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KRASNOSELSKII-TYPE THEOREMS FOR MULTIVALUED OPERATORS

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Abstract. The aim of this paper is to present some fixed point theorems of Krasnoselskii-type for the sum of two multivalued operators.
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1. Introduction

Let (X, d) be a metric space and $F : X \to P_{b,cl}(Y)$ a multivalued operator. Denote by H_d the Pompeiu-Hausdorff metric on $P_{b,cl}(X)$.

Then, F is a $\varphi-$*contraction* if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (i.e. φ is increasing and $\varphi^n(t) \to 0$, as $n \to \infty$, for all $t \to 0$) and

$$H(F(x_1), F(x_2)) \leq \varphi(d(x_1, x_2)), \text{ for all } x_1, x_2 \in X.$$

The aim of this paper is to present some fixed point theorems of Krasnoselskii-type for the sum of two multivalued operators.

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2. Notations and auxiliary results

The aim of this section is to present some notions and symbols used in the paper.

Let us consider the following families of subsets of a metric space (X, d):

$$P(X) := \{Y \in \mathcal{P}(X) | \ Y \neq \emptyset\}; P_b(X) := \{Y \in P(X) | \ Y \text{ is bounded }\};$$

$$P_{cp}(X) := \{Y \in P(X) | \ Y \text{ is compact }\}; \ P_{cl}(X) := \{Y \in P(X) | \ Y \text{ is closed }\};$$

$$P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$$

Let us define the following generalized functionals:

(1) $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \ D(A, B) = \inf\{d(a, b) \mid a \in A, \ b \in B\}$.

D is called the gap functional between A and B. In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.

(2) $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\delta(A, B) = \sup\{d(a, b) | \ a \in A, \ b \in B\}$.

(3) $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\rho(A, B) \ = \sup\{D(a, B) \mid a \in A\}$.
ρ is called the (generalized) excess functional.

(4) $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$.
H is the (generalized) Pompeiu-Hausdorff functional.

(5) $\delta : P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, $\delta A := sup\{d(a, b) | a, b \in A\}$.

Definition 2.1. *Let (X, d) be a metric space. If $F : X \to P(X)$ is a multivalued operator, then:*

(1) $x \in X$ is called fixed point for F if and only if $x \in F(x)$;

(2) $x \in X$ is called strict fixed point for F if and only if $\{x\} = F(x)$.

The set $FixF := \{x \in X | x \in F(x)\}$ is called the fixed point set of F. The set $SFixF := \{x \in X | \{x\} = F(x)\}$ is called the strict fixed point set of F. Also, a sequence of successive approximations of F starting from $x \in X$ is a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X with $x_0 = x, \ x_{n+1} \in F(x_n)$, for $n \in \mathbb{N}$.

Definition 2.2. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$ a multivalued operator. F is said to be upper semi-continuous in $x_0 \in X$ (briefly u.s.c.) if and only if for each open subset U of Y with $F(x_0) \subset U$ there exists an open neighborhood V of x_0 such that for all $x \in V$ we have $F(x) \subset U$.*

F is u.s.c. on X if it is u.s.c in each $x_0 \in X$.

Definition 2.3. *Let X, Y two metric spaces and $F : X \to P(Y)$ a mutivalued operator. Then F is called H-upper semicontinuous in $x_0 \in X$ (briefly H-u.s.c.) if and only if for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x \in B(x_0; \eta)$ we have $F(x) \subset V(F(x_0); \varepsilon)$.*

F is H-u.s.c. on X if it is H-u.s.c. in each $x_0 \in X$.

Definition 2.4. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$ a multivalued operator. Then F is said to be lower semi-continuous in $x_0 \in X$ (briefly l.s.c.) if and only if for each open subset $U \subset Y$ with $F(x_0) \cap U \neq \emptyset$ there exists an open neighborhood V of x_0 such that for all $x \in V$ we have $F(x) \cap U \neq \emptyset$.*

F is l.s.c. on X if it is l.s.c in each $x_0 \in X$.

