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# COMMON FIXED POINT THEOREMS FOR ASYMPTOTICALLY $I$-CONTRACTIVE MAPPINGS WITHOUT CONVEXITY

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**Abstract.** This paper aims to present common fixed point theorems for $I$-nonexpansive mappings from $(I, T)$-star shaped subset of a uniformly convex Banach space $X$ into $X$ under some asymptotic $I$-contraction assumptions. These results extend and generalize results valid for bounded convex sets or asymptotically compact sets.
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In this note we extend and generalize a famous result by Browder [2], Göhde [5] and Kirk [7] recently extended by Luc in [13] and a recent result of Penot[20] by using the notion of asymptotically $I$-compact subset of a Banach space. However, it may be remarked that here no compactness assumption is involved. Instead we use asymptotic contractiveness concepts; a comparison of this concept with other notions of asymptotic conditions (e.g., uniform asymptotic introduced in [18] and asymptotic contractiveness for single map

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introduced in [20]) will be made later on. Note that the meaning of the word "asymptotic" is subtle. Indeed, the word "asymptotic" is not related to the iterations of the map, as in [7], [8], [9], [21] but refers to the behaviour of the map at infinity.

Recall that a subset $C$ of a linear space $X$ is star shaped with respect to $q$ ( or, briefly, star shaped) if there exists a $q \in C$ such that

$$kx + (1-k)q \in C$$

for any $k \in [0,1]$ and $x \in C$. Of course, if $C$ is convex, then it is star shaped with respect to any $q \in C$. Here $q$ is called the star centre of $C$.

**Definition 1.** Let $C$ be a subset of a linear space $X$ and let $T, I : C \to X$. Then $C$ is said to be $(I,T)$-star shaped with respect to $q$ ( or, briefly, $(I,T)$-star shaped) if there exists a $q \in C$ such that

$$kT(x) + (1-k)q \in I(C)$$

for any $k \in [0,1]$ and $x \in C$.

If $I$ and $T$ both are identity maps on $C$, then the definition of $(I,T)$-star shaped reduces to the ordinary definition of star shaped.

**Definition 2.** Let $X$ be a Banach space and let $C$ be a subset of $X$. Let $T, I : C \to X$. Let $C$ be an $(I,T)$-star shaped subset of $X$. We say that $T$ is asymptotically $I$-contractive on $C$ if, for some $q \in C$,

$$\lim_{n\to\infty} \sup_{x\in C,\, \|x\|>n,\, \|I(x)\|>n} \frac{\|T(x)-T(q)\|}{\|I(x)-I(q)\|} < 1. \tag{1}$$

Note that this condition is independent of the choice of $q \in C$. To see this, let us consider $q' \in C$ such that $q' \neq q$, then

$$\lim_{n\to\infty} \sup_{x\in C,\, \|x\|>n,\, \|I(x)\|>n} \frac{\|T(x)-T(q')\|}{\|I(x)-I(q')\|}$$

$$\leq \lim_{n\to\infty} \sup_{x\in C,\, \|x\|>n,\, \|I(x)\|>n} \Big[\frac{\|T(x)-T(q)\| + \|T(q')\|}{\|I(x)-I(q)\|} \cdot \frac{\|I(x)-I(q)\|}{\|I(x)-I(q')\|}\Big]$$

$$= \lim_{n\to\infty} \sup_{x\in C,\, \|x\|>n,\, \|I(x)\|>n} \Big[\Big\{\frac{\|T(x)-T(q)\|}{\|I(x)-I(q)\|} + \frac{\|T(q')\|}{\|I(x)-I(q)\|}\Big\} \cdot \frac{\|I(x)-I(q)\|}{\|I(x)-I(q')\|}\Big]$$

$$< 1.$$

If $I$ is the identity map on $C$, then $T$ is said to be asymptotically contractive on $C$ if, for some $q \in C$,

$$\mathop{lim\,sup}_{x\in\ C,\ \|x\|\to\infty} \frac{\|T(x)-T(q)\|}{\|x-q\|} < 1. \tag{1'}$$

