Melnikov functions.** In this section we consider a system (2.1) with a non-degenerated singular point of center or focus type, which we assume to be at the origin. Such a system takes the form:

$$\dot{x} = -y + p(x,y), \ \dot{y} = x + q(x,y), \tag{2.12}$$

where p and q are analytic functions in a neighborhood of the origin without constant nor linear terms. Taking a transversal section through the origin, we can define the Poincaré map $\Pi(\sigma)$ associated to this singular point (which corresponds to $\sigma = 0$) analogously as before. It can be shown that, in case the origin is not a center, this Poincar return map always takes the form $\Pi(\sigma) = \sigma + V_{2k-1}\sigma^{2k-1} + \mathcal{O}(\sigma^{2k})$ with $V_{2k-1} \neq 0$, $k \geq 1$. In such a case, we say that the origin of system (2.12) is a weak focus *of order* k. The value V_{2k-1} is called a Liapunov constant. The origin of system (2.12) is a center if, and only if, $\Pi(\sigma) \equiv \sigma$.

Françoise, in [10], is the first author who determines the order of a weak focus using Melnikov functions. He considers the following system:

$$\dot{x} = -y + \varepsilon P_n, \, \dot{y} = x + \varepsilon Q_n,$$

where P_n and Q_n are homogeneous polynomials of degree n, which can be seen as a perturbation of the Hamiltonian system with $H = (x^2 + y^2)/2$.

We remark that, in this section, n denotes a natural number. We do this remark to avoid possible confusion with the n which we have used in the section of the curvilinear coordinates.

Gasull and Torregrosa in [14], see also the references therein, study any system of the form (2.12) with the aim to determine the order of the weak focus using the Melnikov functions. In order to correctly define the perturbations which lead to this method, we write system (2.12) in the form

$$\dot{x} = -y + P_1 + P_2 + P_3 + \ldots, \, \dot{y} = x + Q_1 + Q_2 + Q_3 + \ldots, \tag{2.13}$$

where P_i and Q_i are homogeneous polynomials of degree $i+1$. They prove the following result:

Theorem 2.14. [14] *We write system* (2.13) *as the* 1*-form,*

$$dH + \omega_1 + \omega_2 + \cdots = 0,$$

where ω_i *is the* 1*-form* $\omega_i = -Q_i dx + P_i dy$, *with* P_i *and* Q_i *homogeneous polynomials of degree* $i+1$. *If* $V_2 = V_3 = \cdots = V_{n-1} = 0$, *then*

$$V_n = \frac{1}{2^{\frac{n+1}{2}}} \frac{1}{\rho^{\frac{n+1}{2}}} \int_{H=\rho} \sum_{i=1}^{n-1} \omega_i h_{n+1-i},$$

where $h_0 \equiv 1$, *and* h_m *is such that*

$$d\left(\sum_{l=1}^{m} \omega_l h_{m-l}\right) = -d(h_m dH).$$

We remark that if n *is even then* $V_n = 0$.

The main idea to prove this theorem is to consider system (2.13) as the following perturbation of the Hamiltonian system with $H = (x^2+y^2)/2$.

$$\dot{x} = -y + \varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + \ldots,\ \dot{y} = x + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \ldots, \quad (2.14)$$

The relationship between system (2.13) and the perturbation (2.14) is the change of coordinates $(x, y) \mapsto (\varepsilon x, \varepsilon y)$, which ensures that the Poincaré map associated to the origin of system (2.13) coincides with the one defined for (2.14).

There are two basic ideas which allow to make the relationship between the two following problems: the study of the order of the weak focus at the origin of system (2.13) and the number of limit cycles which persist under the perturbation described in (2.14). The first idea is that the order in ε of the perturbation in (2.14) must reflect the multiplicity of vanishing at the origin of the corresponding terms. Since we relate system (2.13) with (2.14) by the change of coordinates $(x, y) \mapsto (\varepsilon x, \varepsilon y)$, we have that the perturbative terms (P_i, Q_i) must go with ε^i. On the other side, Melnikov functions, in this case, are always a monomial in ρ (the level curve $H = \rho$ is parameterized with $\rho > 0$ in the notation used in [14]) multiplied by the corresponding Liapunov constant, as we have seen in Theorem 2.14.

2.6. **The inverse problem for the circumference.** Given a planar differential system of the form (2.1), a direct problem is to determine the invariant algebraic curves that it possesses. This problem is related with the integrability problem. As an inverse problem, it has also been studied which planar polynomial differential systems possess a certain fixed invariant algebraic curve, see [7], [8] and the references therein.

We are interested in the problem of studying the multiplicity of the circumference as limit cycle of a planar polynomial differential system. Thus, we aim to know the systems which possess the circumference Γ as invariant algebraic curve. Throughout this section, we write $f = (x^2 + y^2 - 1)/2$ and $\Gamma := \{f = 0\}$. This characterization is a corollary of the following result due to Christopher, Llibre, Pantazi and Zhang in [7]. This result has also been encountered by [3].