Definition 2.5. *Let X, Y two metric spaces and $F : X \to P(Y)$ a mutivalued operator. Then F is called H-lower semicontinuous in $x_0 \in X$ (briefly H-l.c.s.) if and only if for all $\varepsilon > 0$ there exists $\eta > 0$ such that we have $F(x_0) \subset V(F(x); \varepsilon)$, for all $x \in B(x_0; \eta)$.*

F is H-l.s.c. on X if it is H-l.s.c. in each $x_0 \in X$

Definition 2.6. *Let X, Y be Hausdorff topological spaces and $F : X \to P(Y)$ a multivalued operator. Then F is said to be continuous in $x_0 \in X$ if and only if it is l.s.c and u.s.c. in $x_0 \in X$.*

Definition 2.7. *Let X, Y two metric spaces and $F : X \to P(Y)$ a mutivalued operator. Then F is called H-continuous in $x_0 \in X$ (briefly H-c.) if and only if for all it is H-l.s.c. and H-u.s.c. in $x_0 \in X$.*

Definition 2.8. *(Kuratowski)*

Let X be a Banach space and $A \in P_b(X)$. By the real number $\alpha(A)$ we denote the infimum of all numbers $\varepsilon > 0$ such that A admits a finite covering consisting of subsets of diameter less than ε.

Remark 2.1. *It easy to see that, for $A, B \in P_b(X)$:*

a) $\alpha(A) \leq \delta(A)$, where $\delta(A)$ is the diameter of the set A;

b) $\alpha(A) = 0$ iff A is paracompact;

c) $\alpha(A \cup B) = max\{\alpha(A), \alpha(B)\}$;

d) $\alpha(B(A, \varepsilon)) \leq \alpha(A) + 2\varepsilon$, where $B(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$;

e) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$;

f) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;

Definition 2.9. *(Furi, Vignoli* [2]*)*

Let X be a Banach space and $D \in P(X)$. Then $T : D \to P_{cl}(X)$ is called densifying if is H-continuous and for every bounded set $A \subset D$, such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$.

Definition 2.10. *A function $\varphi : [0, \infty) \to [0, \infty)$ is called an L-function if $\varphi(s) > 0$, for all $s > 0$ and for every $s > 0$ there exists $u > s$ such that $\varphi(t) \leq s$, for $t \in [s, u]$.*

Every L-function satisfies $\varphi(s) \leq s$, for all $s \geq 0$.

Definition 2.11. *Let (X,d) be a metric space. The operator $T : X \to X$ satisfies the Meir-Keeler condition if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(T(x), T(y)) < \varepsilon$.*

Let (X,d) be a metric space. The operator $T : X \to P_{cl}(X)$ satisfies the Meir-Keeler condition if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow H(T(x), T(y)) < \varepsilon$.

Theorem 2.1. *(Lim* [1]*)*

Let X be a metric space and let $T : X \to X$. The following are equivalent:

i) T satisfies the Meir-Keeler's condition;

ii) There exists an L-function $\varphi : [0, \infty) \to [0, \infty)$ nondecreasing and right continuous such that: $d(T(x), T(y)) < \varphi(d(x, y))$, for all $x \neq y \in X$.

Theorem 2.2. *(Lim* [1]*)*

Let X be a metric space and let $T : X \to P_{cl}(X)$. The following are equivalent:

i) T satisfies the Meir-Keeler's condition;

ii) There exists an L-function $\varphi : [0, \infty) \to [0, \infty)$ nondecreasing and right continuous such that: $H(T(x), T(y)) < \varphi(d(x, y))$, for all $x \neq y \in X$.

Theorem 2.3. *(Reich [5])*

Let (X,d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued operator satisfying the Meir-Keeler condition. Then $FixT \neq 0$.

3. Main results

We begin this section by presenting two auxiliary results. We need first a definition.

Definition 3.1. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping. Then:*

(i) φ is called a strict comparison function if φ is monotone increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, for all $t > 0$.

(ii) φ is called a strong strict comparison function if the function $s_\varphi(t) := \sum_{n=1}^{\infty} \varphi^k(t)$ is increasing;

(iii) φ is called an expansive function if $\varphi(t) > t$, for all $t > 0$ and φ is increasing.

Lemma 3.1. *Let $Y, Z \in P_{cl}(X)$ and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an expansive function. Then for all $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq \psi(\rho(Y, Z))$.*

Proof. We suppose by contradiction that there exists $y \in Y$ such that for all $z \in Z$ we have that: $d(y, z) > \psi(\rho(Y, Z))$. Taking the infimum of $z \in Z$ we obtain that there exists $y \in Y$ such that $D(y, Z) \geq \psi(\rho(Y, Z))$. But $\rho(Y, Z) \geq D(y, Z)$. Hence $\rho(Y, Z) \geq \psi(\rho(Y, Z))$ which is a contradiction with the definition of an expansive function. □

In order to prove the main theorems in this article we need the following lema:

Lemma 3.2. *Let (X, d) be a complete metric space, $T_1, T_2 : X \to P_{cl}(X)$ two (φ, ψ)-contractions. Then $\rho(FixT_1, FixT_2) \leq s_\phi(\psi(\sup_{y\in X} \rho(T_1(y), T_2(y))))$, where $s_\phi(t) = \sum_k \phi^k(t)$ and $\phi = \psi \circ \varphi$.*

Proof. Denote by $\delta := s_\phi(\psi(\sup_{y\in X} \rho(T_1(y), T_2(y))))$.

