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STRONG CONVERGENCE THEOREMS FOR ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, we introduce a composite iteration scheme for approximating a zero point of accretive operators in the framework of uniformly smooth Banach spaces and reflexive Banach spaces which have a weakly continuous duality mapping, respectively. Our results improve and extend the corresponding results announced by many others.
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1. Introduction and Preliminaries

Let E be a real Banach space. Recall that a (possibly multivalued) operator A with domain $D(A)$ and range $R(A)$ in E is accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i (i = 1, 2)$, there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \geq 0,$$

where J is the duality map from E to the dual space E^* give by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x^2\| = \|x^*\|^2\}, \quad x \in E.$$

Let C be a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. An accretive operator A is m-accretive if $R(I + rA) = E$ for each $r > 0$. Throughout this article we always assume that A is m-accretive and has a zero (i.e., the inclusion $0 \in A(z)$ is solvable). The set of zeros of A is denoted by F. Hence,

$$F = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = F$ for all $r > 0$. We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is known that J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$ which will be assumed convex. One classical way to study nonexpansive mappings is to use contractions to approximate a fixed point of nonexpansive mappings ([2], [9]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad x \in C, \tag{1.1}$$

where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [2] proved that if X is a Hilbert space, then x_t converges strongly to a fixed point of T that is nearest to u. Reich [9] extended Browder's result to the setting of Banach spaces and proved that if X is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Recently, Kim and Xu [6] studied the sequence generated by the algorithm

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0, \tag{1.2}$$

and proved strongly convergence of scheme (1.2) in the framework of uniformly smooth Banach spaces and reflexive Banach spaces which have a weak continuous duality map, respectively.

Inspired and motivated by the iterative scheme (1.2) given by Kim and Xu [6], this paper introduces the following iterative algorithm

$$\begin{cases} y_n^{m-1} = \beta_n^{m-1} x_n + (1-\beta_n^{m-1}) J_{r_n} x_n, \\ \quad\quad \vdots \\ y_n^1 = \beta_n^1 x_n + (1-\beta_n^1) J_{r_n} y_n^1, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n^1, \quad n \geq 0, \end{cases} \tag{1.3}$$

where J_{r_n} is the resolvent of m-accretive operator A and $u \in C$ is an arbitrary (but fixed) element in C and sequences $\{\alpha_n\}$ in (0,1), $\{\beta_n^i\}$, $i = 1,\ 2,\ \ldots,\ m-1$ in [0,1]. Under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n^i\}$ and $\{r_n\}$, that $\{x_n\}$ defined by the above iteration scheme converges to a zero point of A is proved.

It is our purpose in this paper to introduce this composite iteration scheme for approximating a zero point of accretive operators in the framework of uniformly smooth Banach spaces and the reflexive Banach spaces which have a weak continuous duality map, respectively. We establish the strong convergence of the composite iteration scheme $\{x_n\}$ defined by (1.3). The results improve and extend results of Kim and Xu [6] and some others.

We need the following definitions and lemmas for the proof of our main results.

The norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.4}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit in (1.4) is attained uniformly for $(x, y) \in U \times U$.

Lemma 1.1. *A Banach space E is uniformly smooth if and only if the duality map J is the single-valued and norm-to-norm uniformly continuous on bounded sets of E.*

Lemma 1.2. (The resolvent Identity [1]) *For $\lambda > 0$ and $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda} x + (1 - \frac{\mu}{\lambda}) J_\lambda x\Big).$$

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny([5], [8]) provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [4, 5, 8]: if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0 \quad \text{for all} \quad x \in C \quad \text{and} \quad y \in D.$$

Reich [9] showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Lemma 1.3. (Reich [9]) *Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t \to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies the property*

$$\langle u - Qu, J(z - Qu)\rangle \leq 0, \quad u \in C, z \in F(T). \tag{1.5}$$

Recall that a gauge is a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : E \to E^*$ defined by

$$J_\varphi(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\},\ x \in E.$$

Following Browder [3], we say that a Banach space E has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weak*ly to J_φ). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d_\tau, \quad t \geq 0.$$