Theorem 2.15. [3, 7] *Let $f = 0$ be a nonsingular algebraic curve, then all polynomial systems with $f = 0$ as an invariant algebraic curve, are of the form:*

$$\dot{x} = -f_y c(x, y) + f\, a(x, y), \qquad \dot{y} = f_x c(x, y) + f\, b(x, y),$$

where $a(x, y)$, $b(x, y)$, $c(x, y)$ are polynomials.

Moreover, if the system is of degree d and the curve $f = 0$ is of degree n and we assume that the curve $f = 0$ does not contain any singular point at infinity, we have that $deg(c) \leq d - n + 1$, $deg(a, b) \leq d - n$, where $deg(\cdot)$ denotes the degree.

In the particular case where we look for the systems with the circumference as invariant algebraic curve, we have the following result.

Theorem 2.16. *All the systems with the circumference as invariant algebraic curve are of the form:*

$$\dot{x} \; = \; -y\, c(x, y) + f(x, y)\, a(x, y), \quad \dot{y} \; = \; x\, c(x, y) + f(x, y)\, b(x, y),$$

with $f(x, y) = (x^2 + y^2 - 1)/2$ and $a(x, y)$, $b(x, y)$, $c(x, y)$ are polynomials.

We remark that if the system has degree d, then $\deg(c) \leq d - 1$ and $\deg(a, b) \leq d - 2$, which is proved in [7].

For the sake of completeness, we are going to prove this theorem, using different techniques to the ones described in [3, 7].

Lemma 2.17. *We consider a polynomial system* (2.1) *with the circumference $f = 0$ as periodic orbit. We write $\Gamma := \{\gamma(t) \, : \, 0 \leq t < T\}$, where T is the minimal positive period. Then, there exists a function $\tau(t)$ with $\tau(0) = 0, \tau(T) = 2\,\pi$ and such that $\gamma(t) = (\cos \tau(t), \sin \tau(t))$.*

Proof. We consider the parametrization of the periodic orbit Γ in terms of the time t of the system which is $\gamma(t) = (x_0(t), y_0(t))$ and we choose t such that $x_0(0) = 1$, $y_0(0) = 0$. We consider the function $G(t,\tau) = y_0(t) - \sin\tau$, and we apply the Implicit Function Theorem at the point $(t,\tau) = (0,0)$:

$$G(0,0) = 0 \quad \text{and} \quad \frac{\partial G(t,\tau)}{\partial \tau} = -\cos\tau\Big|_{t=0,\tau=0} = -1 \neq 0.$$

Therefore, there exists $\tau(t)$ such that $\tau(0) = 0$ and $y_0(t) = \sin\tau(t)$. Since we know that $x_0^2(t) + y_0^2(t) = 1$, we deduce that $x_0^2(t) = 1 - y_0^2(t) = 1 - \sin^2\tau(t) = \cos^2(\tau(t))$. The hypothesis $x_0(0) = 1$, $y_0(0) = 0$ ensures that $x_0(t) = \cos(\tau(t))$.

Since the curve $\Gamma := \{\gamma(t) : 0 \leq t < T\}$ is a compact set, there is no problem in extending $\tau(t)$ over all the interval $[0,T]$ and $\tau(T) = 2\pi$ because the minimal positive period of the functions $\cos\tau$ and $\sin\tau$ is 2π. □

We are going to prove Theorem 2.16 using Bézout Theorem, see for instance [12]:

Theorem 2.18. [12] *Let $f_1 = 0$ and $f_2 = 0$ be two algebraic curves with f_1 and f_2 square-free polynomials. We assume that the two curves share an infinity number of intersection points, then either f_1 divides f_2 or f_2 divides f_1.*

Proof of Theorem 2.16. We consider system (2.1) and we define the function $c = -yP + xQ$ which is a polynomial. Since the circumference $\Gamma := \{\gamma(t) : 0 \leq t < T\}$ is a periodic orbit of the system, we have that $\dot\gamma(t) = (P(\gamma(t)), Q(\gamma(t)))$. Using Lemma 2.17, we deduce that $P(\gamma(t)) = -\sin\tau(t)\,\dot\tau(t)$ and $Q(\gamma(t)) = \cos\tau(t)\,\dot\tau(t)$. Thus,

$$c(\gamma(t)) = \sin^2\tau(t)\,\dot\tau(t) + \cos^2\tau(t)\,\dot\tau(t) = \dot\tau(t).$$

We consider the polynomial $A = P + y\,c$ and we evaluate it in $\gamma(t)$:

$$A(\gamma(t)) = -\sin\tau(t)\,\dot\tau(t) + \sin\tau(t)\,\dot\tau(t) \equiv 0.$$

Using Theorem 2.18 and the fact that f is irreducible, we have that the polynomial f divides A. Thus, there exists a polynomial a such that $A = af$. Using that $A = P + y\,c$, we have $P + y\,c = af$ and, so, $P = -y\,c + af$. Reasoning as before, we consider the function $B = Q - x\,c$ and we obtain the result. □

3. Proofs of the main results

In this section, we are going to prove the main results that we have stated in the first section. The first Theorem 1.1 characterizes when Γ is a hyperbolic limit cycle of system (1.2). We recall the statement of this theorem.