We want to prove that for every $x_0 \in FixT_1$ there exists $x_2^* \in FixT_2$ such that $d(x_0, x_2^*) \leq \delta$.

Let $x_0 \in X$ such that $x_0 \in T_1(x_0)$. Applying Lemma 3.1 for $Y = T_1(x_0)$ and $Z = T_2(x_0)$ we obtain that there exists $x_1 \in T_2(x_0)$ such that $d(x_0, x_1) \leq \psi(\rho(T_1(x_0), T_2(x_0))) \leq \psi(\underset{x \in X}{sup}\rho(T_1(x), T_2(x))) := \psi(\eta)$.

Applying once again Lemma 3.1 for $Y = T_2(x_0)$, $Z = T_2(x_1)$ and $x_1 \in T_2(x_0)$ we have that there exists $x_2 \in T_2(x_1)$ such that $d(x_1, x_2) \leq \psi(\rho(T_2(x_0), T_2(x_1))) \leq \psi(\varphi(d(x_0, x_1))) = (\psi \circ \varphi)(d(x_0, x_1))$.

Proceeding this way we obtain inductively the sequence $(x_n)_{n \in \mathbb{N}}$ with the following properties:

(i) $x_{n+1} \in T_2(x_n)$, for all $n \in \mathbb{N}$;

(ii) $d(x_n, x_{n+1}) \leq (\psi \circ \varphi)^n(d(x_0, x_1))$.

From (ii) since $(\psi \circ \varphi)^n(t) \to 0$ we have that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, so $x_n \to x^* \in X$, for all $t > 0$, as $n \to \infty$.

From (i) and from $x_n \to x^*$ and from the fact that T_2 is closed (it is a contraction) we get that $x_2^* \in FixT_2$.

Using that (x_n) is Cauchy we have: $d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + ... + d(x_{n+p-1}, x_{n+p}) \leq (\psi \circ \varphi)^n(d(x_0, x_1)) + ... + (\psi \circ \varphi)^{n+p-1}(d(x_0, x_1)) \leq \sum_{k \geq 0}(\psi \circ \varphi)^k(d(x_0, x_1)) = s_\phi(d(x_0, x_1))$, for all $n \in \mathbb{N}, p \in \mathbb{N}^*$. For $p \to \infty$ we have that $d(x_n, x_2^*) \leq s_\phi(d(x_0, x_1))$, for all $n \geq 0$. Taking $n = 0$ and using the fact that s_ϕ is increasing we obtain $d(x_0, x_2^*) \leq s_\phi(d(x_0, x_1)) \leq s_\phi(\psi(\eta))$. □

Theorem 3.1. *Let X be a Banach space, $Y \in P_{cl,cv}(Y)$. Let $A : Y \to P_{b,cl,cv}(X)$ and $B : Y \to P_{cp,cv}(X)$ two multivalued operators such that:*

i) $A(y_1) + B(y_2) \subset Y$, for all $y_1, y_2 \in Y$;

ii) A is a multivalued (φ, ψ)-contraction, i.e. there exist two continuous functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi \circ \varphi$ is a strong strict comparison function, φ is a comparison function and ψ is an expansive function such that: $H(A(x), A(y)) \leq \varphi(\|x - y\|)$, for all $x, y \in Y$;

iii) B is l.s.c and compact;

iv) For all $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that $s_\phi(\psi(R(\varepsilon))) \leq \varepsilon$, where $s_\phi(t) = \sum_{k=0}^{\infty} \phi^k(t)$, with $\phi = \psi \circ \varphi$.

Then $Fix(A + B) \neq \emptyset$.

