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ITERATIVE SEQUENCES FOR NONLIPSCHITZIAN NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. The purpose of this paper to establish some strong convergence theorems of the Mann and Ishikawa iteration processes for fixed point of nearly asymptotically nonexpansive mappings in convex metric spaces.
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1. Introduction

Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X, if for all x, y in X and $\lambda \in [0, 1]$, the following condition is satisfied:

$$d(u, W(x, y, \lambda)) \leq (1 - \lambda)d(u, x) + \lambda d(u, y) \text{ for all } u \in X.$$

A metric space with convex structure is called a *convex metric space*. Banach spaces and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which cannot be imbedded in any Banach spaces, see [9, 22]. A subset D of convex metric space X is said to be *convex* if $W(x, y, \lambda) \in D$ for all $x, y \in D$ and $\lambda \in [0, 1]$.

Let D be a nonempty subset of a Banach space X and $T : D \to D$ a mapping. We denote $F(T)$, the set of fixed points of T. The mapping T is said to be

(i) *uniformly L-Lipschitzian* if for each $n \in \mathbb{N}$, there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in D$;

(ii) *asymptotically nonexpansive* if for each $n \in \mathbb{N}$, there exists a constant $u_n \geq 0$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$ for all $x, y \in D$;

(iii) *quasi L-Lipschitzian* if $F(T) = \{x \in D : Tx = x\} \neq \phi$ and there exists a constant $L > 0$ such that $\|Tx - y\| \leq L\|x - y\|$ for all $x \in D$ and $y \in F(T)$;

(iv) *asymptotically quasi-nonexpansive* if $F(T) \neq \phi$ and for each $n \in \mathbb{N}$, there exists a constant $u_n \geq 0$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - y\| \leq (1 + u_n)\|x - y\|$ for all $x \in D$ and $y \in F(T)$.

The class of asymptotically nonexpansive mappings as a natural extension to that of nonexpansive mappings, was introduced by Goebel and Kirk [7] in 1972. They proved the following existence theorem:

Theorem GK. *If D is a closed convex bounded subset of a uniformly convex Banach space X, then every asymptotically nonexpansive self-mapping T of D has a fixed point.*

This has been generalized to a nearly uniformly convex Banach space by Xu [25] for nonlipschizian mappings. There appear in the literature two definitions of nonlipschitzian asymptotically nonexpansive mappings. One is due to Kirk [12] : T is said to be a *mapping of asymptotically nonexpansive type* if

$$\limsup_{n\to\infty}(\sup_{y\in D}(\|T^n x - T^n y\| - \|x - y\|)) \leq 0 \text{ for all } x \in D.$$

Another nonlipschitzian mapping between these two classes was introduce by Bruck, Kuczumow and Reich [4]: T is said to be an *asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n\to\infty}(\sup_{x,y\in D}(\|T^n x - T^n y\| - \|x - y\|)) \leq 0.$$

Recently, Sahu [19] introduce another nonlipschitzian mapping: Let D be a nonempty subset of a Banach space X and fix a sequence $\{a_n\}$ in $[0, \infty)$ with

$a_n \to 0$. T is said to be a *nearly asymptotically nonexpansive* if for each $n \in \mathbb{N}$, there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $u_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\| + a_n \text{ for all } x, y \in D.$$

T will be called *nearly asymptotically quasi-nonexpansive with respect to* $\{a_n\}$ if $F(T) = \{x : Tx = x\} \neq \phi$ and there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $u_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\| + a_n \text{ for all } x \in D \textit{ and } y \in F(T).$$

The class of nearly asymptotically nonexpansive mappings is an intermediate class between class of asymptotically nonexpansive mappings and that of mappings of asymptotically nonexpansive type.

Example. Let $X = \mathbb{R}, C = [0, 1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{1}{2} & \text{if} \quad x \in [0, \frac{1}{2}], \\ 0 & \text{if} \quad x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, T is discontinuous and non-Lipschitzian. However, it is nearly nonexpansive. Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we have

$$\|Tx - Ty\| \leq \|x - y\| + a_1 \text{ for all } x, y \in C$$

and

$$\|T^n x - T^n y\| \leq \|x - y\| + a_n \text{ for all } x, y \in C \text{ and } n \geq 2,$$

since

$$T^n x = \frac{1}{2} \text{ for all } x \in [0, 1] \text{ and } n \geq 2.$$

The iterative approximation problems for fixed points of asymptotically nonexpansive mappings and mappings of asymptotically nonexpansive type were extensively studied by Bose [3]; Gornicki [8]; Jung, Cho and Sahu [10]; Lim and Xu [13]; and Xu [25]. Recently, Agrawal, Regan and Sahu [1] introduced a new iteration process namely S-iteration process and studied the iterative approximation problems for fixed points of nearly asymptotically nonexpansive mappings.