Theorem 3.1. *Γ is a limit cycle of multiplicity* 1 *of system* (1.2) *if, and only if,* $\displaystyle\oint_{f=0} \frac{\omega_0}{c} \neq 0$.

Proof. As proved in Theorem 2.16, we consider the system (1.2) with the circumference as a periodic orbit:

$$\dot{x} = -y\, c(x,y) + f(x,y)\, a(x,y), \quad \dot{y} = x\, c(x,y) + f(x,y)\, b(x,y)$$

where $f = (x^2 + y^2 - 1)/2$ and a, b, c are polynomials. We write the aforementioned system as a 1-form:

$$\omega = c\, df + f\omega_0, \quad \text{where} \quad \omega_0 = b\, dx - a\, dy.$$

Since $f = 0$ is a periodic orbit, we have that the polynomial c does not vanish in a neighborhood of the circumference. Therefore, in such a neighborhood we can write $\omega \sim df + f\frac{\omega_0}{c}$.

Theorem 2.5 ensures that Γ is a limit cycle of multiplicity 1 if, and only if, $\int_0^T k_0(\gamma(t))dt \neq 0$, where k_0 is the cofactor of f. We observe that this cofactor is $k_0 = ax + by$ as the following computations show: $\dot{x}f_x + \dot{y}f_y = (-yc + fa)x + (xc + fb)y = fax + fby = f(ax + by)$.

These computations can also be performed using the equivalent Pfaffian form: $\omega \wedge df = (cdf + f\omega_0) \wedge df = f(\omega_0 \wedge df) = f(by + ax)dx \wedge dy$. This last equality comes from $\omega_0 \wedge df = (bdx - ady) \wedge (xdx + ydy) = bydx \wedge dy - axdy \wedge dx = (by + ax)dx \wedge dy$. We deduce that $\omega \wedge df = f(by + ax)dx \wedge dy$, from which we conclude that $k_0 = (ax + by)$.

We parameterize using polar coordinates, or equivalently the parameterization described in Lemma 2.17.

$$\int_0^T k_0(\gamma(t))dt \; = \; \int_0^{2\pi} \left. \frac{ax + by}{c(x,y)} \right|_{x=\cos\tau,\, y=\sin\tau} d\tau \; = \; \oint_{f=0} \frac{\omega_0}{c},$$

where we have used that $c(\gamma(t)) \; = \; \dot{\tau}(t)$. □

We consider the particular case in which c is a constant different from zero and by scaling we take $c \equiv 1$. The second main Theorem 1.3 characterizes when Γ is a limit cycle of multiplicity 2, provided that $c \equiv 1$. The statement of this theorem is the following.

Theorem 3.2. *Consider the particular case $c(x,y) \equiv 1$. Then, Γ is a limit cycle of multiplicity 2 of system (1.2) if, and only if, $\omega_0 = g_0 df + dS_0 + f\omega_1$ and $\oint_{f=0} e^{-S_0}(\omega_1 - g_0 dS_0) \neq 0$.*

Proof. This proof is based on the result given by Proposition 1.2. We have that there exist polynomials g_0, S_0 and a polynomial ψ such that $\omega_0 = g_0\, df + dS_0 + \psi(f)\,(y\,dx - x\,dy)$. Since $c \equiv 1$, the previous theorem ensures that Γ has multiplicity ≥ 1 if, and only if, $\oint_{f=0} \omega_0 = 0$. Therefore, we deduce that:

$$0 = \oint_{f=0} g_0 df + dS_0 + \psi(f)(y\,dx - x\,dy) = \psi(0) \oint_{f=0} y\,dx - x\,dy = \psi(0)\, 2\,\pi.$$

This identity implies that $\psi(0) = 0$. Therefore, $\psi(f) = f\,\phi(f)$ for a certain polynomial ϕ. We conclude that $\omega_0 = g_0\,df + dS_0 + f\,\omega_1$, where ω_1 is the following polynomial 1-form $\omega_1 = \phi(f)\,(ydx - xdy)$.

Theorem 2.5 gives that Γ has multiplicity 2 if, and only if, there exists a generalized exponential factor of order 1 whose cofactor k_1 satisfies that $\int_0^T k_1(\gamma(t))\,dt \neq 0$. We are going to show that the function $F_1 = \exp\{-e^{-S_0}/f\}$ is a generalized exponential factor of order 1 with cofactor $k_1 = *\left(e^{-S_0}\left((\omega_1 - g_0 dS_0) \wedge df + f\omega_1 \wedge dS_0\right)\right)$ where $*$ is the Hodge operator. We recall that the Hodge operator gives the equivalence between 2-forms and functions over the real plane as follows. If $\xi(x,y)(dx \wedge dy)$ is a 2-form, then $*\,(\xi(x,y)(dx \wedge dy)) = \xi(x,y)$ which is the equivalent function. On the other way round, given a function $\xi(x,y)$ then $*\,(\xi(x,y)) = \xi(x,y)(dx \wedge dy)$ which is the corresponding 2-form.

