STRONG CONVERGENCE THEOREMS FOR ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, we introduce a composite iteration scheme for approximating a zero point of accretive operators in the framework of uniformly smooth Banach spaces and reflexive Banach spaces which have a weakly continuous duality mapping, respectively. Our results improve and extend the corresponding results announced by many others.

Key Words and Phrases: Nonexpansive mapping, sunny and nonexpansive retraction, accretive operator.

2000 Mathematics Subject Classification: 47H09, 47H10.

1. INTRODUCTION AND PRELIMINARIES

Let E be a real Banach space. Recall that a (possibly multivalued) operator A with domain D(A) and range R(A) in E is accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i$ ($i = 1, 2$), there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1) \rangle \geq 0,$$

where $J$ is the duality map from E to the dual space $E^*$ give by

$$J(x) = \{ x^* \in E^*: \langle x, x^* \rangle = \| x \|^2 = \| x^* \|^2 \}, \quad x \in E.$$
Let $C$ be a nonempty closed convex subset of $E$, and $T : C \to C$ a mapping. Recall that $T$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in C.$$ 

A point $x \in C$ is a fixed point of $T$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{x \in C : Tx = x\}$. An accretive operator $A$ is $m$-accretive if $R(I + rA) = E$ for each $r > 0$. Throughout this article we always assume that $A$ is $m$-accretive and has a zero (i.e., the inclusion $0 \in A(z)$ is solvable). The set of zeros of $A$ is denoted by $F$. Hence,

$$F = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0).$$

For each $r > 0$, we denote by $J_r$ the resolvent of $A$, i.e., $J_r = (I + rA)^{-1}$. Note that if $A$ is $m$-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = F$ for all $r > 0$. We also denote by $A_r$ the Yosida approximation of $A$, i.e.,

$$A_r = \frac{1}{r}(I - J_r).$$

It is known that $J_r$ is a nonexpansive mapping from $X$ to $C := D(A)$ which will be assumed convex. One classical way to study nonexpansive mappings is to use contractions to approximate a fixed point of nonexpansive mappings ([2], [9]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$(1.1) \quad T_t x = tu + (1 - t)Tx, \quad x \in C,$$

where $u \in C$ is a fixed point. Banach’s Contraction Mapping Principle guarantees that $T_t$ has a unique fixed point $x_t$ in $C$. It is unclear, in general, what is the behavior of $x_t$ as $t \to 0$, even if $T$ has a fixed point. However, in the case of $T$ having a fixed point, Browder [2] proved that if $X$ is a Hilbert space, then $x_t$ converges strongly to a fixed point of $T$ that is nearest to $u$. Reich [9] extended Browder’s result to the setting of Banach spaces and proved that if $X$ is a uniformly smooth Banach space, then $x_t$ converges strongly to a fixed point of $T$ and the limit defines the (unique) sunny nonexpansive retraction from $C$ onto $F(T)$.

Recently, Kim and Xu [6] studied the sequence generated by the algorithm

$$(1.2) \quad x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)J_r x_n, \quad n \geq 0,$$

and proved strongly convergence of scheme (1.2) in the framework of uniformly smooth Banach spaces and reflexive Banach spaces which have a weak continuous duality map, respectively.
Inspired and motivated by the iterative scheme (1.2) given by Kim and Xu [6], this paper introduces the following iterative algorithm

\[
\begin{align*}
\beta_{n}^{m-1} x_n &= (1 - \beta_{n}^{m-1}) J_{\nu_n} x_n, \\
\vdots \\
\beta_{n}^{1} x_n &= (1 - \beta_{n}^{1}) J_{\nu_n} y_n, \\
x_{n+1} &= \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \\
\end{align*}
\]

where $J_{\nu_n}$ is the resolvent of $m$-accretive operator $A$ and $u \in C$ is an arbitrary (but fixed) element in $C$ and sequences $\{\alpha_n\}$ in $(0,1)$, $\{\beta_{n}^{i}\}$, $i = 1, 2, \ldots, m-1$ in $[0,1]$. Under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_{n}^{i}\}$ and $\{\nu_n\}$, that $\{x_n\}$ defined by the above iteration scheme converges to a zero point of $A$ is proved.

It is our purpose in this paper to introduce this composite iteration scheme for approximating a zero point of accretive operators in the framework of uniformly smooth Banach spaces and the reflexive Banach spaces which have a weak continuous duality map, respectively. We establish the strong convergence of the composite iteration scheme $\{x_n\}$ defined by (1.3). The results improve and extend results of Kim and Xu [6] and some others.

We need the following definitions and lemmas for the proof of our main results.