Then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in E,$$

where ∂ denotes the sub-differential in the sense of convex analysis, The first first part of the next lemma is an immediate consequence of the sub-differential inequality an the proof of the second part can be found in [7]

Lemma 1.4. *Assume that E has a weakly continuous duality map J_φ with gauge φ.*

(i) *For all $x, y \in E$, there holds the inequality*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle.$$

(ii) *Assume a sequence x_n in X is weakly convergent to a point x. The there holds the identity*

$$\limsup \lim_{n\to\infty} \Phi(\|x_n - y\|) = \limsup \lim_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|),\ x, y \in E.$$

Notation: " $\rightharpoonup$ " stands for weak convergence and " $\rightarrow$ " for strong convergence.

Lemma 1.5. [12] *Let X be a reflexive Banach space and has a weakly continuous duality map $J_\varphi(x)$ with gauge φ. Let C be closed convex subset of X and let $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique solution in C to Eq.(1.1). Then T has a fixed point if and only if x_t remains bounded as $t \to 0^+$, and in this case, x_t converges as $t \to 0^+$ strongly to a fixed point of T.*

Under the condition of Lemma 1.5, we define a map $Q : C \to F(T)$ by

$$Q(u) := \lim_{t\to 0} x_t, \quad u \in C.$$

from [12 Theorem 3.2] we know Q is the sunny nonexpansive retraction from C onto $F(T)$.

Lemma 1.6. *In a Banach space E, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 1.7. (Xu [10], [11]) *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that

(i) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n = \infty$,
(ii) *either* $\limsup_{n\to\infty}\sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

2. Main results

Theorem 2.1. *Assume that E is a uniformly smooth Banach space and A is an m-accretive operator in E. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ in (0,1) $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1, 2, \ldots, m-1$ in [0, 1], the following conditions are satisfied*

(i) $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$;
(ii) $r_n \geq \epsilon$, $\forall n \geq 0$ *and* $\beta_n^1 + (1+\beta_n^1)\sum_{k=2}^{m-1}\prod_{i=2}^{k}(1-\beta_n^i) < a < 1$, *for some* $a \in (0,1)$;
(iii) $\sum_{n=0}^{\infty}|\beta_{n+1}^i - \beta_n^i| < \infty$, *for* $i = 1, \cdots, m-1$ *and* $\sum_{n=0}^{\infty}|r_n - r_{n-1}| < \infty$.

Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by

$$\begin{cases} y_n^{m-1} = \beta_n^{m-1}x_n + (1-\beta_n^{m-1})J_{r_n}x_n, \\ \qquad\vdots \\ y_n^1 = \beta_n^1 x_n + (1-\beta_n^1)J_{r_n}y_n^1, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n^1, \quad n \geq 0. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a zero point of A.

Proof. First we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, taking a fixed point p of T, we have

$$\|y_n^{m-1} - p\| \leq \beta_n^{m-1}\|x_n - p\| + (1-\beta_n^{m-1})\|J_{r_n}x_n - p\| \leq \|x_n - p\|. \tag{2.1}$$

It follows from (1.3) and (2.1) that

$$\begin{aligned} \|y_n^{m-2} - p\| &\le \beta_n^{m-2}\|x_n - p\| + (1 - \beta_n^{m-2})\|J_{r_n} y_n^{m-1} - p\| \\ &\le \beta_n^{m-2}\|x_n - p\| + (1 - \beta_n^{m-2})\|y_n^{m-1} - p\| \\ &\le \beta_n^{m-2}\|x_n - p\| + (1 - \beta_n^{m-2})\|x_n - p\| \\ &\le \|x_n - p\| \end{aligned} \tag{2.2}$$

In a similar way, we obtain

$$\begin{aligned} \|y_n^{i} - p\| &\le \beta_n^{i}\|x_n - p\| + (1 - \beta_n^{i})\|J_{r_n} y_n^{i+1} - p\| \\ &\le \beta_n^{i}\|x_n - p\| + (1 - \beta_n^{i})\|y_n^{i+1} - p\| \\ &\le \beta_n^{i}\|x_n - p\| + (1 - \beta_n^{i})\|x_n - p\| \\ &\le \|x_n - p\|, \quad \text{for } i = 1, \cdots,\ m-2. \end{aligned} \tag{2.3}$$

