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A GENERAL SELECTION THEOREM FOR MULTIVALUED FUNCTIONS SATISFYING AN IMPLICIT RELATION

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Abstract. J.R. Jachymski [3] initiated the study of Caristi selection. Some generalizations of J.R. Jachymski result are proved in [9], [10], [11]. In this paper a general selection theorem for multivalued functions satisfying an implicit relation which generalizes the results by [9], [10] and [11] is proved.
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1. Introduction

J. Caristi's fixed point theorem [1] states that each operator f from a complete metric space (X, d) into itself satisfying the condition:

(*) there exists a lower semicontinuous $g : X \to \mathbb{R}_+$ such that $d(x, f(x)) \leq g(x) - g(f(x))$, for each $x \in X$,

has at least a fixed point.

For the multivalued case, there exist several results involving multivalued Caristi type conditions (see for example [2], [4], [5]). There are several extensions and generalizations of these important principles of nonlinear analysis (see for example the references listed in [8], [10]).

Let (X, d) be a metric space and $P(X)$ the space of all nonempty subsets of X. We denote by $P_{cl}(X)$ the space of all nonempty closed subsets of X. We consider the following functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A,B) = \inf\{d(a,b) | a \in A, b \in B\},$$

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A,B) = \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\}.$$

H is called the Hausdorff-Pompeiu generalized functional and it is well known that if (X,d) is a complete metric space, then $(P_{cl}(X), H)$ is also a complete metric space.

2. Preliminaries

If X, Y are nonempty sets and $F : X \to P(Y)$ is a multivalued operator, then a selection of F is a single valued operator $f : X \to Y$ such that $f(x) \in F(x)$, for each $x \in X$.

First result concerning the existence of a selection which satisfies the Caristi condition (*) (briefly called Caristi selection) was established by J.R. Jachymski [3] for a multivalued operator with closed valued.

Theorem 2.1 ([3]). *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued contraction.*

Then there exists $f : X \to X$ a Caristi selection (with a Lipschitz map g) of F.

Some extensions of Theorem 2.1 are proved in [9], [10], [11].

Theorem 2.2 ([9]). *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ such that*

$$H(F(x), F(y)) \le a\, d(x,y) + b\, D(x, F(x)) + c\, D(y, F(y)),$$

for each $x, y \in X$, where $a, b, c \in \mathbb{R}_+$ and $a + b + c < 1$.

Then there exists $f : X \to X$ a Caristi selection of F.

Theorem 2.3 ([11]). *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ such that*

$$H(F(x), F(y)) \le a_1\, d(x,y) + a_2\, D(x, F(x)) + a_3\, D(y, F(y)) +$$

$$+ a_4\, D(x, F(y)) + a_5\, D(y, F(x)),$$

for each $x, y \in X$, where $a_1, \ldots, a_5 \in \mathbb{R}_+$ and $a_1 + a_2 + a_3 + 2a_4 < 1$.

Then there exists $f : X \to X$ a Caristi selection of F.

Theorem 2.4 ([10]). *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ such that*

$$H(F(x),F(y)) \le q \max\{d(x,y), D(x,F(x)), D(y,F(y)), D(x,F(y)), D(y,F(x))\},$$

where $0 < q < 1$.

Then there exists $f : X \to X$ a Caristi selection of F.

The present author considered in [6], [7] the study of fixed point for mappings satisfying implicit relations.

3. Implicit relation

Let $\mathcal{F}$ be the set of all real functions $F(t_1,\dots,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is non-decreasing in variable t_1 and non-increasing in variables t_5 and t_3;

(F_2) there exists $h \in (0,1)$ such that for every $u, v \ge 0$ with $F(u,v,v,u,u+v,0) \le 0$ we have $u \le hv$.

Example 3.1. *$F(t_1,\dots,t_6) = t_1 - a_1t_2 - a_2t_3 - a_3t_4 - a_4t_5 - a_5t_6$, where $a_1,\dots,a_5 \in \mathbb{R}_+$, $a_1 + a_2 + a_4 > 0$ and $a_1 + a_2 + a_3 + 2a_4 < 1$.*

(F_1) *Obviously.*

(F_2) *Let $F(u,v,v,u,u+v,0) = u - a_1v - a_2v - a_3u - a_4(u+v) \le 0$. Then $u \le hv$, where $0 < h = (a_1 + a_2 + a_4)/(1 - a_3 - a_4) < 1$.*

Example 3.2. *$F(t_1,\dots,t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, where $0 < k < 1$.*