The norm of $E$ is said to be Gâteaux differentiable (and $E$ is said to be smooth) if

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for each $x, y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth ) if the limit in (1.4) is attained uniformly for $(x, y) \in U \times U$.

**Lemma 1.1.** A Banach space $E$ is uniformly smooth if and only if the duality map $J$ is the single-valued and norm-to-norm uniformly continuous on bounded sets of $E$.

**Lemma 1.2.** (The resolvent Identity [1]) For $\lambda > 0$ and $\mu > 0$ and $x \in E$,

\[
J_{\lambda} x = J_{\mu} \left( \frac{\mu}{\lambda} x + (1 - \frac{\mu}{\lambda}) J_{\lambda} x \right).
\]
Recall that if \( C \) and \( D \) are nonempty subsets of a Banach space \( E \) such that \( C \) is nonempty closed convex and \( D \subset C \), then a map \( Q : C \to D \) is sunny([5], [8]) provided \( Q(x + t(x - Q(x))) = Q(x) \) for all \( x \in C \) and \( t \geq 0 \) whenever \( x + t(x - Q(x)) \in C \). A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [4, 5, 8]: if \( E \) is a smooth Banach space, then \( Q : C \to D \) is a sunny nonexpansive retraction if and only if there holds the inequality

\[
\langle x - Qx, J(y - Qx) \rangle \leq 0 \quad \text{for all } x \in C \quad \text{and} \quad y \in D.
\]

Reich [9] showed that if \( E \) is uniformly smooth and if \( D \) is the fixed point set of a nonexpansive mapping from \( C \) into itself, then there is a sunny nonexpansive retraction from \( C \) onto \( D \) and it can be constructed as follows.

**Lemma 1.3.** (Reich [9]) Let \( E \) be a uniformly smooth Banach space and let \( T : C \to C \) be a nonexpansive mapping with a fixed point. For each fixed \( u \in C \) and \( t \in (0, 1) \), the unique fixed point \( x_t \in C \) of the contraction \( C \ni x \mapsto tu + (1 - t)x \) converges strongly as \( t \to 0 \) to a fixed point of \( T \). Define \( Q : C \to F(T) \) by \( Qu = s - \lim_{t \to 0} x_t \). Then \( Q \) is the unique sunny nonexpansive retract from \( C \) onto \( F(T) \); that is, \( Q \) satisfies the property

\[
(1.5) \quad \langle u - Qu, J(z - Qu) \rangle \leq 0, \quad u \in C, \quad z \in F(T).
\]

Recall that a gauge is a continuous strictly increasing function \( \varphi : [0, \infty) \to [0, \infty) \) such that \( \varphi(0) = 0 \) and \( \varphi(t) \to \infty \) as \( t \to \infty \). Associated to a gauge \( \varphi \) is the duality map \( J_\varphi : E \to E^* \) defined by

\[
J_\varphi(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \| x \| \varphi(\| x \|), \quad \| x^* \| = \varphi(\| x \|) \}, \quad x \in E.
\]

Following Browder [3], we say that a Banach space \( E \) has a weakly continuous duality map if there exists a gauge \( \varphi \) for which the duality map \( J_\varphi \) is single-valued and weak-to-weak* sequentially continuous (i.e., if \( \{ x_n \} \) is a sequence in \( E \) weakly convergent to a point \( x \), then the sequence \( J_\varphi(x_n) \) converges weak*ly to \( J_\varphi(x) \)). It is known that \( l^p \) has a weakly continuous duality map for all \( 1 < p < \infty \). Set

\[
\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad t \geq 0.
\]
Then
\[ J_\varphi(x) = \partial \Phi(\|x\|), \quad x \in E, \]
where \( \partial \) denotes the sub-differential in the sense of convex analysis. The first part of the next lemma is an immediate consequence of the sub-differential inequality and the proof of the second part can be found in [7].

**Lemma 1.4.** Assume that \( E \) has a weakly continuous duality map \( J_\varphi \) with gauge \( \varphi \).

(i) For all \( x, y \in E \), there holds the inequality
\[ \Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y) \rangle. \]

(ii) Assume a sequence \( x_n \) in \( X \) is weakly convergent to a point \( x \). The there holds the identity
\[ \limsup_{n \to \infty} \lim_{t \to 0^+} \Phi(\|x_n - y\|) = \limsup_{n \to \infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad x, y \in E. \]

Notation: "\( \rightharpoonup \)" stands for weak convergence and "\( \to \)" for strong convergence.