Therefore, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(u - p) + (1 - \alpha_n)(y_n^1 - p)\| \\ &\le \alpha_n\|u - p\| + (1 - \alpha_n)\|y_n^1 - p\|\| \\ &\le \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|\| \\ &\le \max\{\|u - p\|, \|x_n - p\|\}. \end{aligned}$$

Now, an induction yields

$$\|x_n - p\| \le \max\{\|u - p\|,\ \|x_0 - p\|\}, \quad n \ge 0. \tag{2.4}$$

This implies that $\{x_n\}$ is bounded, so are $\{y_n^i\}, \quad i = 1, \ldots, m-1$. It follows from (1.3) and condition (i) that

$$\|x_{n+1} - y_n^1\| \le \alpha_n\|u - y_n^1\| \to 0, \quad \text{as } n \to 0. \tag{2.5}$$

Next, we claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.6}$$

In order to prove (2.6), we consider

$$\begin{cases} y_n^{m-1} = \beta_n^{m-1} x_n + (1 - \beta_n^{m-1}) J_{r_n} x_n, \\ y_{n-1}^{m-1} = \beta_{n-1}^{m-1} x_{n-1} + (1 - \beta_{n-1}^{m-1}) J_{r_{n-1}} x_{n-1}. \end{cases}$$

It follows that

$$y_n^{m-1} - y_{n-1}^{m-1} = (1-\beta_n^{m-1})(J_{r_n}x_n - J_{r_{n-1}}x_{n-1}) + \beta_n^{m-1}(x_n - x_{n-1}) + (\beta_n^{m-1} - \beta_{n-1}^{m-1})(x_{n-1} - J_{r_{n-1}}x_{n-1}).$$

It follows that

$$\|y_n^{m-1} - y_{n-1}^{m-1}\| \leq (1-\beta_n^{m-1})\|J_{r_n}x_n - J_{r_{n-1}}x_{n-1}\| + \beta_n^{m-1}\|x_n - x_{n-1}\| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}|\|x_{n-1} - J_{r_{n-1}}x_{n-1}\|. \tag{2.7}$$

From Lemma 1.2, the resolvent identity implies that

$$J_{r_n}x_n = J_{r_{n-1}}(\frac{r_{n-1}}{r_n}x_n + (1-\frac{r_{n-1}}{r_n})J_{r_n}x_n).$$

If $r_{n-1} \leq r_n$, which in turn implies that

$$\begin{aligned}\|J_{r_n}x_n - J_{r_{n-1}}x_{n-1}\| &\leq \|\frac{r_{n-1}}{r_n}x_n + (1-\frac{r_{n-1}}{r_n})J_{r_n}x_n - x_{n-1}\| \\ &\leq \|\frac{r_{n-1}}{r_n}(x_n - x_{n-1}) + (1-\frac{r_{n-1}}{r_n})(J_{r_n}x_n - x_{n-1})\| \\ &\leq \|x_n - x_{n-1}\| + (\frac{r_n - r_{n-1}}{r_n})\|J_{r_n}x_n - x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + (\frac{r_n - r_{n-1}}{\epsilon})\|J_{r_n}x_n - x_{n-1}\|.\end{aligned} \tag{2.8}$$

Substitute (2.8) into (2.7) yields that

$$\begin{aligned}&\|y_n^{m-1} - y_{n-1}^{m-1}\| \\ &\leq (1-\beta_n^{m-1})(\|x_n - x_{n-1}\| + (\frac{r_n - r_{n-1}}{\epsilon})\|J_{r_n}x_n - x_{n-1}\|) \\ &\quad + \beta_n^{m-1}\|x_n - x_{n-1}\| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}|\|x_{n-1} - J_{r_{n-1}}x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}|),\end{aligned} \tag{2.9}$$

where M_1 is an appropriate constant such that

$$M_1 > \max\{\frac{\|J_{r_n}x_n - x_{n-1}\|}{\epsilon},\ \|x_{n-1} - J_{r_{n-1}}x_{n-1}\|\}.$$

