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A FIXED POINT THEOREM FOR MATKOWSKI CONTRACTIONS

SIMEON REICH* AND ALEXANDER J. ZASLAVSKI**

*Department of Mathematics
The Technion-Israel Institute of Technology
32000 Haifa, Israel
E-mail: sreichtx.technion.ac.il

**Department of Mathematics
The Technion-Israel Institute of Technology
32000 Haifa, Israel
E-mail: ajzasltx.technion.ac.il

Abstract. We establish a fixed point theorem for Matkowski contractions. Our result is concerned with the case where such mappings take a nonempty, closed subset of a complete metric space X into X.
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Let (X, ρ) be a complete metric space. According to Banach's fixed point theorem [1], the iterates of any strict contraction on X converge to its unique fixed point. This classical theorem has found numerous important applications and has also been extended in several directions. See [2] for a comprehensive survey of the results available in the literature regarding various types of contractive mappings up to 2001. Another important topic in fixed point theory is the search for fixed points of nonself-mappings (see, for example, [4] and the references mentioned therein).

In the present paper we combine these two themes by presenting a new sufficient condition for the existence and approximation of the unique fixed point of a Matkowski contraction [3, p. 8] which maps a nonempty, closed subset of X into X.

Theorem. *Let K be a nonempty, closed subset of a complete metric space (X, ρ). Assume that $T : K \to X$ satisfies*

$$\rho(Tx, Ty) \leq \phi(\rho(x, y)) \text{ for each } x, y \in K, \tag{1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is increasing and satisfies $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. Assume that $K_0 \subset K$ is a nonempty, bounded set with the following property:

(P1) For each natural number n, there exists $x_n \in K_0$ such that $T^n x_n$ is defined.

Then the following assertions hold.

(A) There exists a unique $\bar{x} \in K$ such that $T\bar{x} = \bar{x}$.

(B) Let $M, \epsilon > 0$. Then there exists a natural number k such that for each sequence $\{x_i\}_{i=0}^n \subset K$ with $n \geq k$ satisfying

$$\rho(x_0, \bar{x}) \leq M \text{ and } Tx_i = x_{i+1},\ i = 0, \dots, n-1,$$

the inequality $\rho(x_i, \bar{x}) \leq \epsilon$ holds for all $i = k, \dots, n$.

Proof. For each $x \in X$ and $r > 0$, set

$$B(x, r) = \{y \in X : \ \rho(x, y) \leq r\}.$$

(A) Since $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$, and since ϕ is increasing, we have

$$\phi(t) < t \text{ for all } t > 0. \tag{2}$$

This implies the uniqueness of $\bar{x}$. Clearly, $\phi(0) = 0$.

For each natural number n, let x_n be as guaranteed by property (P1). Fix $\theta \in K$. Since K_0 is bounded, there is $c_0 > 0$ such that

$$\rho(\theta, z) \leq c_0 \text{ for all } z \in K_0. \tag{3}$$

Let $\epsilon > 0$. We will show that there exists a natural number k such that the following property holds:

(P2) If the integers i and n satisfy $k \leq i < n$, then

$$\rho(T^i x_n, T^{i+1} x_n) \leq \epsilon.$$

By (1) and (3), for each $z \in K_0$,

$$\begin{aligned} \rho(z, Tz) &\leq \rho(z, \theta) + \rho(\theta, T\theta) + \rho(T\theta, Tz) \\ &\leq 2\rho(z, \theta) + \rho(\theta, T\theta) \leq 2c_0 + \rho(\theta, T\theta). \end{aligned} \tag{4}$$

Clearly, there is a natural number k such that

$$\phi^k(2c_0 + \rho(\theta, T\theta)) < \epsilon. \tag{5}$$

Assume now that the integers i and n satisfy $k \le i < n$.

By (1), (2), (4), the choice of x_n, and (5),

$$\rho(T^i x_n, T^{i+1} x_n) \le \rho(T^k x_n, T^{k+1} x_n) \le \phi^k(\rho(x_n, Tx_n))$$
$$\le \phi^k(2c_0 + \rho(\theta, T\theta)) < \epsilon.$$

Thus property (P2) holds for this k.

Let $\delta > 0$. Next, we claim that there exists a natural number k such that the following property holds:

(P3) If the integers i, j and n satisfy $k \le i < j < n$, then

$$\rho(T^i x_n, T^j x_n) \le \delta.$$

Indeed, by (2),

$$\phi(\delta) < \delta. \tag{6}$$

By (P2) and (6), there is a natural number k such that (P2) holds with $\epsilon = \delta - \phi(\delta)$.

Assume now that the integers i and n satisfy $k \le i < n$. In view of the choice of k, and property (P2) with $\epsilon = \delta - \phi(\delta)$,

$$\rho(T^i x_n, T^{i+1} x_n) \le \delta - \phi(\delta). \tag{7}$$

Now let

$$x \in K \cap B(T^i x_n, \delta). \tag{8}$$

It follows from (1), (7) and (8) that

$$\rho(Tx, T^i x_n) \le \rho(Tx, T^{i+1} x_n) + \rho(T^{i+1} x_n, T^i x_n) \le \phi(\rho(x, T^i x_n)) + \delta - \phi(\delta) \le \delta.$$

Thus

$$T(K \cap B(T^i x_n, \delta)) \subset B(T^i x_n, \delta),$$

and if an integer j satisfies $i < j < n$, then $\rho(T^i x_n, T^j x_n) \le \delta$. Hence property (P3) does hold, as claimed.