**Lemma 1.5.** [12] Let \( X \) be a reflexive Banach space and has a weakly continuous duality map \( J_\varphi(x) \) with gauge \( \varphi \). Let \( C \) be closed convex subset of \( X \) and let \( T : C \to C \) be a nonexpansive mapping. Fix \( u \in C \) and \( t \in (0, 1) \). Let \( x_t \in C \) be the unique solution in \( C \) to Eq.(1.1). Then \( T \) has a fixed point if and only if \( x_t \) remains bounded as \( t \to 0^+ \), and in this case, \( x_t \) converges as \( t \to 0^+ \) strongly to a fixed point of \( T \).

Under the condition of Lemma 1.5, we define a map \( Q : C \to F(T) \) by
\[ Q(u) := \lim_{t \to 0^+} x_t, \quad u \in C. \]

from [12 Theorem 3.2] we know \( Q \) is the sunny nonexpansive retraction from \( C \) onto \( F(T) \).

**Lemma 1.6.** In a Banach space \( E \), there holds the inequality
\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad x, y \in E, \]
where \( j(x + y) \in J(x + y) \).
Lemma 1.7. (Xu [10], [11]) Let \( \{\alpha_n\} \) be a sequence of nonnegative real numbers satisfying the condition
\[
\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n \sigma_n, \quad n \geq 0,
\]
where \( \{\gamma_n\}_{n=0}^\infty \subset (0, 1) \) and \( \{\sigma_n\}_{n=0}^\infty \) such that
\[\begin{align*}
(i) & \quad \lim_{n \to \infty} \gamma_n = 0 \text{ and } \sum_{n=0}^\infty \gamma_n = \infty, \\
(ii) & \quad \text{either } \limsup_{n \to \infty} \sigma_n \leq 0 \text{ or } \sum_{n=0}^\infty |\gamma_n\sigma_n| < \infty.
\end{align*}\]
Then \( \{\alpha_n\}_{n=0}^\infty \) converges to zero.

2. Main results

Theorem 2.1. Assume that \( E \) is a uniformly smooth Banach space and \( A \) is an \( m \)-accretive operator in \( E \). Given a point \( u \in C \), the initial guess \( x_0 \in C \) is chosen arbitrarily and given sequences \( \{\alpha_n\}_{n=0}^\infty \in (0, 1) \) \( \{\beta_i^n\}_{n=0}^\infty \), \( i = 1, 2, \ldots, m - 1 \) in \([0, 1]\), the following conditions are satisfied
\[\begin{align*}
(i) & \quad \sum_{n=0}^\infty \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty; \\
(ii) & \quad r_n \geq \epsilon, \forall n \geq 0 \text{ and } \beta_i^n + (1 + \beta_i^n) \sum_{k=2}^{m-1} \prod_{i=2}^k (1 - \beta_i^n) < a < 1, \quad \text{for some } a \in (0, 1); \\
(iii) & \quad \sum_{n=0}^\infty |\beta_i^{n+1} - \beta_i^n| < \infty, \text{ for } i = 1, \ldots, m-1 \text{ and } \sum_{n=0}^\infty |r_n - r_{n-1}| < \infty.
\end{align*}\]

Let \( \{x_n\}_{n=1}^\infty \) be the composite process defined by
\[
\begin{cases}
y_n^{m-1} = \beta_n^{m-1}x_n + (1 - \beta_n^{m-1})J_{r_n}x_n, \\
\vdots \\
y_1^n = \beta_1^n x_n + (1 - \beta_1^n)J_{r_n}y_1^n, \\
x_{n+1} = \alpha_n u + (1 - \alpha_n)y_1^n, \quad n \geq 0.
\end{cases}
\]

Then \( \{x_n\}_{n=1}^\infty \) converges strongly to a zero point of \( A \).