Proof. Let $C : Y \to P(Y)$ a multivalued operator defined as follows:

a) For all $x \in Y$ let $T_x : Y \to P_{cp,cv}(Y)$ be defined by $T_x(y) = A(y) + B(x)$. Then $H(T_x(y_1), T_x(y_2)) = H(A(y_1) + B(x), A(y_2) + B(x)) \leq H(A(y_1), A(y_2)) \leq \varphi(\|y_1 - y_2\|) \leq (\psi \circ \varphi)(\|y_1 - y_2\|)$. From Wegrzyk fixed point theorem (see [7]) we have that: $FixT_x \neq \emptyset$, for all $x \in Y$.

Next we will prove that the set $FixT_x$ is closed, for each $x \in Y$. Recall that $FixT_x$ is closed if and only if for all $y_n \subset FixT_x$ with $y_n \to y$, as $n \to \infty$ we have that $y \in FixT_x$. Since $y_n \subset FixT_x$ we have that $y_n \in T_x(y_n)$. Thus $D(y, T_x(y)) \leq d(y, y_n) + D(y_n, T_x(y)) \leq d(y, y_n) + H(T_x(y_n), T_x(y)) \leq d(y, y_n) + \varphi(\|y_n - y\|) \to 0$ as $n \to \infty$. We have that $y \in T_x(y)$.

b) Let $F : Y \times Y \to P_{cp,cv}(Y)$, $F(x, y) = A(y) + B(x)$, for all $(x, y) \in Y \times Y$. F satisfies the hypothesis of Theorem 1 in Rybinski [6]. Thus, we have that there exists $f : Y \times Y \to Y$ continuous such that $f(x, y) \in A(f(x, y)) + B(x)$.

Let $C(x) = FixT_x$ be given by $C : Y \to P_{cl}(Y)$ and let $c : Y \to Y$ defined by $c(x) = f(x, x)$ for all $x \in Y$. Then c is a continuous function and we have that: $c(x) = f(x, x) \in A(f(x, x)) + B(x) = A(c(x)) + B(x) = T_x(c(x))$ for all $x \in Y$. We will prove that $c(Y)$ is relatively compact. It is enough to prove that $C(Y)$ is relatively compact.

We show that $C(Y)$ is totally bounded. From the fact that B is compact we have that $B(Y)$ is relatively compact and thus totally bounded. So for all $\varepsilon > 0$ there exists $Z = \{x_1, x_2, ..., x_n\} \subset Y$ such that $B(Y) \subset \{z_1, ..., z_n\} + B(0, R(\varepsilon)) \subset \bigcup_{i=1}^{n} B(x_i) + B(0, R(\varepsilon))$, where $z_i \in B(x_i), i = 1, ..., n$. We have that for all $x \in Y$, $B(x) \subset \bigcup_{i=1}^{n} B(x_i) + B(0, R(\varepsilon))$ and hence there exists $x_k \in Z$ such that $\rho(B(x), B(x_k)) < R(\varepsilon)$. So $\rho(C(x), C(x_k)) = \rho(FixT_x, FixT_{x_k}) \overset{(*)}{\leq} s_\phi(\psi(\sup_{y \in Y} \rho(T_x(y), T_{x_k}(y)))) \leq \varepsilon$. The inequality (*) follows from Lemma 3.2. From the fact that $\rho(T_x(y), T_{x_k}(y)) = \rho(A(y) + B(x), A(y) + B(x_k)) \leq \rho(B(x), B(x_k)) < R(\varepsilon)$ we have that $s_\phi(\psi(R(\varepsilon))) \leq \varepsilon$. It implies that for each $u \in C(x)$ there exists $v_k \in C(x_k)$ such that $\|u - v_k\| < \varepsilon$. So for all $x \in Y$, $C(x) \subset Q + B(0, \varepsilon)$, where $Q = \{v_1, ..., v_k, ..., v_n\}$ with $v_i \in C(x_i), i = 1, ..., n$. Since in a Banach space a totally bounded set is relatively compact we get that $C(Y)$ is relatively compact.

Thus $c : Y \to Y$ satisfies the hypothesis in Schauder's theorem. Let $x^* \in Y$ a fixed point for c. We have that $x^* = c(x^*) \in A(c(x^*))+B(x^*) = A(x^*)+B(x^*)$. $\square$

Theorem 3.2. *Let X be a Banach space, $Y \in P_{cl,cv}(Y)$. Let $A : Y \to P_{b,cl,cv}(X)$ and $B : Y \to P_{cp,cv}(X)$ two multivalued operators such that:*

i) If $y \in A(y) + B(x) \subset Y$, for all $x \in Y$ then $y \in Y$;

ii) A is a multivalued (φ, ψ)-contraction, i.e. there exist two continuous functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi \circ \varphi$ is a strong strict comparison function, φ is a comparison function and ψ is an expansive function such that: $H(A(x), A(y)) \leq \varphi(\|x - y\|)$, for all $x, y \in Y$;

iii) B is l.s.c and compact;

iv) For all $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that $s_\phi(\psi(R(\varepsilon))) \leq \varepsilon$, where $s_\phi(t) = \sum_{k=0}^{\infty} \phi^k(t)$, with $\phi = \psi \circ \varphi$.