The purpose of this paper to establish a few strong convergence theorems of the Mann and Ishikawa iteration processes for fixed point of nearly asymptotically nonexpansive mappings in convex metric spaces. The result presented in this paper extend the results due to Chang [5], Khan and Takahashi [11], Liu and Kang [14], Osilike and Aniagbosor [15], Rhoades [18], Schu [20, 21], Tan and Xu [23, 24] and others.

2. Main Results

We begin with the following lemma which is a generalization of Lemma 1 of Osilike and Aniagbosor [15].

Lemma 2.1. *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative numbers such that*

(i) $a_{n+1} \leq (1 + Mb_n)a_n + M'c_n$ *for all* $n \in \mathbb{N}$, *and for some* $M, M' \geq 0$;

(ii) $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$.

Then we have the following:

(a) $\lim_{n\to\infty} a_n$ *exists.*

(b) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Proof. (a) Let $K = e^{\sum_{n=1}^{\infty} b_n}$. Observe that

$$\begin{aligned}
a_{n+m} &\leq (1 + Mb_{n+m-1})a_{n+m-1} + M'c_{n+m-1} \\
&\leq e^{Mb_{n+m-1}} a_{n+m-1} + M'c_{n+m-1} \\
&\leq e^{Mb_{n+m-1}} (e^{Mb_{n+m-2}} a_{n+m-2} + M'c_{n+m-2}) + M'c_{n+m-1} \\
&\leq e^{M(b_{n+m-1}+b_{n+m-2})} a_{n+m-2} + (e^{Mb_{n+m-1}} c_{n+m-2} + c_{n+m-1})M' \\
&\leq \dots \\
&\leq e^{M\sum_{i=n}^{n+m-1} b_i} a_n + (e^{M\sum_{i=n}^{n+m-1} b_i} \sum_{i=n}^{n+m-1} c_i)M' \text{ for all } n, m \in \mathbb{N}.
\end{aligned}$$

It follows that

$$a_{n+1} \leq e^{MK} a_1 + \left(e^{MK} \sum_{i=1}^{\infty} c_i \right) M',$$

i.e., $\{a_n\}$ is bounded. Thus $a_n \leq M''$ for all $n \in \mathbb{N}$ and some $M'' \geq 0$. Hence

$$\begin{aligned}
a_{n+1} &\leq a_n + MM''b_n + M'c_n \\
&\leq a_n + K'(b_n + c_n) \text{ for all } n \in \mathbb{N},
\end{aligned}$$

where $K' = \max\{MM'', M'\}$. Therefore, $\lim_{n\to\infty} a_n$ exists by Lemma 1 of Tan and Xu [23, p. 303].

(b) It follows easily from part (a). $\square$

Lemma 2.2. *Let D be a nonempty convex subset of a convex metric space X and $T : D \to D$ a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Define Ishikawa iterative sequence $\{x_n\}$ iteratively by*

$$x_1 \in D, x_{n+1} = W(x_n, T^n y_n, \alpha_n), y_n = W(x_n, T^n x_n, \beta_n), n \in \mathbb{N}. \qquad (I)$$

Then for each $p \in F(T)$

$$d(x_{n+1}, p) \leq (1 + (1 + \beta_n)u_n + \beta_n u_n^2) d(x_n, p) + (1 + \beta_n(1 + u_n))a_n, \ n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} d(x_n, p) \text{ exists.}$$

Proof. For $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T^n y_n, \alpha_n), p) \\ &\leq \alpha_n(1 + u_n)d(y_n, p) + (1 - \alpha_n)d(x_n, p) + a_n \qquad (1) \end{aligned}$$

and

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T^n x_n, \beta_n), p) \\ &\leq \beta_n d(T^n x_n, p) + (1 - \beta_n)d(x_n, p) \\ &\leq (1 + \beta_n u_n)d(x_n, p) + a_n\beta_n. \qquad (2) \end{aligned}$$

From (1) and (2), we obtain

$d(x_{n+1}, p) \leq (1 + (1 + \beta_n)u_n + \beta_n u_n^2) d(x_n, p) + (1 + \beta_n(1 + u_n))a_n$ for all $n \in \mathbb{N}$.