Proof. First we observe that \( \{x_n\}_{n=0}^\infty \) is bounded. Indeed, taking a fixed point \( p \) of \( T \), we have
\[
\|y_n^{m-1} - p\| \leq \beta_n^{m-1}\|x_n - p\| + (1 - \beta_n^{m-1})\|J_{r_n}x_n - p\| \leq \|x_n - p\|.
\]
It follows from (1.3) and (2.1) that
\[\|y^m_n - p\| \leq \beta_n^{m-2}\|x_n - p\| + (1 - \beta_n^{m-2})\|J_n y_n^{m-1} - p\|\]
\[\leq \beta_n^{m-2}\|x_n - p\| + (1 - \beta_n^{m-2})\|y_n^{m-1} - p\|\]
\[\leq \beta_n^{m-2}\|x_n - p\| + (1 - \beta_n^{m-2})\|x_n - p\|\]
\[\leq \|x_n - p\|\]
(2.2)
In a similar way, we obtain
\[\|y^i_n - p\| \leq \beta_n^i\|x_n - p\| + (1 - \beta_n^i)\|J_n y_n^{i+1} - p\|\]
\[\leq \beta_n^i\|x_n - p\| + (1 - \beta_n^i)\|y_n^{i+1} - p\|\]
\[\leq \beta_n^i\|x_n - p\| + (1 - \beta_n^i)\|x_n - p\|\]
\[\leq \|x_n - p\|, \quad \text{for } i = 1, \cdots, m - 2.\]
(2.3)
Therefore, we have
\[\|x_{n+1} - p\| = \|\alpha_n(u - p) + (1 - \alpha_n)(y^1_n - p)\|\]
\[\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|y^1_n - p\|\]
\[\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|\]
\[\leq \max\{\|u - p\|, \|x_n - p\|\}.
\]
Now, an induction yields
(2.4) \[\|x_n - p\| \leq \max\{\|u - p\|, \|x_0 - p\|\}, \quad n \geq 0.\]
This implies that \(\{x_n\}\) is bounded, so are \(\{y^i_n\}\), \(i = 1, \ldots, m - 1\). It follows from (1.3) and condition (i) that
\[\|x_{n+1} - y^1_n\| \leq \alpha_n\|u - y^1_n\| \to 0, \quad \text{as } n \to 0.\]
(2.5)
Next, we claim that
(2.6) \[\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.\]
In order to prove (2.6), we consider
\[
\begin{cases}
y^m_n = \beta_n^{m-1}x_n + (1 - \beta_n^{m-1})J_n x_n, \\
y^m_{n-1} = \beta_n^{m-1}x_{n-1} + (1 - \beta_n^{m-1})J_n x_{n-1}.
\end{cases}
\]
It follows that
\[ y_n^{m-1} - y_{n-1}^{m-1} = (1 - \beta_n^{m-1})(J_{r_n}x_n - J_{r_n-1}x_{n-1}) + \beta_n^{m-1}(x_n - x_{n-1}) + (\beta_n^{m-1} - \beta_n^{m-1})(x_{n-1} - J_{r_n-1}x_{n-1}). \]

It follows that
\[ (2.7) \quad \|y_n^{m-1} - y_{n-1}^{m-1}\| \leq (1 - \beta_n^{m-1})\|J_{r_n}x_n - J_{r_n-1}x_{n-1}\| + \beta_n^{m-1}\|x_n - x_{n-1}\| + |\beta_n^{m-1} - \beta_n^{m-1}||x_{n-1} - J_{r_n-1}x_{n-1}|. \]

From Lemma 1.2, the resolvent identity implies that
\[ J_{r_n}x_n = J_{r_n-1}(\frac{r_n-1}{r_n}x_n + (1 - \frac{r_n-1}{r_n})J_{r_n}x_n). \]

If \( r_{n-1} \leq r_n \), which in turn implies that
\[ (2.8) \quad \|J_{r_n}x_n - J_{r_n-1}x_{n-1}\| \leq \|\frac{r_n-1}{r_n}x_n + (1 - \frac{r_n-1}{r_n})J_{r_n}x_n - x_{n-1}\| \]
\[ \leq \|\frac{r_n-1}{r_n}(x_n - x_{n-1}) + (1 - \frac{r_n-1}{r_n})J_{r_n}x_n - x_{n-1}\| \]
\[ \leq \|x_n - x_{n-1}\| + \frac{(r_n - r_n-1)}{r_n}\|J_{r_n}x_n - x_{n-1}\| \]
\[ \leq \|x_n - x_{n-1}\| + \frac{(r_n - r_n-1)}{\epsilon}\|J_{r_n}x_n - x_{n-1}\|. \]

Substitute (2.8) into (2.7) yields that
\[ (2.9) \quad \|y_n^{m-1} - y_{n-1}^{m-1}\| \]
\[ \leq (1 - \beta_n^{m-1})(\|x_n - x_{n-1}\| + \frac{(r_n - r_n-1)}{\epsilon}\|J_{r_n}x_n - x_{n-1}\|) + \beta_n^{m-1}\|x_n - x_{n-1}\| + |\beta_n^{m-1} - \beta_n^{m-1}||x_{n-1} - J_{r_n-1}x_{n-1}| \]
\[ \leq \|x_n - x_{n-1}\| + M_1(|r_n - r_n-1| + |\beta_n^{m-1} - \beta_n^{m-1}|), \]
where \( M_1 \) is an appropriate constant such that
\[ M_1 > \max\{\frac{\|J_{r_n}x_n - x_{n-1}\|}{\epsilon}, \|x_{n-1} - J_{r_n-1}x_{n-1}\|\}. \]