We consider $F_1 = \exp\{-e^{-S_0}/f\}$ and we have that the Pfaffian form ω reads for $\omega = df + f\,(g_0\,df + dS_0) + f^2\,\omega_1$.

Then we have:

$$\begin{aligned}
\omega \wedge dF_1 &= F_1 \left[\left(df + f\,(g_0\,df + dS_0) + f^2\,\omega_1\right) \wedge e^{-S_0} \left(\frac{f\,dS_0 + df}{f^2} \right) \right] \\
&= \frac{F_1\,e^{-S_0}}{f^2} \left[f df \wedge dS_0 + f^2\,g_0\,df \wedge dS_0 + f\,dS_0 \wedge df + \right. \\
&\qquad \left. + f^3 \omega_1 \wedge dS_0 + f^2\,\omega_1 \wedge df \right] \\
&= F_1\,e^{-S_0} \left[(\omega_1 - g_0\,dS_0) \wedge df + f\,\omega_1 \wedge dS_0 \right].
\end{aligned}$$

Using the same parameterization as in the proof of the previous theorem, we see that

$$\int_0^T k_1(\gamma(t))\, dt \;=\; \oint_{f=0} e^{-S_0}(\omega_1 - g_0 dS_0),$$

and the claim follows. □

In order to study a polynomial case, we are going to restrict ourselves to the particular case in which $c \equiv 1$ and S_0 is a constant, that is, we are going to determine when the circumference is a limit cycle of multiplicity m, with $m \geq 2$, of the following 1-form:

$$\omega \;=\; df + f\, g_0\, df \;+\; f^2\, \omega_1,$$

where, $f = (x^2+y^2-1)/2$, g_0 is a polynomial in $\mathbb{R}[x, y]$ and ω_1 is the following 1-form $\omega_1 \;=\; \psi(f)\,(y\, dx - x\, dy)$, with ψ a polynomial in $\mathbb{R}[x]$, where we have used the result of Françoise stated in Proposition 1.2.

The statement of Theorem 1.4 is the following.

Theorem 3.3. *Consider the* 1*-form* (1.3)*. Then,* Γ *is a limit cycle of multiplicity* m *if, and only if,* $\omega \;=\; df \;+\; f\, g_0\, df \;+\; f^m\, \phi(f)\,(ydx - xdy)$*, with* ϕ *a polynomial in* $\mathbb{R}[x]$ *such that* $\phi(0) \neq 0$.

Proof. We suppose that $\omega = df + f g_0 df + f^m\, \phi(f)\,(ydx - xdy)$, with $\phi(0) \neq 0$, and we aim to see that Γ is a limit cycle of multiplicity m of system (1.3).

We consider $F_j = \exp\{1/f^j\}$, and we see that for $j = 1, 2, \ldots, m-1$, it is an exponential factor associated to $f = 0$ of order j and with cofactor $k_j = -\, f^{m-j-1}\, \phi(f)\,(x^2+y^2)$ as the following computations show. We have that $dF_j = -j\, F_j\, df/f^{j+1}$,

$$\begin{aligned}\omega \wedge dF_j &= -f^m \phi(f)\, \frac{j}{f^{j+1}}\, F_j\,(ydx - xdy) \wedge df\\ &= -j\, f^{m-j-1}\phi(f)(x^2+y^2)F_j\, dx \wedge dy.\end{aligned}$$

Since $\phi(0) \neq 0$ and the term (x^2+y^2) is equal to 1 over $\gamma(t)$ we deduce that $\int_0^{2\pi} k_j(\gamma(t))dt = 0$ for $j = 1, 2, \ldots, m-2$ and $\int_0^{2\pi} k_{m-1}(\gamma(t))dt \neq 0$. Therefore, using Theorem 2.5 we deduce that $f = 0$ is an analytic m-solution and, thus, Γ is a limit cycle of multiplicity m.

Reciprocally, we suppose that Γ is a limit cycle of multiplicity m of system (1.3) and we will see that ω reads for $\omega = df + f g_0 df +\, f^m\, \phi(f)\,(ydx - xdy)$, with $\phi(0) \neq 0$.