Let $\epsilon > 0$. We will show that there exists a natural number k such that the following property holds:

(P4) If the integers n_1, n_2 and i satisfy $k \le i \le \min\{n_1, n_2\}$, then

$$\rho(T^i x_{n_1}, T^i x_{n_2}) \le \epsilon.$$

Indeed, there exists a natural number k such that

$$\phi^i(2c_0) < \epsilon \text{ for all integers } i \geq k. \tag{9}$$

Assume now that the natural numbers n_1, n_2 and i satisfy

$$k \leq i \leq \min\{n_1, n_2\}. \tag{10}$$

By (1), (3) and (9),

$$\rho(T^i x_{n_1}, T^i x_{n_2}) \leq \phi^i(\rho(x_{n_1}, x_{n_2})) \leq \phi^i(2c_0) < \epsilon.$$

Thus property (P4) indeed holds.

Let $\epsilon > 0$. By (P4), there exists a natural number k_1 such that

$$\rho(T^i x_{n_1}, T^i x_{n_2}) \leq \epsilon/4 \text{ for all integers } n_1, n_2 \geq k_1$$
$$\text{and all integers } i \text{ satisfying } k_1 \leq i \leq \min\{n_1, n_2\}. \tag{11}$$

By property (P3), there exists a natural number k_2 such that

$$\rho(T^i x_n, T^j x_n) \leq \epsilon/4 \text{ for all natural numbers } n, i, j \text{ satisfying } k_2 \leq i, j < n. \tag{12}$$

Assume that the natural numbers n_1, n_2, i and j satisfy

$$n_1, n_2 > k_1 + k_2,\ i, j \geq k_1 + k_2,\ i < n_1, j < n_2. \tag{13}$$

We claim that

$$\rho(T^i x_{n_1}, T^j x_{n_2}) \leq \epsilon.$$

By (1), (6), (11) and (13),

$$\rho(T^{k_1+k_2} x_{n_1}, T^{k_1+k_2} x_{n_2}) \leq \rho(T^{k_1} x_{n_1}, T^{k_1} x_{n_2}) \leq \epsilon/4. \tag{14}$$

In view of (12) and (13),

$$\rho(T^{k_1+k_2} x_{n_1}, T^i x_{n_1}) \leq \epsilon/4 \text{ and } \rho(T^{k_1+k_2} x_{n_2}, T^j x_{n_2}) \leq \epsilon/4.$$

When combined with (14), this implies that

$$\rho(T^i x_{n_1}, T^j x_{n_2}) \leq \rho(T^i x_{n_1}, T^{k_1+k_2} x_{n_1}) + \rho(T^{k_1+k_2} x_{n_1}, T^{k_1+k_2} x_{n_2})$$
$$+\rho(T^{k_1+k_2} x_{n_2}, T^j x_{n_2}) \leq \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon.$$

Thus we have shown that the following property holds:

(P5) For each $\epsilon > 0$, there exists a natural number $k(\epsilon)$ such that

$$\rho(T^i x_{n_1}, T^j x_{n_2}) \leq \epsilon$$

for all natural numbers $n_1, n_2 > k(\epsilon)$, $i \in [k(\epsilon), n_1)$ and $j \in [k(\epsilon), n_2)$.

Consider now the sequences $\{T^{n-2}x_n\}_{n=3}^{\infty}$ and $\{T^{n-1}x_n\}_{n=3}^{\infty}$. Property (P5) implies that these sequences are Cauchy sequences and that

$$\lim_{n\to\infty} \rho(T^{n-2}x_n, T^{n-1}x_n) = 0.$$

Hence there exists $\bar{x} \in K$ such that

$$\lim_{n\to\infty} \rho(\bar{x}, T^{n-2}x_n) = \lim_{n\to\infty} \rho(\bar{x}, T^{n-1}x_n) = 0.$$

Since the mapping T is continuous, $T\bar{x} = \bar{x}$ and part (A) is proved.

(B) Since T is a Matkowski contraction, there is a natural number k such that $\phi^k(M) < \epsilon$.

Assume that a point $x_0 \in B(\bar{x}, M)$, an integer $n \geq k$, and that $T^i x_0$ is defined for all $i = 0, \ldots, n$. Then $T^i x_0 \in K$, $i = 0, \ldots, n-1$, and by (1),

$$\rho(T^k x_0, \bar{x}) \leq \phi^k(\rho(x_0, \bar{x})) \leq \phi^k(M) < \epsilon.$$

By (1) and (2), we have for $i = k, \ldots, n$,

$$\rho(T^i x_0, \bar{x}) \leq \rho(T^k x_0, \bar{x}) \leq \epsilon.$$

Thus part (B) of our theorem is also proved. □

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