It may be observed that the notion of asymptotically $I$-contractive map enables us to extend to unbounded sets the result of [2], [5], [7] valid for $I$-nonexpansive self-mappings on closed star-shaped bounded subsets of uniformly convex Banach spaces. Note that every convex subset of a Banach space is star shaped but the converse need not be true. For example, one may observe that $C = \{(x,0) : x \in [0,\infty)\} \bigcup \{(0,y) : y \in [0,\infty)\}$ is a star-shaped subset of $\mathbb{R}^2$ with respect to $(0,0)$, but it is neither bounded nor convex. Define $T, I : C \to X$ by $T(x,0) = (\frac{x}{2},0)$, if $x \in [0,1], T(x,0) = (0,0)$ if $x > 1$ and $T(0,y) = (0,0)$ if $y \geq 0$; $I(x,0) = (\frac{x}{2},0)$, if $x \in [0,1], I(x,0) = (0,0)$ if $x > 1$ and $I(0,y) = (\frac{y}{2},0)$, if $y \in [0,1], I(0,y) = (0,0)$ if $y > 1$. Clearly, $C$ is $(I,T)$-star shaped with respect to $q = (0,0)$. Observe that $I(C)$ is bounded, closed and convex.

Recall that a mapping $T$ is $I$-nonexpansive in $C$, if $\|T(x) - T(y)\| \leq \|Ix - Iy\|$ for any $x, y$ in $C$.

**Definition 3.** Let $X$ be a Banach space and let $C$ be a subset of $X$. Let $T, I : C \to X$. Let $C$ be an $(I,T)$-star shaped subset of $X$ with respect to some $q \in C$( or, briefly, $(I,T)$-star shaped). Then $T$ is said to be *radiallyasymptoticallyI − contractive* with respect to some $q \in C$ in the sense that for any unit vector $u$ in the asymptotic cone (or horizon cone)

$$C_\infty := \lim_{t\to\infty} sup\, t^{-1}C := \{v \in X : \exists (t_n) \to \infty, (v_n) \to v, t_n v_n \in C \forall n \in \mathbb{N}\}$$

of $C$ one has

$$\lim_{t\to\infty,\ q+tu\in C} sup \frac{\|T(q+tu) - T(q)\|}{\|I(q+tu) - I(q)\|} < 1.$$

If $I$ is the identity map on $C$, then the above inequality reduces to

$$\lim_{t\to\infty,\ q+tu\in C} sup \frac{1}{t}\|T(q+tu) - T(q)\| < 1.$$

In such case, $T$ is said to be *radiallyasymptoticallycontractive* with respect to some $q \in C$.

Recall that two mappings $T : C \to X$ and $I : C \to X$ are said to be weakly compatible in $C$ if $TI(v) = IT(v)$ whenever $T(v) = I(v)$ for some $v$ in $C$. We now prove the following variant of the main result of Jungck [6].

**Proposition 4.** *Let $C$ be a subset of a Banach space $(X, \|\cdot\|)$ and let $T, I : C \to X$ be two non-self maps satisfying the inequality*

$$\|Tx - Ty\| \leq \lambda\|Ix - Iy\| \tag{2}$$

*for all $x, y \in C$, where $0 < \lambda < 1$. If $T(C) \subset I(C)$ and $I(C)$ is closed, then $T$ and $I$ have a coincidence point $v$ in $C$. Further, if $I^2v = Iv$, and $T$ and $I$ are weakly compatible in $C$, then $T$ and $I$ have a unique common fixed point.*

**Proof.** Let $x_0 \in C$ be arbitrary. Since $Tx_0 \in I(C)$, there is some $x_1 \in C$ such that $Ix_1 = Tx_0$. Then choose $x_2 \in C$ such that $Ix_2 = Tx_1$. In general, after having chosen $x_n \in C$ we choose $x_{n+1} \in C$ such that $Ix_{n+1} = Tx_n$. We now show that $\{Ix_n\}$ is a Cauchy sequence. From (2) we have

$$\|Ix_n - Ix_{n+1}\| = \|Tx_{n-1} - Tx_n\| \leq \lambda\|Ix_{n-1} - Ix_n\|.$$

Repeating the above argument n-times we get

$$\|Ix_n - Ix_{n+1}\| \leq \lambda\|Ix_{n-1} - Ix_n\| \leq \ldots \leq \lambda^n\|Ix_0 - Ix_1\|.$$

It then follows that, for any $m > n$,

$$\|Ix_n - Ix_m\| \leq \frac{\lambda^n}{1-\lambda}\|Ix_0 - Ix_1\| \to 0 \text{ as } m > n \to \infty.$$

Thus $\{Ix_n\}$ is a Cauchy sequence. Since $I(C)$ is closed in $X$ and so complete, there is some $u \in I(C)$ such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_{n-1} = u.$$

Since $u \in I(C)$, there exists a $v \in C$ such that $Iv = u$. From (2) we get

$$\|Tx_n - Tv\| \leq \lambda\|Ix_n - Iv\|.$$

Taking the limit as $n \to \infty$ we obtain

$$\|u - Tv\| \leq \lambda\|u - Iv\|.$$

Hence $Tv = u$; i.e., $v$ is a coincidence point of $T$ and $I$. Further, if $I^2v = Iv$ and $T$ and $I$ weakly compatible in $C$ we get

$$u = Iv = I^2v = Iu = ITv = TIv = Tu.$$

Hence $u$ is a common fixed point of $T$ and $I$. The uniqueness of the common fixed point $u$ follows from (2).

