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MULTIPLICITY RESULTS FOR A DISCRETE BOUNDARY VALUE PROBLEM VIA CRITICAL POINT THEORY

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Abstract. This paper is a survey on some recent multiplicity results, contained in [11], for a discrete boundary value problem involving the p-Laplacian via critical point theory. An overview on the abstract critical points results used to obtain them it is also given.
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In the last few years, many authors have paid more attention to discrete problems, also to the influence of the advent of computers which have made it easy to implement numerical methods in order to solve differential equations. In this connection various approaches have been followed to obtain existence and multiplicity results for several types of boundary value problems. As you can see, a complete overview on these subjects is contained in the monographs [1], [4] and [15] and in the references given therein. Nevertheless, it seems that the variational approach to studying difference equations is more recent. On this direction, in my opinion, two basic papers are [2] and [3]. We also mention the papers [5], [14], [16], [20].

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The main purpose of the present paper is to give an overview on recent multiplicity results, contained in [11], to the following problem (P_λ),

$$(P_\lambda)\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \lambda f(k,u(k)), & k\in[1,T],\\ u(0)=u(T+1)=0, \end{cases}$$

where, T is a fixed positive integer, $[1,T]$ is the discrete interval $\{1,2,...,T\}$, λ is a positive real parameter, $\Delta u(k) := u(k+1)-u(k)$ is the forward difference operator, $\phi_p(s) := |s|^{p-2}s$, $1<p<+\infty$ and $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Now, we remind you the variational formulation of (P_λ). On the T-dimensional Banach space

$$W := \{u:[0,T+1]\to\mathbb{R} : u(0)=u(T+1)=0\},$$

equipped with the norm

$$\|u\| := \left(\sum_{k=1}^{T+1}|\Delta u(k-1)|^p\right)^{1/p}, \quad u\in W,$$

we define the functional $I_\lambda : W\to\mathbb{R}$ by putting, for every $u\in W$,

$$I_\lambda(u) := \sum_{k=1}^{T+1}\left[\frac{1}{p}|\Delta u(k-1)|^p - \lambda F(k,u(k))\right],$$

where, $F(k,t) := \int_0^t f(k,\xi)d\xi$ for every $(k,t)\in[1,T]\times\mathbb{R}$. An easy computation ensures that $I_\lambda\in C^1(W)$ with

$$I'_\lambda(u)(v) = \sum_{k=1}^{T+1}\left[\phi_p\left(\Delta u(k-1)\right)\Delta v(k-1) - \lambda f(k,u(k))v(k)\right] \quad \forall v\in W.$$

Therefore, taking into account that, for every $u,v\in W$, it results

$$-\sum_{k=1}^{T+1}\Delta(\phi_p(\Delta u(k-1)))v(k) = \sum_{k=1}^{T+1}\phi_p\left(\Delta u(k-1)\right)\Delta v(k-1),$$

it's clear that the critical points of I_λ are exactly the solutions of problem (P_λ).

Let c and d be two positive constants, write

$$\Theta(c) := \frac{\sum_{k=1}^{T}\sup_{|\xi|\le c}F(k,\xi)}{c^p} \qquad \Gamma(d) := \frac{\sum_{k=1}^{T}\left[F(k,d)-\sup_{|\xi|\le c}F(k,\xi)\right]}{d^p},$$

we have

Theorem 1. *Assume that there exist four positive constants a, c, d and s with $c < d$ and $s < p$ such that*

(b_1) $\Theta(c) < \left(\frac{2}{T+1}\right)^{p-1} \Gamma(d)$;

(b_2) $F(k, \xi) \leq a(1 + |\xi|^s)$ *for all* $(k, \xi) \in [1, T] \times \mathbb{R}$.

Then, for every $\lambda \in \left]\dfrac{2}{p\Gamma(d)}, \dfrac{2^p}{p\Theta(c)(T+1)^{p-1}}\right[$, *problem* (P_λ) *admits at least three solutions.*

Remark 1. We explicitly observe that each of the assumptions of Theorem 1 ensures at least one solution to problem (P_λ) separately.

It is understood that the novelty introduced by Theorem 1 in the study of discrete boundary value problems lies in the assumption (b_1). Therefore, in order to identify a class of functions satisfying such hypothesis, we give the following result.
Let $h : \mathbb{R} \to \mathbb{R}$, $q : [1, T] \to \mathbb{R}$ be two nonnegative functions such that h be continuous and $Q := \sum_{k=1}^{T} q(k) > 0$. Put, for every $t \in \mathbb{R}$, $H(t) := \int_0^t h(\xi)d\xi$, we get

Theorem 2. *Assume that there exist four positive constants ρ, c, d and s with $c < d$ and $s < p$ such that*

(b_3) $\frac{H(c)}{c^p} < \frac{2^{p-1}}{2^{p-1}+(T+1)^{p-1}} \frac{H(d)}{d^p}$;

(b_2') $H(\xi) \leq \rho(1 + |\xi|^s)$, *for all* $\xi \in \mathbb{R}$.