Similarly, From (1.3) we obtain
\[
\begin{align*}
\begin{cases}
y_n^{m-2} = \beta_n^{m-2}x_n + (1 - \beta_n^{m-2})J_{r_n}y_n^{m-1}, \\
y_{n-1}^{m-2} = \beta_n^{m-2}x_{n-1} + (1 - \beta_n^{m-2})J_{r_n-1}y_{n-1}^{m-1}.
\end{cases}
\end{align*}
\]
It follows that
\[ y_n^{m-2} - y_{n-1}^{m-2} = (1 - \beta_n^{m-2})(J_{r_n}y_n^{m-1} - J_{r_{n-1}}y_{n-1}^{m-1}) + \beta_n^{m-2}(x_n - x_{n-1}) + (\beta_n^{m-2} - \beta_{n-1}^{m-2})(x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-1}), \]
which yields that
\[
\|y_n^{m-2} - y_{n-1}^{m-2}\| \\
\leq (1 - \beta_n^{m-2})\|J_{r_n}y_n^{m-1} - J_{r_{n-1}}y_{n-1}^{m-1}\| + \beta_n^{m-2}\|x_n - x_{n-1}\| \\
+ |\beta_n^{m-2} - \beta_{n-1}^{m-2}|\|x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-1}\|. 
\]
(2.10)
Similar to (2.8), we can get
\[
\|J_{r_n}y_n^{m-1} - J_{r_{n-1}}y_{n-1}^{m-1}\| \leq \|y_n^{m-1} - y_{n-1}^{m-1}\| + \left(\frac{r_n - r_{n-1}}{\epsilon}\right)\|J_{r_n}y_n^{m-1} - y_{n-1}^{m-1}\|. 
\]
(2.11)
Combine (2.9) with (2.11) yields that
\[
\|J_{r_n}y_n^{m-1} - J_{r_{n-1}}y_{n-1}^{m-1}\| \\
\leq \|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}|) \\
+ \left(\frac{r_n - r_{n-1}}{\epsilon}\right)\|J_{r_n}y_n^{m-1} - y_{n-1}^{m-1}\|. 
\]
(2.12)
Substituting (2.12) into (2.10), we obtain
\[
\|y_n^{m-2} - y_{n-1}^{m-2}\| \\
\leq (1 - \beta_n^{m-2})(\|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}|) \\
+ \left(\frac{r_n - r_{n-1}}{\epsilon}\right)\|J_{r_n}y_n^{m-1} - y_{n-1}^{m-1}\| + \beta_n^{m-2}\|x_n - x_{n-1}\| \\
+ |\beta_n^{m-2} - \beta_{n-1}^{m-2}|\|x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-1}\| \\
\leq \|x_n - x_{n-1}\| + M_2(2|r_n - r_{n-1}| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}| + |\beta_n^{m-2} - \beta_{n-1}^{m-2}|),
\]
where \(M_2\) is an appropriate constant such that
\[
M_2 > \max\left\{\frac{\|J_{r_n}y_n^{m-1} - y_{n-1}^{m-1}\|}{\epsilon}, \|x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-1}\|, \ M_1\right\}.
\]
In this fashion, it is easy to get that
\[
\|y_n^{m-1} - y_{n-1}^{m-1}\| \leq \|x_n - x_{n-1}\| + M_1(\sum_{j=1}^{i} |\beta_n^{m-j} - \beta_{n-1}^{m-j}| + i|r_n - r_{n-1}|),
\]
(2.14)
where $M_i$ is an appropriate constant such that

$$
M_i > \max\left\{ \frac{\| J_r y_n^{m-(i-1)} - y_{n-1}^m \|}{\epsilon}, \| x_{n-1} - J_{n-1} y_{n-1}^{m-(i-1)} \|, M_{i-1} \right\}
$$

for all $2 \leq i \leq (m-1)$. Therefore, one can easily see that

(2.15)

$$
\| y_n^1 - y_{n-1}^1 \| \leq \| x_n - x_{n-1} \|
$$

$$
+ M_{m-1} \left( \sum_{j=1}^{m-1} | \beta_n^{m-j} - \beta_{n-1}^{m-j} | + (m-1)|r_n - r_{n-1}| \right),
$$

On the other hand, observe that

$$
x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n^1, \quad x_n = \alpha_{n-1} u + (1 - \alpha_{n-1})y_{n-1}^1.
$$