Following essentially the same reasoning as in Propositions 10.1 and 10.2 in Goebel and Kirk [5], we can easily prove the following Propositions 5 and 6, respectively.

**Proposition 5.** *Suppose $C$ is a subset of a uniformly convex Banach space $X$ and suppose $T : C \to X$ and $I : C \to X$ are two non-self maps such that mapping $T$ is $I$-nonexpansive in $C$ and $I(C)$ is bounded and convex. Then for $\{u_n\}, \{v_n\}, \{z_n\}$ in $C$ and $Iz_n = \frac{1}{2}(Iu_n + Iv_n)$,*

$$\lim_{n\to\infty} \|Iu_n - Tu_n\| = 0, \ \lim_{n\to\infty} \|Iv_n - Tv_n\| = 0$$

*implies*

$$\lim_{n\to\infty} \|Iz_n - Tz_n\| = 0.$$

**Proposition 6.** *Suppose $C$ is a subset of a uniformly convex Banach space $X$ and suppose $T : C \to X$ and $I : C \to X$ are two non-self maps such that $T$ is $I$-nonexpansive in $C$, $I(C)$ is bounded, closed and convex and satisfy $inf\{\|Ix - Tx\| : x \in C\} = 0$. Then $T$ and $I$ have a coincidence point in $C$.*

The following proposition is an easy consequence of Propositions 5 and 6.

**Proposition 7.** *Let $X$ be a uniformly convex Banach space, $C$ a subset of $X$, $T : C \to X$ and $I : C \to X$ be two non-self maps such that mapping $T$ is $I$-nonexpansive in $C$, $I(C)$ is bounded, closed and convex subset of $X$. Then the mapping $f = I - T$ is demiclosed on $X$.*

In [20], Penot prove the following result.

**Proposition 8.** *Let $X$ be a uniformly convex Banach space and let $C$ be a closed convex subset of $X$. Let $T : C \to X$ be a nonexpansive map which is asymptotically contractive on $C$ and such that $T(C) \subset C$. Then $T$ has a fixed point.*

We now extend and generalize the above result of Penot [20] for a pair of maps in the following.

**Theorem 9.** *Let $X$ be a uniformly convex Banach space, $C$ a subset of $X$. Let $T, I : C \to X$ and assume that $C$ an $(I, T)$-star shaped subset of $X$. Let $T$ be an $I$-nonexpansive map which is asymptotically $I$-contractive in $C$ and such that $kT(C) + (1-k)C \subset I(C)$ for any $k \in [0, 1]$, $I(C)$ is bounded, closed and convex and $I$ is continuous. Then $T$ and $I$ have a coincidence point $\bar{x}$ in*

*$C$. Further, if $I^2\bar{x} = I\bar{x}$, and $T$ and $I$ are weakly compatible in $C$, then $T$ and $I$ have a unique common fixed point.*

**Proof.** Let $(t_n)$ be a sequence in $(0, 1)$ with limit 0 and let $C$ be an $(I, T)$-star shaped subset of $X$ with respect to some $q \in C$. For $n \in \mathbb{N}$, define $T_n : C \to X$ by

$$T_n(x) := (1 - t_n)T(x) + t_n\, q \ (3)$$

so that, by the $(I, T)$-star shaped property of $C$, $T_n(x) \in I(C)$ for each $x \in C$. Since $T$ is $I$-nonexpansive, it follows that

$$\|T_n(x) - T_n(y)\| = (1 - t_n)\|T(x) - T(y)\|$$
$$\leq (1 - t_n)\|I(x) - I(y)\|$$

i.e., $T_n$ is $I$-contractive with rate $(1 - t_n)$. Proposition 4 ensures that $T_n$ and $I$ have a coincidence point $x_n \in C$.