Then $Fix(A + B) \neq \emptyset$.

Proof. Let $C : Y \to P(Y)$ a multivalued operator defined as follows:

a) For all $x \in Y$ let $T_x : Y \to P_{cp,cv}(Y)$ be defined by $T_x(y) = A(y) + B(x)$. Then $H(T_x(y_1), T_x(y_2)) = H(A(y_1) + B(x), A(y_2) + B(x)) \leq H(A(y_1), A(y_2)) \leq \varphi(\|y_1 - y_2\|) \leq (\psi \circ \varphi)(\|y_1 - y_2\|)$. From Wegrzyk fixed point theorem (see [7]) we have that: $FixT_x \neq \emptyset$, for all $x \in Y$.

Next we will prove that the set $FixT_x$ is closed, for each $x \in Y$. Recall that $FixT_x$ is closed if and only if for all $y_n \subset FixT_x$ with $y_n \to y$, as $n \to \infty$ we have that $y \in FixT_x$. Since $y_n \subset FixT_x$ we have that $y_n \in T_x(y_n)$. Thus $D(y, T_x(y)) \leq d(y, y_n) + D(y_n, T_x(y)) \leq d(y, y_n) + H(T_x(y_n), T_x(y)) \leq d(y, y_n) + \varphi(\|y_n - y\|) \to 0$ as $n \to \infty$. We have that $y \in T_x(y)$. From i) we have that $FixT_x \subset Y$.

b) Let $F : Y \times Y \to P_{cp,cv}(Y)$, $F(x, y) = A(y)+B(x)$, for all $(x, y) \in Y \times Y$. F satisfies the hypothesis of Theorem 1 in Rybinski [6]. Thus, we have that there exists $f : Y \times Y \to Y$ continuous such that $f(x, y) \in A(f(x, y)) + B(x)$.

Let $C(x) = FixT_x$ be given by $C : Y \to P_{cl}(Y)$ and let $c : Y \to Y$ defined by $c(x) = f(x, x)$ for all $x \in Y$. Then c is a continuous function and we have that: $c(x) = f(x, x) \in A(f(x, x)) + B(x) = A(c(x)) + B(x) = T_x(c(x))$ for all $x \in Y$. We will prove that $c(Y)$ is relatively compact. It is enough to prove that $C(Y)$ is relatively compact.

We show that $C(Y)$ is totally bounded. From the fact that B is compact we have that $B(Y)$ is relatively compact and thus totally bounded. So for all $\varepsilon > 0$ there exists $Z = \{x_1, x_2, ..., x_n\} \subset Y$ such that $B(Y) \subset \{z_1, ..., z_n\} + B(0, R(\varepsilon)) \subset \bigcup_{i=1}^{n} B(x_i) + B(0, R(\varepsilon))$,where $z_i \in B(x_i), i = 1, ..., n$. We have that for all $x \in Y$, $B(x) \subset \bigcup_{i=1}^{n} B(x_i) + B(0, R(\varepsilon))$ and hence there exists $x_k \in Z$ such that $\rho(B(x), B(x_k)) < R(\varepsilon)$. So $\rho(C(x), C(x_k)) = \rho(FixT_x, FixT_{x_k}) \overset{(*)}{\leq} s_\phi(\psi(\sup_{y \in Y} \rho(T_x(y), T_{x_k}(y)))) \leq \varepsilon$. The inequality (*) follows from Lemma 3.2. From the fact that $\rho(T_x(y), T_{x_k}(y)) = \rho(A(y) + B(x), A(y) + B(x_k)) \leq \rho(B(x), B(x_k)) < R(\varepsilon)$ we have that $s_\phi(\psi(R(\varepsilon))) \leq \varepsilon$. It implies that for each $u \in C(x)$ there exists $v_k \in C(x_k)$ such that $\|u - v_k\| < \varepsilon$. So for all $x \in Y$, $C(x) \subset Q + B(0, \varepsilon)$, where $Q = \{v_1, ..., v_k, ..., v_n\}$ with $v_i \in C(x_i), i = 1, ..., n$. Since in a Banach space a totally bounded set is relatively compact we get that $C(Y)$ is relatively compact.