It follows that

$$d(x_{n+1}, p) \leq (1 + Mu_n)d(x_n, p) + M'a_n \qquad (3)$$

for some $M, M' \geq 0$, because $\{u_n\}$ is bounded. Hence $\lim_{n\to\infty} d(x_n, p)$ exists by Lemma 2.1 (a). □

Lemma 2.3. *Let D be a nonempty subset of a metric space X and $T : D \to D$ a quasi L-Lipschitzian. If $\{x_n\}$ is a sequence in D such that $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\lim_{n\to\infty} x_n = v \in D$. Then v is a fixed point of T.*

($d(x, D)$ denotes the distance of x to set D, i.e., $d(x, D) = inf_{y\in D} d(x, y)$)

Proof. Since $\lim_{n\to\infty} x_n = v$, then for each $\varepsilon > 0$, there exists a natural number N_1 such that

$$d(x_n, v) < \frac{\varepsilon}{2(1+L)} \text{ for all } n \geq N_1.$$

By $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a natural number $N_2 \geq N_1$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{3(1+3L)} \text{ for all } n \geq N_2$$

and hence

$$d(x_{N_2}, F(T)) < \frac{\varepsilon}{3(1+3L)}.$$

It follows that there exists a point $z \in F(T)$ such that

$$d(x_{N_2}, z) < \frac{\varepsilon}{2(1+3L)}.$$

Thus, we have

$$\begin{aligned} d(Tv, v) &\leq d(Tv, z) + d(z, Tx_{N_2}) + d(Tx_{N_2}, z) + d(z, x_{N_2}) + d(x_{N_2}, v) \\ &\leq Ld(v, z) + (1+2L)d(x_{N_2}, z) + d(x_{N_2}, v) \\ &\leq (1+L)d(x_{N_2}, v) + (1+3L)d(x_{N_2}, z) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, v is a fixed point of T since ε is arbitrary. $\square$

We now prove the following theorems:

Theorem 2.1. *Let D be a nonempty closed convex subset of a complete convex metric space $X, T : D \to D$ a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n,\ u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$ and let $F(T)$ be a closed set. For arbitrary $x_1 \in D$, define the Ishikawa iterative sequence $\{x_n\}$ by*

$$x_{n+1} = W(x_n, T^n y_n, \alpha_n), y_n = W(x_n, T^n x_n, \beta_n), n \in \mathbb{N}, \qquad (I)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. Then $\{x_n\}$ converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T) = 0$.

Proof. The necessity of the condition is obvious. Thus, we will only prove the sufficiency. From (3), we have

$$d(x_{n+1}, p) \leq (1 + Mu_n)d(x_n, p) + M'a_n \text{ for all } n \in \mathbb{N},$$

which implies that

$$d(x_{n+1}, F(T)) \leq (1 + Mu_n)d(x_n, F(T)) + M'a_n \text{ for all } n \in \mathbb{N}.$$

It follows from Lemma 2.1(b) that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. For each $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F(T)) < \frac{\varepsilon}{3}$ for all $n \geq n_0$. Then there exists a $p' \in F(T)$ such that $d(x_m, p') < \frac{\varepsilon}{2}$ for all $n \geq n_0$. Hence for $n, m \geq n_0$, we have

$$d(x_n, x_m) \leq d(x_n, p') + d(x_m, p') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence in D. Let $\lim_{n\to\infty} x_n = v \in D$. By $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and closedness of $F(T)$, we have that $v \in F(T)$. This completes the proof. □

Corollary 2.1. *Let D be a nonempty closed convex subset of a complete convex metric space X, and let $T : D \to D$ be a nearly asymptotically nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Let $F(T)$ be a nonempty closed set and let $\{x_n\}$ be the Ishikawa iterative sequence defined by (I). Then $\{x_n\}$ converges to a fixed point of T if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Theorem 2.2. *Let D be a nonempty closed convex subset of a complete convex metric space X and $T : D \to D$ an asymptotically quasi-nonexpansive mapping with sequence $\{u_n\}$ such that $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be the Ishikawa iterative sequence defined by (I), then $\{x_n\}$ converges to a fixed point of T if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof. As in proof of Theorem 2.1, it can easily shown that $\{x_n\}$ is Cauchy sequence in D. Let $\lim_{n\to\infty} x_n = v$. Since every asymptotically quasi-nonexpansive mapping is quasi L-Lipschitzian, it follows from Lemma 2.3 that v is a fixed point of T. □

Remark 2.1. Theorem 2.1 extends and improves Theorem 3.1 of Ghosh and Debnath [6], Theorem 1.1 and 1.1' of Petryshyn and Williamson [16], and Theorem 1 of Qihou [17] in the more general space setting.