It follows that

(2.16)

$$
x_{n+1} - x_n = (1 - \alpha_n)(y_n^1 - y_{n-1}^1) + (\alpha_n - \alpha_{n-1})(u - y_{n-1}^1).
$$

It follows from (2.15) that

(2.17)

$$
\| x_{n+1} - x_n \|
$$

$$
\leq (1 - \alpha_n)\| y_n^1 - y_{n-1}^1 \| + |\alpha_n - \alpha_{n-1}|\| u - y_{n-1}^1 \|
$$

$$
\leq (1 - \alpha_n)(\| x_n - x_{n-1} \| + M_{m-1} \left( \sum_{j=1}^{m-1} | \beta_n^{m-j} - \beta_{n-1}^{m-j} | + (m-1)|r_n - r_{n-1}| \right))
$$

$$
+ |\alpha_n - \alpha_{n-1}|\| u - y_{n-1}^1 \|
$$

$$
\leq (1 - \alpha_n)\| x_n - x_{n-1} \|
$$

$$
+ M(|\alpha_n - \alpha_{n-1}| + \sum_{j=1}^{m-1} | \beta_n^{m-j} - \beta_{n-1}^{m-j} | + (m-1)|r_n - r_{n-1}|),
$$

where $M$ is an appropriate constant such that

$$
M \geq \max\{ \| u - y_{n-1}^1 \|, M_{m-1} \}
$$

for all $n$. Similarly we can prove (2.12) if $r_{n-1} \geq r_n$, by assumptions(i)-(iii), we have that

$$
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.
$$
and
\[ \sum_{n=1}^{\infty} \left( |\alpha_n - \alpha_{n-1}| + \sum_{j=1}^{m-1} |\beta_n^{m-j} - \beta_{n-1}^{m-j}| + (m-1)|r_n - r_{n-1}| \right) < \infty. \]

Hence, Lemma 5 is applicable to (2.17) and we obtain
\[ (2.18) \quad \|x_{n+1} - x_n\| \to 0, \quad \text{as } n \to \infty \]

On the other hand, from (1.3) we have
\[ \|y_n - J_{r_n}x_n\| \leq \|x_n - J_{r_n}y_n\| + \|y_n - J_{r_n}x_n\| \]
\[ \leq \beta_n \|x_n - J_{r_n}y_n\| + \|y_n - x_n\| \]
\[ \leq \beta_n \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_{r_n}y_n\| + \|y_n - x_n\| \]
\[ \leq \beta_n \|x_n - J_{r_n}x_n\| + \beta_n \|J_{r_n}x_n - J_{r_n}y_n\| + \|y_n - x_n\| \]
\[ \leq \beta_n \|x_n - J_{r_n}x_n\| + (1 + \beta_n) \|x_n - y_n\| \]
\[ \leq \beta_n \|x_n - J_{r_n}x_n\| + (1 + \beta_n)(1 - \beta_n) \|x_n - J_{r_n}y_n\| \]
\[ \vdots \]
\[ \leq (\beta_n + (1 + \beta_n) m - 1 \prod_{i=2}^{m-1} (1 - \beta_n))^n \|x_n - J_{r_n}x_n\|. \]

It follows that
\[ (2.19) \quad \|J_{r_n}x_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - J_{r_n}x_n\| \]
\[ \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \]
\[ + (\beta_n + (1 + \beta_n) m - 1 \prod_{i=2}^{m-1} (1 - \beta_n))^n \|x_n - J_{r_n}x_n\|. \]

From condition (ii), (2.5) and (2.18) we obtain
\[ (2.20) \quad \lim_{n \to \infty} \|J_{r_n}x_n - x_n\| = 0. \]