Similarly, From (1.3) we obtain

$$\begin{cases} y_n^{m-2} = \beta_n^{m-2}x_n + (1-\beta_n^{m-2})J_{r_n}y_n^{m-1}, \\ y_{n-1}^{m-2} = \beta_{n-1}^{m-2}x_{n-1} + (1-\beta_{n-1}^{m-2})J_{r_{n-1}}y_{n-1}^{m-1}. \end{cases}$$

It follows that

$$y_n^{m-2} - y_{n-1}^{m-2} = (1-\beta_n^{m-2})(J_{r_n}y_n^{m-1} - J_{r_{n-1}}y_{n-1}^{m-1}) + \beta_n^{m-2}(x_n - x_{n-1}) + (\beta_n^{m-2} - \beta_{n-1}^{m-2})(x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-1}),$$

which yields that

$$\begin{aligned}&\|y_n^{m-2} - y_{n-1}^{m-2}\| \\ &\le (1-\beta_n^{m-2})\|J_{r_n}y_n^{m-1} - J_{r_{n-1}}y_{n-1}^{m-1}\| + \beta_n^{m-2}\|x_n - x_{n-1}\| \\ &\quad + |\beta_n^{m-2} - \beta_{n-1}^{m-2}|\|x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-1}\|.\end{aligned} \tag{2.10}$$

Similar to (2.8), we can get

$$\|J_{r_n}y_n^{m-1} - J_{r_{n-1}}y_{n-1}^{m-1}\| \le \|y_n^{m-1} - y_{n-1}^{m-1}\| + (\frac{r_n - r_{n-1}}{\epsilon})\|J_{r_n}y_n^{m-1} - y_{n-1}^{m-1}\|. \tag{2.11}$$

Combine (2.9) with (2.11) yields that

$$\begin{aligned}&\|J_{r_n}y_n^{m-1} - J_{r_{n-1}}y_{n-1}^{m-1}\| \\ &\le \|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}|) \\ &\quad + (\frac{r_n - r_{n-1}}{\epsilon})\|J_{r_n}y_n^{m-1} - y_{n-1}^{m-1}\|.\end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.10), we obtain

$$\begin{aligned}&\|y_n^{m-2} - y_{n-1}^{m-2}\| \\ &\le (1-\beta_n^{m-2})(\|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}|) \\ &\quad + (\frac{r_n - r_{n-1}}{\epsilon})\|J_{r_n}y_n^{m-1} - y_{n-1}^{m-1}\|) + \beta_n^{m-2}\|x_n - x_{n-1}\| \\ &\quad + |\beta_n^{m-2} - \beta_{n-1}^{m-2}|\|x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-1}\| \\ &\le \|x_n - x_{n-1}\| + M_2(2|r_n - r_{n-1}| + |\beta_n^{m-1} - \beta_{n-1}^{m-1}| + |\beta_n^{m-2} - \beta_{n-1}^{m-2}|),\end{aligned} \tag{2.13}$$

where M_2 is an appropriate constant such that

$$M_2 > \max\{\frac{\|J_{r_n}y_n^{m-1} - y_{n-1}^{m-1}\|}{\epsilon},\ \|x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-1}\|,\ M_1\}.$$

In this fashion, it is easy to get that

$$\|y_n^{m-i} - y_{n-1}^{m-i}\| \le \|x_n - x_{n-1}\| + M_i(\sum_{j=1}^{i}|\beta_n^{m-j} - \beta_{n-1}^{m-j}| + i|r_n - r_{n-1}|), \tag{2.14}$$

where M_i is an appropriate constant such that

$$M_i > \max\{\frac{\|J_{r_n}y_n^{m-(i-1)} - y_{n-1}^{m-(m-i)}\|}{\epsilon},\ \|x_{n-1} - J_{r_{n-1}}y_{n-1}^{m-(i-1)}\|,\ M_{i-1}\}$$

for all $2 \leq i \leq (m-1)$. Therefore, one can easily see that

$$\|y_n^1 - y_{n-1}^1\| \leq \|x_n - x_{n-1}\| + M_{m-1}(\sum_{j=1}^{m-1} |\beta_n^{m-j} - \beta_{n-1}^{m-j}| + (m-1)|r_n - r_{n-1}|), \tag{2.15}$$