Using Proposition 1.2, we can write $\omega = df + f g_0 df + f^2\psi(f)(ydx - xdy)$ and we only need to show that $\psi(f) = f^{m-2}\phi(f)$ with ϕ a polynomial such that $\phi(0) \neq 0$. We know, by Theorem 2.5, that Γ is a limit cycle of multiplicity m if, and only if, there exists $m-1$ generalized exponential factors $F_1, F_2, \ldots, F_{m-1}$ associated to $f = 0$ of consecutive orders from 1 to $m-1$ and whose respectively associated cofactors $k_1, k_2, \ldots, k_{m-1}$ are such that $\int_0^T k_j(\gamma(t))dt = 0$, for $j = 0, 1, \ldots, m-2$ and $\int_0^T k_{m-1}(\gamma(t))dt \neq 0$.

We remark that $F_1 = \exp\{1/f\}$ is a generalized exponential factor associated to $f = 0$ of order 1 as the following computations show: $dF_1 = -F_1\, df/f^2$ and

$$\omega \wedge dF_1 = -f^2\psi(f)\frac{1}{f^2}F_1(ydx - xdy) \wedge df = -\psi(f)(x^2 + y^2)F_1.$$

We see that its cofactor is $k_1 = -\psi(f)(x^2 + y^2)$ so $\int_0^{2\pi} k_1(\gamma(t))dt = -\psi(0)2\pi$. Since we assume that Γ has multiplicity m with $m \geq 2$, we deduce that Γ has multiplicity 2 if, and only if, $\psi(0) \neq 0$, and the claim for $m = 2$ follows.

We show the inductive step over m. We suppose that $\psi(f) = f^{m-3}\tilde{\phi}(f)$, we will see that $\tilde{\phi}(0) = 0$ and, therefore, $\psi(f) = f^{m-2}\phi(f)$. We see that $F_{m-2} = \exp\{1/f^{m-2}\}$ is a generalized exponential factor associated to $f = 0$ of order $m-2$ with cofactor $k_{m-2} = (2-m)\tilde{\phi}(f)(x^2+y^2)$. The computations to show this fact are the same performed in the previous paragraph. Since Γ is a limit cycle of multiplicity m by assumption, we deduce that $\int_0^T k_{m-2}(\gamma(t))\, dt = 0$. Therefore, $\tilde{\phi}(0) = 0$ and thus we deduce that $\psi(f) = f^{m-2}\phi(f)$ for a certain polynomial ϕ. To end the proof, we only need to see that $\phi(0) \neq 0$. We see that $F_{m-1} = \exp\{1/f^{m-1}\}$ is a generalized exponential factor associated to $f = 0$ of order $m - 1$ with cofactor $k_{m-1} = (1 - m)\phi(f)(x^2 + y^2)$, using the same computations as before. By Theorem 2.5, we conclude that Γ is a limit cycle of multiplicity m if, and only if, $\int_0^T k_{m-1}(\gamma(t))\, dt \neq 0$. Therefore $\phi(0) \neq 0$ and the claim follows. □

We are going to prove the following corollary, stated as Corollary 1.5, which specifies when the circumference Γ belongs to a period annulus for the 1-form described in (1.3).

Corollary 3.4. *Consider the* 1*-form* (1.3)*. Then,* Γ *belongs to a continuum of periodic orbits if, and only if,* $\omega = df + f\, g_0\, df$*, where* g_0 *is any real polynomial.*

Proof. If Γ belongs to a continuum of periodic orbits, then Γ is a periodic orbit with any multiplicity m for any natural number m. Therefore, in Theorem 1.4 we have that $\phi(f) \equiv 0$ and the claim follows.

If $\omega = df + f g_0 df$, we see that f is a first integral of this 1-form, so Γ belongs to a continuum of periodic orbits. $\square$

We end this section with the proof of Theorem 1.6. We recall that we are under the hypothesis that, in the Pfaffian form $\omega = c\, df + f(g_0\, df + dS_0) + f^2\, \omega_1$, $c \equiv 1$ and S_0 is a constant. Thus, we can consider the 1-form (1.3).

Theorem 3.5. *Consider the* 1*-form* (1.3) *and assume that it is of degree* d*. Then, the cyclicity of* Γ *as a limit cycle is* $\lfloor (d-1)/2 \rfloor$*.*

Proof. Note that if ω is a polynomial 1-form of degree d of the form (1.3), where $d = 1, 2, 3, 4$, then the result is obvious or it is a consequence of Theorem 1.1.

We assume that $d \geq 5$ from now on. If ω is the polynomial 1-form given in (1.3) and it is of degree d, we have $\omega = df + f g_0 df + f^2 \psi(f)(y dx - x dy)$, and $\psi(f)$ is a polynomial in f of degree $d-5$, i.e, $\psi(s)$ is a polynomial in s of degree $\lfloor (d-5)/2 \rfloor$.

We remark that by Theorem 1.4 and its Corollary 1.5 we deduce that the maximum multiplicity of Γ as limit cycle of ω is when $\psi(f) = f^{\lfloor (d-5)/2 \rfloor}\, \tilde{\nu}$ where $\tilde{\nu}$ needs to be a nonzero constant. Then, using Theorem 1.4 again, we get that this maximum multiplicity, i.e. the cyclicity, is $\lfloor (d-5)/2 \rfloor + 2$.