Then, for each

$$\lambda \in \left] \frac{2}{pQ} \frac{2^{p-1} + (T+1)^{p-1}}{(T+1)^{p-1}} \frac{d^p}{H(d)}, \frac{2^p}{pQ(T+1)^{p-1}} \frac{c^p}{H(c)} \right[,$$

the problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \lambda q(k) h(u(k)), & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

admits at least three nonnegative solutions.

Example 1. Write, for each $k \in [1, T]$, $q(k) = k$ and let $h : \mathbb{R} \to \mathbb{R}$ be defined by putting

$$h(t) = \begin{cases} e^t, & \text{if } t \leq 12; \\ e^{12}, & \text{if } t > 12. \end{cases}$$

By choosing for instance $\rho = e^{12}$, $c = 1$, $d = 12$, $s = 1$ and $p = 3$, the assumptions of Theorem 2 are satisfied. Therefore, for each $\lambda \in \left]\frac{164}{10^6}, \frac{344}{10^6}\right[$, the problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|\Delta u(k-1))) = \lambda k h(u(k)), & k \in [1,T], \\ u(0) = u(10) = 0, \end{cases}$$

has at least three positive solutions.

Remark 2. It is worthwhile observing that a sufficient condition in order that (b_3) holds is the following:

(b'_3) *there exist a positive constant d such that* $H(d) > 0$ *and* $\lim\limits_{c \to 0^+} \frac{H(c)}{c^p} = 0$.

Of course, in this case, Theorem 2 ensures the existence of at least to non-trivial solutions only.

Now we discuss the case $s = p$ inside the growth condition (b_2).

Theorem 3. *Assume that there exist three positive constants a, c, d with $c < d$ such that (b_1) holds and in addition suppose that*

(b_4) $F(k,\xi) \le a(1+|\xi|^p)$ *for all* $(k,\xi) \in [1,T] \times \mathbb{R}$, *with* $a < \frac{2^{p-1}}{T(T+1)^{p-1}}\Gamma(d)$.

Then, for every

$$\lambda \in \left] \frac{2}{p\Gamma(d)}, \frac{2^p}{p(T+1)^{p-1}} \min\left\{ \frac{1}{\Theta(c)}, \frac{1}{aT} \right\} \right[,$$

problem (P_λ) *admits at least three solutions.*

Remark 3. We point out that if (b_4) holds with $a < \frac{2^p}{pT(T+1)^{(p-1)}}$ we get the existence of at least one solution to (P_1).

Moreover we present another situation where condition (b_1) is verified.

Theorem 4. *Assume that there exist four positive constants a, c, d and s with $c < d$ and $s < p$ such that (b_2) holds and moreover, suppose that*

(b_5) $\max_{|\xi| \le c} F(k,\xi) \le 0$ *for all* $k \in [1,T]$;

(b_6) *there exists* $\overline{k} \in [1,T]$ *such that* $\int_0^d f(\overline{k},\xi)d\xi > 0$.

Then, for every $\lambda \in \left] \dfrac{2d^p}{p\int_0^d f(\overline{k},\xi)d\xi}, +\infty \right[$, *problem* (P_λ) *admits at least two positive solutions.*

Now, we give a brief overview on the variational framework used to achieve the previous results. In particular we have adopted some recent critical points theorems for functionals $I_\lambda : X \to \mathbb{R}$, where X is a real Banach space and I_λ satisfies the structural hypothesis:

(Λ) $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ *for all* $u \in X$, *where* $\Phi, \Psi : X \to \mathbb{R}$ *are two functions (of class* C^1*) on* X *with* Φ *coercive, i.e.* $\lim_{\|u\|\to+\infty} \Phi(u) = +\infty$, *and* λ *is a positive parameter.*

For a treatment of the general case on this subject we mention, for instance, [6], [7] and [8]. The scientific background of the above mentioned papers is located, chiefly, in the seminal paper [19].
Finally we wish to stress that other results on problem (P_λ) are contained in the following work **G. Bonanno** and **P. Candito**, *Nonlinear difference equations via critical point methods, submitted.* However, on this setting, the abstract framework is furnished by [10], which is based on the classical papers on nonsmooth analysis due to [12], [13], [18] as well as on some critical points results for functionals of type (Λ) established in [17] and [9].

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