We shall show that the sequence $(x_n)$ is bounded. If this is not the case, taking a subsequence if necessary, we may assume that $(||x_n||) \to \infty$. Let $\alpha \in (0, 1)$ and $\rho > 0$ be such that $||T(x) - T(q)|| \leq \alpha||I(x) - I(q)||$ for $x \in C$ satisfying $||x|| \geq \rho$. Then, for sufficiently large n, we have

$$||x_n|| = ||(1 - t_n)T(x_n) + t_n\, q||$$
$$\leq (1 - t_n)(\alpha||I(x_n) - I(q)|| + ||T(q)||) + t_n||q||.$$

Noting that $I(C)$ is bounded, dividing both sides by $||x_n||$ and taking limits, we get $1 \leq \alpha$, a contradiction, Thus, $(x_n)$ and hence $(T(x_n))$ is bounded, and

$$||I(x_n) - T(x_n)|| = t_n||q - T(x_n)|| \to 0, \text{ as } n \to \infty.$$

Since $X$ is reflexive, taking a subsequence if necessary, we may assume that $(x_n)$ has a weak limit, say, $\bar{x}$. Since $I - T$ is demi-closed (i.e. its graph is sequentially closed in the product of the weak topology with the norm topology), we get that $I(\bar{x}) - T(\bar{x})$ =0; i.e., $\bar{x}$ is a coincidence point of $I$ and $T$. Further, if $I^2\bar{x} = I\bar{x}$, and $T$ and $I$ weakly commute in $C$, we have

$$I\overline{x} = I^2\overline{x} = IT\overline{x} = TI\overline{x}$$

showing, that $I\overline{x}$ is a common fixed point of $T$ and $I$. □

**Remarks.** a) One can add that the set of common fixed points is closed, convex and bounded. The first two properties are proved in the usual way; the boundedness property follows immediately from (1).

b) The preceding result can also be deduced from the classical result of [2], [6], [8] by applying it to the restriction of $T$ to a sufficiently large ball in $X$. This direct way can be deduced from the preceding proof. It also follows from the observation that $T$ is asymptotically $I$-contractive on $C$ iff there exists some $c \in (0,1)$ and $r > 0$ such that

$$||T(x)|| \leq c||I(x)|| \; \forall x \in C \backslash rB_X,$$

whenever $\|x\| \to \infty$ implies $\|I(x)\| \to \infty$, where $B_X$ is the closed unit ball of $X$, so that $T(C \cap rB_X) \subset I(C \cap rB_X)$. □

If $I$ is the identity map on $C$ , then $T$ is asymptotically contractive on $C$ iff there exists some $c \in (0,1)$ and an $r > 0$ such that

$$\|T(x)\| \leq c\|x\| \; \forall x \in C \backslash rB_X,$$

where $B_X$ is the closed unit ball of $X$, so that $T(C \cap rB_X) \subset C \cap rB_X$. □

**Definition 10.** A subset $C$ of a uniformly convex Banach space $X$ is said to be *asymptotically I-compact* if, for any sequence $(x_n)$ of $C$ such that $\|x\| \to \infty$ implies $\|I(x)\| \to \infty$ and that $(r_n) := (\|I(x_n)\|) \to \infty$, the sequence $(r_n^{-1} I(x_n))$ has a convergent subsequence. Locally compact convex sets and epigraphs of hyper-coercive functions $T : C \to X$; i.e., *epi* $T = \{(y,t) \in C \times \mathbb{R} : T(y) \leq t\}$ with respect to $I : C \to X$ are asymptotically $I$-compact in the sense that $T(x)/\|I(x)\| \to \infty$ as $\|I(x)\| \to \infty$. If $I$ is the identity map on $C$, then $C$ is called asymptotically compact (see [21]).

We now compare the preceding result with [13] Theorem 5.1. There, $C$ is assumed to be *asymptotically compact* in the sense of [1], [15], [19], [22] (see also [4], [17], [23]); i.e., for any sequence $(x_n)$ of $C$ such that $(r_n) := (\|x_n\|) \to \infty$, the sequence $(r_n^{-1} x_n)$ has a convergent subsequence . Obviously this assumption is satisfied in finite dimensions; but it is a rather restrictive assumption in infinite dimensional spaces. However, locally compact convex sets and epigraphs of hyper-coercive functions (i.e. functions $T$ such that $T(x)/\|x\| \to \infty$ as $\|x\| \to \infty$) are asymptotically compact.