To end the proof, we only need to show the following formula:

$$\left\lfloor \frac{d-5}{2} \right\rfloor + 2 = \left\lfloor \frac{d-1}{2} \right\rfloor,$$

for any natural number d.

• If d is even then there exists k an integer number, such that $d = 2k$, and

$$\lfloor (d-5)/2 \rfloor + 2 = \lfloor (2k-5)/2 \rfloor + 2 = \lfloor k - 5/2 \rfloor + 2 = k - 3 + 2 = k - 1.$$

On the other hand, $\lfloor (d-1)/2 \rfloor = \lfloor (2k-1)/2 \rfloor = \lfloor k - 1/2 \rfloor = k - 1$.

• If d is odd, there exists k an integer number such that $d = 2k+1$, and

$$\lfloor (d-5)/2 \rfloor + 2 = \lfloor (2k+1-5)/2 \rfloor + 2 = \lfloor k - 2 \rfloor + 2 = k - 2 + 2 = k.$$

On the other hand $\lfloor (d-1)/2 \rfloor = \lfloor (2k+1-1)/2 \rfloor = k$. $\square$

4. Examples

Our purpose in this section is to apply the results given in Section 3 to some distinguished systems of the form (1.2). We apply our results to quadratic, cubic and quartic systems with the circumference as algebraic limit cycle.

The following result is due to Ch'in Yuan-shün [6] and characterizes the algebraic limit cycles of degree 2 for a quadratic system.

Theorem 4.1. [6] *If a quadratic system has an algebraic limit cycle of degree* 2*, then after an affine change of variables, the limit cycle becomes the circle*

$$\Gamma := x^2 + y^2 - 1 = 0. \tag{4.1}$$

Moreover, Γ *is the unique limit cycle of the quadratic system which can be written in the form*

$$\begin{aligned} \dot{x} &= -y\,(c_{10}x + c_{01}y + c_{00}) - (x^2 + y^2 - 1), \\ \dot{y} &= x\,(c_{10}x + c_{01}y + c_{00}), \end{aligned} \tag{4.2}$$

with $c_{10} \neq 0$*,* $c_{00}^2 + 4(c_{01} + 1) > 0$ *and* $c_{00}^2 > c_{10}^2 + c_{01}^2$*.*

We state some known results related to quadratic systems with an algebraic limit cycle. It is shown by Evdokimenko that there are no algebraic limit cycles of degree 3 for a quadratic system. Related to the study of algebraic limit cycles of degree 4, Chavarriga, Llibre and Sorolla [4] proved that any quadratic system with an algebraic limit cycle of degree 4 is affine-equivalent to one of four concrete families. There are other examples of quadratic systems with an algebraic limit cycle of degrees 5 and 6, which were encountered by Llibre and Świrszcz. In [15], Giacomini and Grau proved that all the mentioned known algebraic limit cycles of a quadratic system are hyperbolic. For a survey on these results see [4, 15] and the references therein.

In the first example, we characterize all quadratic systems with the circumference as a limit cycle and we study its multiplicity. We first prove a Lemma to describe the systems under study.

Lemma 4.2. *Let us consider all quadratic systems with the circumference as a periodic orbit. By a rotation and a scaling, these systems are of one of the* *following two forms:*

$$\begin{aligned}
&(1) \quad \dot{x} = -y + f\, a_{00}, \quad \dot{y} = x + f\, b_{00} \\
&(2) \quad \dot{x} = -y\,(1 - \nu\, x) + f\, a_{00}, \quad \dot{y} = x\,(1 - \nu\, x) + f\, b_{00},
\end{aligned}$$

where $f(x,y) = (x^2 + y^2 - 1)/2$ *as along all the section, and* a_{00}, b_{00} *and* ν *are real numbers with* $0 < \nu < 1$.

Proof. Since the system has the circumference as a periodic orbit, using Theorem 2.16, it takes the form:

$$\dot{x} \;=\; -y\, c(x,y) + f(x,y)\, a(x,y), \quad \dot{y} \;=\; x\, c(x,y) + f(x,y)\, b(x,y),$$

with $f(x,y) = (x^2+y^2-1)/2$ and $a(x,y)$, $b(x,y)$, $c(x,y)$ polynomials. We also have that the circumference $f(x,y) = 0$ and $c(x,y) = 0$ have no intersection points. Since the system is quadratic, then $a(x,y)$ and $b(x,y)$ are real numbers and there are two cases to choose $c(x,y)$. In the first case, $c(x,y)$ is a nonzero constant which, by scaling, can be taken to be $c(x,y) \equiv 1$. In the second case, $c(x,y)$ is a polynomial of degree 1 which, by rotation, can be taken such that the curve $c(x,y) = 0$ is a straight line perpendicular to the x-axis. Therefore, by a scaling, we can take $c(x,y) = 1 - \nu\, x$ with $0 < \nu < 1$. $\square$

Example 1. We are going to analyze the cyclicity of the circumference in each of the two forms of the system that we have described in the Lemma 4.2:

(1) If we write the system in the Pfaffian form $\omega = 0$, we have $\omega = df + f\,\omega_0$, with $\omega_0 = dS_0$ and $S_0 = b_{00}\, x - a_{00}\, y$. Then we apply Theorem 1.1, and we have that Γ is a limit cycle of multiplicity 1 if, and only if, $\oint_{f=0} \dfrac{\omega_0}{c} \neq 0$. This integral is always equal to zero and, moreover, Γ belongs to a continuum of periodic orbits because we can write $\omega = df + f\, d(S_0)$ and $H = f\, e^{S_0}$ is a first integral.