Theorem 2.3. *Let D be a nonempty closed convex subset of a complete convex metric space $X, T : D \to D$ a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Suppose that $F(T)$ is closed and $\{x_n\}$ is the Ishikawa iterative sequence defined by (I) satisfying the following conditions:*

(i) $\{x_n\}$ *is an approximating fixed point sequence for* T, *i.e.,*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

(ii) *there exists a constant* $c > 0$ *such that*

$$d(x_n, F(T)) \leq cd(x_n, Tx_n), \text{ for all } n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges to a fixed point of* T.

Proof. Since $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F(T)$, it follows from (i) and (ii), we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, by Theorem 2.1, we conclude that $\{x_n\}$ converges to a fixed point of T. □

Finally, we discuss the problem of approximation of fixed points of nearly asymptotically nonexpansive mappings in a uniformly convex metric space.

Theorem 2.4. *Let* D *be a nonempty closed convex subset of a complete and uniformly convex metric space* X *and* $T : D \to D$ *a nearly asymptotically nonexpansive mapping with* $F(T) \neq \phi$ *and sequence* $\{(a_n, u_n)\}$ *satisfying* $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty} u_n < \infty$. *For arbitrary* $x_1 \in D$, *define the Ishikawa iterative sequence* $\{x_n\}$ *by*

$$x_{n+1} = W(T^n y_n, x_n, \frac{1}{2}), y_n = W(T^n x_n, x_n, \frac{1}{2}), n \in \mathbb{N}. \qquad (J)$$

Then the following holds:

(a) $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$

(b) *if* T *is uniformly continuous and* T^m *is demi-compact for some* $m \in \mathbb{N}$, *it follows that* $\{x_n\}$ *converges to a fixed point of* T.

Proof. (a) Let p be a fixed point of T. Since X is uniformly convex, by Beg [2, Theorem 4.2], there exists a number $c > 0$ such that

$$2d^2(W(x, y, \frac{1}{2}), z) \leq d^2(x, z) + d^2(y, z) - cd^2(x, y) \text{ for all } x, y, z \in X. \qquad (4)$$

From (J)

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2(W(T^n y_n, x_n, \frac{1}{2}), p)\\
&\le \frac{1}{2}d^2(T^n y_n, p) + \frac{1}{2}d^2(x_n,p) - \frac{c}{2}d^2(T^n y_n, x_n)\\
&\le \frac{1}{2}[(1+u_n)d(y_n,p)+a_n]^2 + \frac{1}{2}d^2(x_n,p)\\
&= \frac{1}{2}[(1+u_n)d^2(y_n,p) + 2a_n(1+u_n)d(y_n,p) + a_n^2] + \frac{1}{2}d^2(x_n,p).
\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n,p)$ exists by Lemma 2.2, $\{x_n\}$ is bounded. It follows from (2) that $\{y_n\}$ is bounded. Then there exists a constant $K>0$ such that

$$d^2(x_{n+1},p) \le \frac{1}{2}(1+u_n)^2 d^2(y_n,p) + a_nK + \frac{1}{2}d^2(x_n,p).$$

It further implies by (4) that

$$\begin{aligned}
d^2(x_{n+1},p) &\le \frac{1}{2}(1+u_n)^2 d^2(W(T^n x_n, x_n, \frac{1}{2}), p) + \frac{1}{2}d^2(x_n,p) + a_nK\\
&\le \frac{1}{2}(1+u_n)^2[\frac{1}{2}d^2(T^n x_n,p) + \frac{1}{2}d^2(x_n,p) - \frac{c}{2}d^2(x_n, T^n x)]\\
&\quad + \frac{1}{2}d^2(x_n,p) + a_nK\\
&\le \frac{1}{2}(1+u_n)^2[\frac{1}{2}\{(1+u_n)d(x_n,p)+a_n\}^2 + \frac{1}{2}d^2(x_n,p)\\
&\quad - \frac{c}{2}d^2(x_n,T^n x_n)] + \frac{1}{2}d^2(x_n,p) + a_nK\\
&\le \frac{1}{2}(1+u_n)^2[\frac{1}{2}(1+u_n)^2 d^2(x_n,p) + a_nK' + \frac{1}{2}d^2(x_n,p)\\
&\quad - \frac{c}{2}d^2(x_n,T^n x_n)] + \frac{1}{2}d^2(x_n,p) + a_nK\\
&\le \frac{1}{2}(1+u_n)^2[(1+u_n)^2 d^2(x_n,p) + a_nK'\\
&\quad - \frac{c}{2}d^2(x_n,T^n x_n)] + \frac{1}{2}d^2(x_n,p) + a_nK\\
&\le (1+u_n)^4 d^2(x_n,p) + a_nK'' - \frac{c}{2}d^2(x_n,T^n x_n)\\
&\le (1+Mu_n)d^2(x_n,p) + a_nK'' - \frac{c}{2}d^2(x_n,T^n x_n)
\end{aligned}$$