Take a fixed number \( r \) such that \( \epsilon > r > 0 \), from Lemma 1.2 we obtain
\[ \|J_{r_n}x_n - J_rx_n\| = \|J_r(\frac{r}{r_n}x_n + (1 - \frac{r}{r_n})J_{r_n}x_n) - J_rx_n\| \]
\[ \leq (1 - \frac{r}{r_n})\|x_n - J_{r_n}x_n\| \]
\[ \leq \|x_n - J_{r_n}x_n\|. \]
Therefore, we have
\[
\|x_n - J_r x_n\| \leq \|x_n - J_{r_n} x_n\| + \|J_{r_n} x_n - J_r x_n\|
\]
(2.22)
\[
\leq \|J_{r_n} x_n - x_n\| + \|J_{r_n} x_n - x_n\|
\]
\[
\leq 2\|J_{r_n} x_n - x_n\|.
\]
Hence, we obtain
\[
\lim_{n \to \infty} \|x_n - J_r x_n\| = 0.
\]
Since in a uniformly smooth Banach space, the sunny nonexpansive retract \(Q\) from \(E\) onto the fixed point set \(F(J_r)(= F = A^{-1}(0))\) of \(J_r\) is unique, it must be obtained from Lemma 1.3. Namely, \(Q u = s - \lim_{t \to 0} z_t, \quad u \in E\), where \(t \in (0, 1)\) and \(z_t\) solves the fixed point equation
\[
z_t = tu + (1 - t)J_r z_t.
\]
Next, we claim that
(2.23) \[\limsup_{n \to \infty} \langle u - Q(u), J(x_n - Q(u)) \rangle \leq 0.\]
Thus we have
\[
\|z_t - x_n\| = \|(1 - t)(J_r z_t - x_n) + t(u - x_n)\|.
\]
It follows from Lemma 1.6 that
(2.24) \[
\|z_t - x_n\|^2 \leq (1 - t)^2 \|J_r z_t - x_n\|^2 + 2t \langle u - x_n, J(z_t - x_n) \rangle
\]
\[
\leq (1 - 2t + t^2)\|z_t - x_n\|^2 + f_n(t)
\]
\[
+ 2t(u - z_t, J(z_t - x_n)) + 2t\|z_t - x_n\|^2,
\]
where
(2.25) \[
f_n(t) = (2\|z_t - x_n\| + \|x_n - J_r x_n\|)\|x_n - J_r x_n\| \to 0, \text{ as } n \to 0.
\]
It follows that
(2.26) \[\langle z_t - u, J(z_t - x_n) \rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}f_n(t).\]
Letting \(n \to \infty\) in (2.26) and noting (2.25), we obtain
(2.27) \[\limsup_{n \to \infty} \langle z_t - u, J(z_t - x_n) \rangle \leq \frac{t}{2} M,
\]
where \( M > 0 \) is an appropriate constant such that \( M \geq \|z_t - x_n\|^2 \) for all \( t \in (0, 1) \) and \( n \geq 1 \). Letting \( t \to 0 \) and from (2.27), we have

\[
\limsup_{t \to 0} \limsup_{n \to \infty} (z_t - u, J(z_t - x_n)) \leq 0.
\]

So, for any \( \epsilon > 0 \), there exists a positive number \( \delta_1 \) such that, for \( t \in (0, \delta_1) \) we get

\[
\limsup_{n \to \infty} (z_t - u, J(z_t - x_n)) \leq \frac{\epsilon}{2}.
\]

On the other hand, since \( z_t \to q \) as \( t \to 0 \), from Lemma 1.1, there exists \( \delta_2 > 0 \) such that, for \( t \in (0, \delta_2) \) we have

\[
|\langle u - q, J(x_n - q) \rangle - \langle z_t - u, J(z_t - x_n) \rangle| \\
\leq |\langle u - q, J(x_n - q) \rangle - \langle u - q, J(x_n - z_t) \rangle| \\
+ |\langle u - q, J(x_n - z_t) \rangle - \langle z_t - u, J(z_t - x_n) \rangle| \\
\leq |\langle u - q, J(x_n - q) - J(x_n - z_t) \rangle| + |\langle z_t - q, J(x_n - z_t) \rangle| \\
\leq \|u - q\|\|J(x_n - q) - J(x_n - z_t)\| + \|z_t - q\|\|x_n - z_t\| < \frac{\epsilon}{2}.
\]

Choosing \( \delta = \min\{\delta_1, \delta_2\}, \forall t \in (0, \delta) \), we have

\[
\langle u - Q(u), J(x_n - Q(u)) \rangle \leq \langle z_t - u, J(z_t - x_n) \rangle + \frac{\epsilon}{2}.
\]

That is,

\[
\limsup_{n \to \infty} \langle u - Q(u), J(x_n - Q(u)) \rangle \leq \limsup_{n \to \infty} (z_t - u, J(z_t - x_n)) + \frac{\epsilon}{2}.
\]

It follows from (2.28) that

\[
\limsup_{n \to \infty} \langle u - Q(u), J(x_n - Q(u)) \rangle \leq \epsilon.
\]

Since \( \epsilon \) is chosen arbitrarily, we have

\[
\limsup_{n \to \infty} \langle u - Q(u), J(x_n - Q(u)) \rangle \leq 0.
\]

Finally, we show that \( x_n \to Q(u) \) strongly and this concludes the proof. Observe that

\[
\|x_{n+1} - Q(u)\|^2 = \|(1 - \alpha_n)(y_n - Q(u)) + \alpha_n(u - Q(u))\|^2 \\
\leq (1 - \alpha_n)^2\|y_n - Q(u)\|^2 + 2\alpha_n(u - Q(u), J(x_{n+1} - Q(u))) \\
\leq (1 - \alpha_n)\|x_n - Q(u)\|^2 + 2\alpha_n(u - Q(u), J(x_{n+1} - Q(u))).
\]
Now we apply Lemma 1.7 and use (2.29) to see that \( \|x_n - Q(u)\| \to 0 \) as \( n \to \infty \).