(2) If we write the system as a 1-form, then, $\omega = (1 - \nu\, x) df + f\,\omega_0$, where $\omega_0 = b_{00}\, dx - a_{00}\, dy$. Using Theorem 1.1, Γ is a limit cycle of multiplicity 1 if, and only if, $\oint_{f=0} \dfrac{\omega_0}{c} \neq 0$. We have that $\oint_{f=0} \dfrac{\omega_0}{c} = \displaystyle\int_0^{2\pi} \frac{-b_{00} \sin\theta - a_{00} \cos\theta}{1 - \nu\, \cos\theta} d\theta$. If we separate in two integrals, then:

$$\oint_{f=0} \frac{\omega_0}{c} = \frac{-b_{00}}{\nu} \int_0^{2\pi} \frac{\nu \sin\theta}{1 - \nu\, \cos\theta} d\theta - a_{00} \int_0^{2\pi} \frac{\cos\theta}{1 - \nu \cos\theta} d\theta.$$

The first integral is always equal to zero because:

$$\int_0^{2\pi} \frac{\nu \sin\theta}{1-\nu\,\cos\theta} d\theta \;=\; -\frac{b_{00}}{\nu}\;\ln|1-\nu\,\cos\theta|\Big|_0^{2\,\pi} \;=\; 0.$$

Using basic rules of integration, the second one is equal to

$$\int_0^{2\pi} \frac{\cos\theta}{1-\nu\cos\theta} d\theta \;=\; \frac{2\,\pi\nu}{\sqrt{1-\nu^2}\,(1+\sqrt{1-\nu^2})},$$

and, since $0<\nu<1$, we deduce that the integral $\oint_{f=0} \omega_0/c$ is equal to zero only if $a_{00}=0$. Then, Γ is a limit cycle of multiplicity 1 if, and only if, $a_{00}\neq 0$.

If we suppose that $a_{00}=0$, then, the Pfaffian form $\omega=0$ can be written as $\omega = (1-\nu\,x)\,df + f\omega_0$, where $\omega_0 = b_{00}\,dx = d(b_{00}\,x)$. We have $\omega = (1-\nu\,x)df + f\,S_0$ and Γ belongs to a continuum of periodic orbits because a first integral is $H = f(1-\nu\,x)^{-b_{00}/\nu}$.

In summary, in the family of quadratic systems, the cyclicity of the circumference as a limit cycle is 1. In this way, we recover the results described by Theorem 4.1 (except the uniqueness) and also the hyperbolicity.

We note that the cyclicity is 1 which does not contradict Theorem 1.6 because $c\not\equiv 1$, which is one of its hypothesis.

In order to apply the other results given in Section 3, we give the next example. To do this, we must consider the case in which $c\equiv 1$. In this example we study the family of cubic systems with the circumference as a limit cycle.

Example 2. We consider all cubic systems with the circumference as a limit cycle, that is of the form (1.2), and with $c\equiv 1$. By a rotation and a scaling, these systems are:

$$\dot{x} = -y + f\,a(x,y), \quad \dot{y} = x + f\,b(x,y),$$

where $a(x,y) = a_{00} + a_{10}x + a_{01}y$ and $b(x,y) = b_{00} + b_{10}x + b_{01}y$ with a_{ij}, b_{ij} are real numbers for $i,j=0,1$. We write this system as a 1-form as follows, $\omega = df + f\omega_0$, with $\omega_0 = b(x,y)\,dx - a(x,y)\,dy$. Using Theorem 1.1, we need to see when $\oint_{f=0}\frac{\omega_0}{c}\neq 0$. Since

$$\oint_{f=0} \frac{\omega_0}{c} = -(a_{10}+b_{01})\pi.$$

We deduce that if $a_{10}+b_{01}\neq 0$ then, Γ is a limit cycle of multiplicity 1.

We suppose that $b_{01} = -a_{10}$ and we consider $\omega = df + f\,\omega_0$ with $\omega_0 = (b_{00} + b_{10}x - a_{10}y)dx - (a_{00} + a_{10}x + a_{01}y)dy$. We can rewrite ω_0 as follows: $\omega_0 = d(b_{00}x + b_{10}x^2/2 - a_{10}xy - a_{00}y - a_{10}xy - a_{01}y^2/2)$. If we write ω_0 as before, then we have $\omega \sim df + f\,dS_0$, where $S_0 = 2b_{00}x + b_{10}x^2 - 2a_{00}y - 2a_{10}xy - a_{01}y^2$. In this way, the system has a first integral $H = fe^{S_0}$ and therefore Γ belongs to a continuum of periodic orbits.