Thus $c : Y \to Y$ satisfies the hypothesis in Schauder's theorem. Let $x^* \in Y$ a fixed point for c. We have that $x^* = c(x^*) \in A(c(x^*)) + B(x^*) = A(x^*) + B(x^*)$. □

Theorem 3.3. *Let (X, d) be a metric space, D a complete subset of X and let $T : D \to P(X)$ be a densifying multivalued operator. Then any bounded sequence $\{x_n\}$, such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$ is compact and all the limit points of $\{x_n\}$ are fixed for T.*

Proof. Let $\{x_n\}$ be a bounded sequence such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$. Put $M = \{x_n : n = 1, 2, ..\}$, so that $T(M) = \{T(x_n) : n = 1, 2, ...\}$. Given any $\varepsilon > 0$, it follows that $B(T(M), \varepsilon)$ contains all but a finite number of elements of M, since $D(x_n, T(x_n)) \to 0$. Then $\alpha(M) \leq \alpha(B(T(M), \varepsilon)) \leq \alpha(T(M)) + 2\varepsilon$; hence $\alpha(T(M)) \geq \alpha(M)$. Therefore $\{x_n\}$ is compact. By the H-continuity of T all the limit points of $\{x_n\}$ are fixed for T. □

Corollary 3.1. *Let (X, d) be a bounded, complete metric space and $T : x \to P(X)$ be a densifying multivalued operator. If $inf\{D(x, T(x)) : x \in X\} = 0$ then T has at least a fixed point.*

Proof. It follows immediately from the theorem above. □

Corollary 3.2. *Let (X,d) be a complete metric space and $T : X \to P(X)$ be a completely continuous multivalued operator. If $inf\{D(x,T(x)) : x \in X\} = 0$ then T has at least a fixed point.*

Proof. It follows immediately from the corollary above. □

Corollary 3.3. *Let $T : D \to P(F)$ be a multivalued operator defined on a closed subset D of a Frechet space F such that $T = G + H$, where $G : D \to P(F)$ is a completely continuous operator and $H : D \to P(F)$ is contractive. Then any bounded sequence $\{x_n\}$ such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$ is compact and all the limit points of $\{x_n\}$ are fixed for T.*

Proof. It is sufficient to prove that T is densifying. Let $A \subset F$ be a bounded set with $\alpha(A) > 0$. We have $\alpha(T(A)) \leq \alpha(G(A) + H(A)) \leq \alpha(G(A)) + \alpha(H(A)) = \alpha(H(A)) < \alpha(A)$. □

Theorem 3.4. *Let X be a Banach space and $Y \in P_{cl,b,cv}(X)$. Let $A, B : Y \to X$ such that:*

i) $A(x) + B(y) \in Y$, for all $x, y \in Y$;

ii) A satisfies the Meir-Keeler condition;

iii)B is completely continuous.

Then $Fix(A + B) \neq \emptyset$.

Proof. From (i) we have that A satisfies the Meir-Keeler condition. Lim showed in Theorem 2.1 that A is a nonlinear contraction. Applying the main result in Nashed and Wong [3] the conclusion follows. □

Theorem 3.5. *Let X be a Banach space and $Y \in P_{cl,b,cv}(X)$. Let $A, B : Y \to P_{cp,cv}(X)$ such that:*

i) $A(x) + B(y) \in Y$, for all $x, y \in Y$;

ii) A satisfies the Meir-Keeler condition;

iii)B is l.s.c and compact.

If we denote by φ the function from Lim's characterisation theorem, then there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ an expansive function such that $\psi \circ \varphi$ is a comparison function and:

iv) For all $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that $s_\phi(\psi(R(\varepsilon))) \leq \varepsilon$, where $s_\phi(t) = \sum_{k=0}^{\infty} \phi^k(t)$, with $\phi = \psi \circ \varphi$.

Then $Fix(A + B) \neq \emptyset$.

Proof. From the hypothesis we have that A satisfies the Meir-Keeler condition, so from Theorem 2.2 we have that A satisfies the following condition: $H(A(x), A(y)) \leq \varphi(\|x - y\|)$, for all $x, y \in Y$, with $x \neq y$. By Theorem 3.1 the conclusion follows. $\square$

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