**Theorem 2.2.** Suppose that \( E \) is reflexive and has a weakly continuous duality map \( J_\varphi \) with gauge \( \varphi \). Suppose that \( A \) is an \( m \)-accretive operator in \( X \) such that \( C = D(A) \) is convex, \( \{x_n\}_{n=0}^\infty \{\alpha_n\}_{n=0}^\infty \) and \( \{\beta_n^i\}_{n=0}^\infty \), \( i = 1, 2, \ldots, m-1 \) are as Theorem 2.1. Then \( \{x_n\}_{n=1}^\infty \) converges strongly to a zero point of \( A \).

**Proof.** We only include the differences. From Theorem 2.1 we obtain

\[
\|x_{n+1} - J_{r_n}x_n\| = \|x_{n+1} - y_n^1\| + \|y_n^1 - J_{r_n}x_n\| \\
\leq \alpha_n\|u - y_n\| + (\beta_n^1 + (1 + \beta_n^1) \sum_{k=2}^{m-1} \prod_{i=2}^k (1 - \beta_n^i)) \|x_n - J_{r_n}x_n\|.
\]

That is,

\[
(2.30) \quad \lim_{n \to \infty} \|x_{n+1} - J_{r_n}x_n\| = 0.
\]

We next prove that

\[
(2.31) \quad \limsup_{n \to \infty} \langle u - Q(u), J_\varphi(x_n - Q(u)) \rangle \leq 0.
\]

By Lemma 1.5, we have the sunny nonexpansive retraction \( Q : C \to F(T) \).

Take a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that

\[
(2.32) \quad \limsup_{n \to \infty} \langle u - Q(u), J_\varphi(x_n - Q(u)) \rangle = \lim_{k \to \infty} \langle u - Q(u), J_\varphi(x_{n_k} - Q(u)) \rangle.
\]

Since \( X \) is reflexive, we may further assume that \( x_{n_k} \rightharpoonup \tilde{x} \). Moreover, since \( \|x_{n+1} - J_{r_n}\| \to 0 \),

we obtain

\[
J_{r_{n_k-1}}x_{n_k-1} \rightharpoonup \tilde{x}.
\]

Taking the limit as \( k \to \infty \) in the relation

\[
[J_{r_{n_k-1}}x_{n_k-1}, A_{r_{n_k-1}}x_{n_k-1}] \in A,
\]

we get \( [\tilde{x}, 0] \in A \). That is, \( \tilde{x} \in F \). Hence by (2.32) and (1.5) we have

\[
\limsup_{n \to \infty} \langle u - Q(u), J_\varphi(x_n - Q(u)) \rangle = \langle u - Q(u), J_\varphi(\tilde{x} - Q(u)) \rangle \leq 0.
\]
That is (2.31) holds. Finally to prove that $x_n \to p$. It follows from (2.2) and (2.3) that
\[
\Phi(\|y_n^1 - p\|) = \Phi(\|\beta_n^1(x_n - p) + (1 - \beta_n^1)(J_{r_n}y_n^2 - p)\|)
\leq \Phi(\|\beta_n\|x_n - p\| + (1 - \beta_n)\|J_{r_n}y_n^2 - p\|)
\leq \Phi(\|x_n - p\|).
\]
(2.33)

Therefore, from (2.33) we obtain
\[
\Phi(\|x_{n+1} - p\|) = \Phi(\|\alpha_n(u - p) + (1 - \alpha_n)(y_n^1 - p)\|)
\leq \Phi((1 - \alpha_n)\|y_n^1 - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p) \rangle
\leq (1 - \alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p) \rangle.
\]

An application of Lemma 1.3 yields that $\Phi(\|x_n - p\|) \to 0$; that is $\|x_n - p\| \to 0$ as $n \to \infty$. This completes the proof.

**Remark 2.3.** Theorem 2.1 and Theorem 2.2 improve Kim and Xu [6] and Xu [12] as a special case. We note that our theorems in this paper carry over trivially to the so-called viscosity approximation methods.

**References**


*Received: June 27, 2007; Accepted: January 18, 2008.*