(F_1) *Obviously.*

(F_2) *Let $F(u,v,v,u,u+v,0) = u - k \max\left\{u, v, \frac{1}{2}(u+v)\right\} \le 0$. If $u > 0$ and $u \ge v$, then $u(1-k) \le 0$, a contradiction. Hence $u < v$ which implies $u \le hv$, where $0 < h = k < 1$. If $u = 0$, then $u \le hv$.*

Example 3.3. *$F(t_1,\dots,t_6) = t_1^2 + at_1t_2 - bt_3t_4 - ct_5t_6$, where $a, c \ge 0$ and $0 < b < 1$.*

(F_1) *Obviously.*

(F_2) *Let $F(u,v,v,u,u+v,0) = u^2 + auv - buv \le 0$, which implies $u^2 - buv \le 0$. If $u > 0$, then $u \le hv$, where $0 < h = b < 1$. If $u = 0$, then $u \le hv$.*

Example 3.4. $F(t_1,\dots,t_6)=t_1^2+\frac{t_1}{1+t_5+t_6}-(at_2^2+bt_3^2+ct_4^2)$, *where* $a,b,c\ge 0$, $a+b>0$ *and* $a+b+c<1$.

(F_1) *Obviously.*

(F_2) *Let* $F(u,v,v,u,u+v,0)=u^2+\frac{u}{1+u+v}-(av^2+bv^2+cu^2)\le 0$, *which implies* $u^2-(av^2+bv^2+cu^2)\le 0$. *Hence* $u\le hv$, *where* $0<h=\sqrt{\frac{a+b}{1-c}}<1$.

Example 3.5. $F(t_1,\dots,t_6)=\max\{t_1,t_2,t_4\}-\min\{t_5,t_6\}-qt_3$, *where* $0<q<1$.

(F_1) *Obviously.*

(F_2) *Let* $F(u,v,v,u,u+v,0)=\max\{u,v\}-qv\le 0$. *If* $u>0$ *and* $u>v$, *then* $u(1-q)\le 0$, *a contradiction. Hence* $u<v$ *and* $u\le hv$, *where* $0<h=q<1$. *If* $u=0$, *then* $u\le hv$.

The purpose of this paper is to prove a general theorem which generalizes the results from Theorems 2.2, 2.3 and 2.4.

4. Main results

Theorem 4.5. *Let* (X,d) *be a metric space and* $F:X\to P_{cl}(X)$ *a multivalued function such that*

$$\Phi(H(F(x),F(y)),d(x,y),D(x,F(x)),D(y,F(y)),\\ D(x,F(y)),D(y,F(x)))\le 0, \tag{4.1}$$

for each $x,y\in X$, *where* $\Phi\in\mathcal{F}$.

Then there exists $f:X\to X$ *a Caristi selection of* F.

Proof. Let $\varepsilon=\frac{1-h}{2}$ and $g(x)=\frac{1}{\varepsilon}D(x,F(x))$. Then, obviously $\varepsilon+h=\frac{1+h}{2}<1$. Since $\frac{1}{\varepsilon+h}>1$, for each $x\in X$ we can choose $f(x)\in F(x)$ such that

$$d(x,f(x))\le\frac{1}{\varepsilon+h}D(x,F(x)).$$

By 4.1 and (F_1) we have successively

$$\Phi(H(F(x),F(f(x))),d(x,f(x)),D(x,F(x)),D(f(x),F(f(x))),\\ D(x,F(f(x))),D(f(x),F(x)))\le 0,$$

$$\Phi(D(f(x),F(f(x))),d(x,f(x)),d(x,f(x)),D(f(x),F(f(x))),\\ d(x,f(x))+D(f(x),F(f(x))),0)\le 0.$$

By (F_2) we have

$$D(f(x), F(f(x))) \leq h\, d(x, f(x)),$$

for each $x \in X$.

We will prove now that f is a Caristi type operators. Indeed, for each $x \in X$ we have

$$d(x, f(x)) = \frac{1}{\varepsilon}[(\varepsilon + h)d(x, f(x)) - h\, d(x, f(x))] \leq$$

$$\leq \frac{1}{\varepsilon}[D(x, F(x)) - D(f(x), F(f(x)))] = g(x) - g(f(x)). \qquad \square$$

Remark 4.1. Theorems 2.2-2.4 follow from Theorem 4.1 and, respectively Example 3.1, for $a_4 = a_5 = 0$, Example 3.1 and Example 3.2.

Remark 4.2. Other results can be obtained from Theorem 4.1 via Examples 3.3-3.5.

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