for some $K', K'' \ge 0$. By boundedness of $\{x_n\}$, we have

$$d^2(x_{n+1},p) \le d^2(x_n,p) + (a_n+u_n)M' - \frac{c}{2}d^2(x_n,T^n x_n)$$

for all $n \in \mathbb{N}$ and for some $M' \geq 0$.

Therefore,

$$\frac{c}{2} d^2(x_n, T^n x_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p) + (a_n + u_n)M'.$$

This implies that

$$\begin{aligned}\sum_{n=1}^{m} d^2(x_n, T^n x_n) &\leq \frac{2}{c} \sum_{n=1}^{m} [d^2(x_n, p) - d^2(x_{n+1}, p)] + \frac{2M'}{c} \sum_{n=1}^{M} (a_n + u_n) \\ &= \frac{c}{2} [d^2(x_1, p) - d^2(x_{m+1}, p)] + \frac{2M'}{c} \sum_{n=1}^{m} (a_n + u_n),\end{aligned}$$

it follows that $\sum_{n=1}^{\infty} d^2(x_n, T^n x_n) < \infty$. Therefore, $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$.

(b) By uniform continuity of T,

$$d(x_n, T^n x_n) \to 0 \text{ implies that } d(Tx_n, T^{n+1} x_n) \to 0.$$

Observe that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{1}{2} d(x_n, T^n y_n) \\ &\leq \frac{1}{2} [d(x_n, T^n x_n) + d(T^n x_n, T^n y_n)] \\ &\leq d(x_n, T^n x_n) + (1 + u_n) d(x_n, y_n) + a_n \\ &\leq d(x_n, T^n x_n) + (1 + u_n) d(x_n, T^n x_n) + a_n \to 0 \text{ as } n \to \infty. \end{aligned}$$

Also

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^{n+1} x_{n+1}, T^{n+1} x_n) \\ &\quad + d(T^{n+1} x_n, T x_n) \\ &\leq (2 + u_n) d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^{n+1} x_n, T x_n) \\ &\quad + a_{n+1} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Again by uniform continuity of T

$$d(x_n, Tx_n) \to 0 \Rightarrow d(Tx_n, T^2 x_n) \to 0 \Rightarrow \cdots \Rightarrow d(T^i x_n, T^{i+1} x_n) \to 0$$

for $i = 0, 1, 2, \cdots$. It follows that

$$d(x_n, T^m x_n) \leq \sum_{i=0}^{m-1} d(T^i x_n, T^{i+1} x_n) \to 0 \text{ as } n \to \infty,$$

i.e., $\lim_{n\to\infty} d(x_n, T^m x_n) = 0$. Since $d(x_n, T^m x_n) \to 0$, demicompactness of T^m, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} T^m x_{n_j} = v \in D$.

Observe that

$$d(x_{n_j}, v) \leq d(x_{n_j}, T^m x_{n_j}) + d(T^m x_{n_j}, v) \to 0 \text{ as } j \to \infty.$$

Hence $x_{n_j} \to v$, since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we get $v \in F(T)$. Since $\lim_{n\to\infty} d(x_n, v)$ exists by Lemma 2.2 and $\lim_{n\to\infty} d(x_{n_j}, v) = 0$, we conclude that $x_n \to v$. This completes the proof. □

Remark 2.2. Theorem 2.4 extends the results of Beg [2], Chang [5], Khan and Takahashi [11], Liu and Kang [14], Osilike and Aniagbosor [15], Rhoades [18], Schu [20, 21], Tan and Xu [23, 24] and others for a more general class of non-Lipschitzian mappings in metric space setting.

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