On the other hand, observe that

$$x_{n+1} = \alpha_n u + (1-\alpha_n)y_n^1, \quad x_n = \alpha_{n-1}u + (1-\alpha_{n-1})y_{n-1}^1.$$

It follows that

$$x_{n+1} - x_n = (1-\alpha_n)(y_n^1 - y_{n-1}^1) + (\alpha_n - \alpha_{n-1})(u - y_{n-1}^1). \tag{2.16}$$

It follows from (2.15) that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq (1-\alpha_n)\|y_n^1 - y_{n-1}^1\| + |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}^1\| \\
&\leq (1-\alpha_n)(\|x_n - x_{n-1}\| + M_{m-1}(\sum_{j=1}^{m-1} |\beta_n^{m-j} - \beta_{n-1}^{m-j}| + (m-1)|r_n - r_{n-1}|)) \\
&\quad + |\alpha_n - \alpha_{n-1}|\|u - y_{n-1}^1\| \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| \\
&\quad + M(|\alpha_n - \alpha_{n-1}| + \sum_{j=1}^{m-1} |\beta_n^{m-j} - \beta_{n-1}^{m-j}| + (m-1)|r_n - r_{n-1}|),
\end{aligned} \tag{2.17}$$

where M is an appropriate constant such that

$$M \geq \max\{\|u - y_{n-1}^1\|,\ M_{m-1}\}$$

for all n. Similarly we can prove (2.12) if $r_{n-1} \geq r_n$, by assumptions(i)-(iii), we have that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and

$$\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n-1}| + \sum_{j=1}^{m-1} |\beta_n^{m-j} - \beta_{n-1}^{m-j}| + (m-1)|r_n - r_{n-1}|) < \infty.$$

Hence, Lemma 5 is applicable to (2.17) and we obtain

$$\|x_{n+1} - x_n\| \to 0, \quad \text{as } n \to \infty \tag{2.18}$$

On the other hand, from (1.3) we have

$$\begin{aligned}
\|y_n^1 - J_{r_n}x_n\| &\le \|y_n^1 - J_{r_n}y_n^2\| + \|J_{r_n}y_n^2 - J_{r_n}x_n\| \\
&\le \beta_n^1\|x_n - J_{r_n}y_n^2\| + |y_n^2 - x_n\| \\
&\le \beta_n^1\|x_n - J_{r_n}x_n\| + \beta_n^1\|J_{r_n}x_n - J_{r_n}y_n^2\| + |y_n^2 - x_n\| \\
&\le \beta_n^1\|x_n - J_{r_n}x_n\| + \beta_n^1\|x_n - y_n^2\| + |y_n^2 - x_n\| \\
&\le \beta_n^1\|x_n - J_{r_n}x_n\| + (1+\beta_n^1)\|x_n - y_n^2\| \\
&\le \beta_n^1\|x_n - J_{r_n}x_n\| + (1+\beta_n^1)(1-\beta_n^2)\|x_n - J_{r_n}y_n^3\| \\
&\ \ \vdots \\
&\le (\beta_n^1 + (1+\beta_n^1)\sum_{k=2}^{m-1}\prod_{i=2}^{k}(1-\beta_n^i))\|x_n - J_{r_n}x_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|J_{r_n}x_n - x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n^1\| + \|y_n^1 - J_{r_n}x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n^1\| \\
&\quad + (\beta_n^1 + (1+\beta_n^1)\sum_{k=2}^{m-1}\prod_{i=2}^{k}(1-\beta_n^i))\|x_n - J_{r_n}x_n\|.
\end{aligned} \tag{2.19}$$

From condition (ii), (2.5) and (2.18) we obtain

$$\lim_{n\to\infty}\|J_{r_n}x_n - x_n\| = 0. \tag{2.20}$$

Take a fixed number r such that $\epsilon > r > 0$, from Lemma 1.2 we obtain

$$\begin{aligned}
\|J_{r_n}x_n - J_rx_n\| &= \|J_r(\frac{r}{r_n}x_n + (1-\frac{r}{r_n})J_{r_n}x_n) - J_rx_n\| \\
&\le (1-\frac{r}{r_n})\|x_n - J_{r_n}x_n\| \\
&\le \|x_n - J_{r_n}x_n\|.
\end{aligned} \tag{2.21}$$