Example 3. We consider all quartic systems with the circumference as a limit cycle, that is of the form (1.2), and with $c \equiv 1$. By a rotation and a scaling, these systems are:

$$\dot{x} = -y + f\,a(x,y), \quad \dot{y} = x + f\,b(x,y),$$

where $a(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$ and $b(x,y) = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2$ with a_{ij}, b_{ij} are real numbers for $i,j = 0,1,2$. We write this system as a Pfaffian form as follows, $\omega = df + f\omega_0$, with $\omega_0 = b(x,y)\,dx - a(x,y)\,dy$. Reasoning as the previous examples, we use Theorem 1.1 and we get that Γ is a limit cycle of multiplicity 1 if, and only if, $\oint_{f=0} \frac{\omega_0}{c} \neq 0$. Since

$$\oint_{f=0} \frac{\omega_0}{c} = -(a_{10} + b_{01})\pi,$$

we deduce that if $a_{10} + b_{01} \neq 0$ then, Γ is a limit cycle of multiplicity 1.

We suppose that $b_{01} = -a_{10}$ and we know that there exists S_0, g_0 and ω_1 such that $\omega_0 = g_0\,df + dS_0 + f\,\omega_1$. We calculate g_0, S_0 and ω_1 as follows:

$$\begin{aligned}\oint_{f=h} \omega_0 \;=\; & \int_0^{2\pi} \sqrt{1+2h}(-a_{20}(1+2h)\cos^3\theta - \cos^2\theta(a_{10}\sqrt{1+2h} + \\ & +(a_{11}+b_{20})(1+2h)\sin\theta) - \sin\theta(b_{00} - a_{10}\sqrt{1+2h}\sin\theta + \\ & +b_{02}(1+2h)\sin^2\theta) - \cos\theta(a_{00} + (a_{01}+b_{10})\sqrt{1+2h}\sin\theta + \\ & +(a_{02}+b_{11})(1+2h)\sin^2\theta)))\,d\theta \;\equiv\; 0.\end{aligned}$$

This fact implies that $\omega_1 \equiv 0$. To calculate g_0 we need to solve the equation $dg_0 \wedge df - d\omega_1 = 0$ with respect to g_0. We get $g_0(x,y) = -a_{11}x - 2b_{02}x + 2a_{20}y + b_{11}y$. Now, we know that $\omega_0 = g_0\,df + dS_0$ and we replace the value of $g_0(x,y)$ so that $dS_0 = \omega_0 - g_0 df$. Solving this last equation we have that: $S_0(x,y) = (6b_{00}x + 3b_{10}x^2 + 2a_{11}x^3 + 4b_{02}x^3 + 2b_{20}x^3 - 6a_{00}y - 6a_{10}xy - 6a_{20}x^2y - 3a_{01}y^2 + 6b_{02}xy^2 - 2a_{02}y^3 - 4a_{20}y^3 - 2b_{11}y^3)/6$.

Using Theorem 1.3, if the integral $\oint_{f=0} e^{-S_0}(\omega_1 - g_0 dS_0)$ is different from zero, then Γ has multiplicity 2. We see that this integral can be written as the sum of the following two integrals:

$$\begin{aligned}\oint_{f=0} e^{-S_0}(\omega_1 - g_0 dS_0) \;&=\; -(a_{11} + 2b_{02}) \oint_{f=0} x e^{-S_0} d(-S_0) \; + \\ &\qquad + (2a_{20} + b_{11}) \oint_{f=0} y e^{-S_0} d(-S_0).\end{aligned}$$

Using integration by parts we deduce that:

$$\oint_{f=0} e^{-S_0}(\omega_1 - g_0 dS_0) \;=\; (a_{11} + 2b_{02}) \oint_{f=0} e^{-S_0}\, dx \;-\; (2a_{20} + b_{11}) \oint_{f=0} e^{-S_0}\, dy.$$

To be able to apply the Theorem 1.4 we need to suppose that S_0 is constant. We impose S_0 to be a constant and we have that $\omega = df + f\, g_0\, df$. Applying Corollary 1.5, Γ belongs to a continuum of periodic orbits.

In summary, under the hypothesis $c \equiv 1$ and S_0 is a constant, in the family of the quartic systems, the cyclicity of the circumference as a limit cycle is 1. If we consider the case $c \equiv 1$ but S_0 is not a constant then it can occur that, in the family of the quartic systems, the cyclicity of the circumference as a limit cycle is greater or equal to 2.

In future works we will study the more general cases in which either c or S_0 are not constants. We are also addressed the problem of studying algebraic curves of higher degree than the circumference.

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