Therefore, we have

$$\begin{aligned} \|x_n - J_r x_n\| &\le \|x_n - J_{r_n} x_n\| + \|J_{r_n} x_n - J_r x_n\| \\ &\le \|J_{r_n} x_n - x_n\| + \|J_{r_n} x_n - x_n\| \\ &\le 2\|J_{r_n} x_n - x_n\|. \end{aligned} \tag{2.22}$$

Hence, we obtain

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0.$$

Since in a uniformly smooth Banach space, the sunny nonexpansive retract Q from E onto the fixed point set $F(J_r)(= F = A^{-1}(0))$ of J_r is unique, it must be obtained from Lemma 1.3. Namely,

$$Qu = s - \lim_{t\to 0} z_t, \quad u \in E,$$

where $t \in (0, 1)$ and z_t solves the fixed point equation

$$z_t = tu + (1-t)J_r z_t.$$

Next, we claim that

$$\limsup_{n\to\infty} \langle u - Q(u), J(x_n - Q(u))\rangle \le 0. \tag{2.23}$$

Thus we have

$$\|z_t - x_n\| = \|(1-t)(J_r z_t - x_n) + t(u - x_n)\|.$$

It follows from Lemma 1.6 that

$$\begin{aligned} \|z_t - x_n\|^2 &\le (1-t)^2\|J_r z_t - x_n\|^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle \\ &\le (1 - 2t + t^2)\|z_t - x_n\|^2 + f_n(t) \\ &\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2, \end{aligned} \tag{2.24}$$

where

$$f_n(t) = (2\|z_t - x_n\| + \|x_n - J_r x_n\|)\|x_n - J_r x_n\| \to 0, \text{ as } n \to 0. \tag{2.25}$$

It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t} f_n(t). \tag{2.26}$$

Letting $n \to \infty$ in (2.26) and noting (2.25), we obtain

$$\limsup_{n\to\infty} \langle z_t - u, J(z_t - x_n)\rangle \le \frac{t}{2} M, \tag{2.27}$$

where $M > 0$ is an appropriate constant such that $M \geq \|z_t - x_n\|^2$ for all $t \in (0,1)$ and $n \geq 1$. Letting $t \to 0$ and from (2.27), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq 0.$$

So, for any $\epsilon > 0$, there exists a positive number δ_1 such that, for $t \in (0, \delta_1)$ we get

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{\epsilon}{2}. \tag{2.28}$$

On the other hand, since $z_t \to q$ as $t \to 0$, from Lemma 1.1, there exists $\delta_2 > 0$ such that, for $t \in (0, \delta_2)$ we have

$$\begin{aligned}
&|\langle u - q, J(x_n - q)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\
&\leq |\langle u - q, J(x_n - q)\rangle - \langle u - q, J(x_n - z_t)\rangle| \\
&\quad + |\langle u - q, J(x_n - z_t)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\
&\leq |\langle u - q, J(x_n - q) - J(x_n - z_t)\rangle| + |\langle z_t - q, J(x_n - z_t)\rangle| \\
&\leq \|u - q\|\|J(x_n - q) - J(x_n - z_t)\| + \|z_t - q\|\|x_n - z_t\| < \frac{\epsilon}{2}.
\end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}, \forall t \in (0, \delta)$, we have

$$\langle u - Q(u), J(x_n - Q(u))\rangle \leq \langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

That is,

$$\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \leq \limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

It follows from (2.28) that

$$\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \leq \epsilon.$$

Since ϵ is chosen arbitrarily, we have

$$\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \leq 0. \tag{2.29}$$

Finally, we show that $x_n \to Q(u)$ strongly and this concludes the proof. Observe that

$$\begin{aligned}
\|x_{n+1} - Q(u)\|^2 &= \|(1 - \alpha_n)(y_n - Q(u)) + \alpha_n(u - Q(u))\|^2 \\
&\leq (1 - \alpha_n)^2\|y_n - Q(u)\|^2 + 2\alpha_n\langle u - Q(u), J(x_{n+1} - Q(u))\rangle \\
&\leq (1 - \alpha_n)\|x_n - Q(u)\|^2 + 2\alpha_n\langle u - Q(u), J(x_{n+1} - Q(u))\rangle.
\end{aligned}$$

Now we apply Lemma 1.7 and use (2.29) to see that $\|x_n - Q(u)\| \to 0$ as $n \to \infty$.

Theorem 2.2. *Suppose that E is reflexive and has a weakly continuous duality map J_φ with gauge φ. Suppose that A is an m-accretive operator in X such that $C = \overline{D(A)}$ is convex, $\{x_n\}_{n=0}^{\infty}$ $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1,\ 2,\ \ldots,\ m-1$ are as Theorem 2.1. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a zero point of A.*

Proof. We only include the differences. From Theorem 2.1 we obtain

$$\begin{aligned}
&\|x_{n+1} - J_{r_n}x_n\| \\
&= \|x_{n+1} - y_n^1\| + \|y_n^1 - J_{r_n}x_n\| \\
&\le \alpha_n\|u - y_n\| + (\beta_n^1 + (1+\beta_n^1)\sum_{k=2}^{m-1}\prod_{i=2}^{k}(1-\beta_n^i))\|x_n - J_{r_n}x_n\|.
\end{aligned}$$

That is,

$$\lim_{n\to\infty}\|x_{n+1} - J_{r_n}x_n\| = 0. \tag{2.30}$$

We next prove that

$$\limsup_{n\to\infty}\langle u - Q(u), J_\varphi(x_n - Q(u)\rangle \le 0. \tag{2.31}$$

By Lemma 1.5, we have the sunny nonexpansive retraction $Q : C \to F(T)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - Q(u), J_\varphi(x_n - Q(u)\rangle = \lim_{k\to\infty}\langle u - Q(u), J_\varphi(x_{n_k} - Q(u)\rangle. \tag{2.32}$$

Since X is reflexive, we may further assume that $x_{n_k} \rightharpoonup \widetilde{x}$. Moreover, since

$$\|x_{n+1} - J_{r_n}\| \to 0,$$

we obtain

$$J_{r_{n_k-1}}x_{n_k-1} \rightharpoonup \widetilde{x}.$$

Taking the limit as $k \to \infty$ in the relation

$$[J_{r_{n_k-1}}x_{n_k-1}, A_{r_{n_k-1}}x_{n_k-1}] \in A,$$

we get $[\widetilde{x}, 0] \in A$. That is, $\widetilde{x} \in F$. Hence by (2.32) and (1.5) we have

$$\limsup_{n\to\infty}\langle u - Q(u), J_\varphi(x_n - Q(u))\rangle = \langle u - Q(u), J_\varphi(\widetilde{x} - Q(u))\rangle \le 0.$$

That is (2.31) holds. Finally to prove that $x_n \to p$. It follows from (2.2) and (2.3) that

$$
\begin{aligned}
\Phi(\|y_n^1 - p\|) &= \Phi(\|\beta_n^1(x_n - p) + (1-\beta_n^1)(J_{r_n}y_n^2 - p)\|) \\
&\le \Phi(\|\beta_n\|x_n - p\| + (1-\beta_n)\|J_{r_n}y_n^2 - p\|) \\
&\le \Phi(\|x_n - p\|).
\end{aligned} \tag{2.33}
$$

Therefore, from (2.33) we obtain

$$
\begin{aligned}
\Phi(\|x_{n+1} - p\|) &= \Phi(\|\alpha_n(u-p) + (1-\alpha_n)(y_n^1 - p)\|) \\
&\le \Phi((1-\alpha_n)\|y_n^1 - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle \\
&\le (1-\alpha_n)\Phi(\|y_n^1 - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle \\
&\le (1-\alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle.
\end{aligned}
$$

An application of Lemma 1.3 yields that $\Phi(\|x_n - p\|) \to 0$; that is $\|x_n - p\| \to 0$ as $n \to \infty$. This completes the proof.

Remark 2.3. Theorem 2.1 and Theorem 2.2 improve Kim and Xu [6] and Xu [12] as a special case. We note that our theorems in this paper carry over trivially to the so-called viscosity approximation methods.

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