On the other hand, the asymptotic condition imposed on $T$ in [14] is milder than the one considered here. Our asymptotic condition is obviously satisfied if $T$ is asymptotically $I$-contractive on $C$. In fact, if $T$ is $I$-nonexpansive , $T$ is radially asymptotically $I$-contractive if and only if it is directionally asymptotically $I$-contractive in the sense that for any unit vector $u \in C_\infty$ one has

$$\lim_{t\to\infty,\, v\to u,\, q+tv\in C} sup \frac{\|T(q+tu)-T(q)\|}{\|I(q+tu)-I(q)\|} < 1,$$

whenever $\|x\| \to \infty$ implies $\|I(x)\| \to \infty$ and one has the following relationships with our assumption.

**Lemma 11.** *Any asymptotic $I$-contraction $T : C \to X$ is a directional asymptotic $I$-contraction. If $C$ is asymptotically $I$-compact the converse holds.*

**Proof.** The first part of the assertion is immediate. To prove the second part, assume that $T$ is not an asymptotic $I$-contraction; i.e., for any $q \in C$ there exists a sequence$(x_n)$ in $C$ such that $(\|x_n\|) \to \infty$ and $lim_n t_n^- 1\|T(x_n) - T(q)\| \geq 1$ for $t_n := \|I(x_n) - I(q)\|$. Since $C$ is asymptotically $I$-compact, the sequence $(u_n) := (t_n^- 1(I(x_n) - I(q)))$ has a convergent subsequence with limit $u \in C_\infty$. Since $lim_n t_n^- 1\|T(x_n) - T(q)\| \geq 1$, $T$ is not an asymptotic $I$-contraction.

For the case when $I$ is the identity map on $C$, we recover Lemma 3 of Penot [20] in the following .

**Corollary 12 ([20]).** *Any asymptotic contraction $T : C \to X$ is a directional asymptotic contraction. If $C$ is asymptotically compact the converse holds.*

It follows from Lemma 11 above that Theorem 5.1 of [14] is a direct consequence of Theorem 9. We also observe that Corollary 3 of [20] (stated below) is an immediate consequence of a corollary in [8].

**Corollary 13 ([14]).** *Let $X$ be a uniformly convex Banach space and let $C$ be a closed convex subset of $X$. Let $T : C \to X$ be a nonexpansive map which is radially asymptotically contractive on $C$ and such that $T(C) \subset C$. Then $T$ has a fixed point.*

It may be remarked that the assumption of uniform convexity in Corollary 13 above is not needed. It is sufficient to know that bounded closed convex sets have the fixed point property for nonexpansive maps (see, for instance, Kirk [8]). We now present a criterion in order that $T$ be asymptotially $I$-contractive. It relies on the following notion introduced in [19]. Here $X$ is any normed linear space and $B_X$ denotes its closed unit ball.

**Definition 14.** A cone $K$ of $X$ is a firm (outer)asymptotic cone of a subset $C$ of $X$ if for any $\epsilon > 0$ there exists some $r > 0$ such that for any $x \in C \setminus rB_X$ one has $d(x, K) < \epsilon\|x\|$.

We now introduce, in more general form, a variant of concepts due to Krasnoselski [12].

**Definition 15.** Given a firm asymptotic cone $K$ of a subset $C$ of $X$, a positively homogeneous mapping $T_\infty : K \to X$ is said to be a firm (outer) asymptotic derivative of $T : C \to X$ with respect to $I : C \to X$ if for any $\epsilon > 0$, there exists a $\rho > 0$ such that, for any $x \in C \setminus \rho B_X$, there exists a $v \in K$ satisfying $\|x - v\| < \epsilon \|I(x)\|$,

$$\|T(x) - T_\infty(v)\| < \epsilon \|I(x)\|.$$

If $I$ is the identity map on $C$, $T_\infty : K \to X$ is called a firm (outer) asymptotic derivative of $T : C \to X$.

Note that this condition is satisfied when $T : C \to X$ has a firm ( or strong ) asymptotic derivative ( or F-derivative at infinity ) with respect to $I : C \to X$ in the sense that there exists a continuous linear mapping $T_\infty : X \to X$ such that

$$\lim_{r\to\infty} \sup_{x \in C \setminus rB_X} \frac{1}{\|I(x)\|} \|T(x) - T_\infty(x)\| = 0.$$

The following criterion was established in [12, section 3.2.2].

**Lemma 16.** Suppose $T : X \to X$ is G*âteaux* differentiable on $X \setminus rB_X$ for some $r > 0$ and there exists a continuous linear mapping $A : X \to X$ such that $\|T'(x) - A\| \to 0$ as $\|x\| \to \infty$. Then $T$ has a firm (or strong) asymptotic derivative $T_\infty = A$.

A weaker condition than the above is that of asymptotable. A map $T : C \to X$ is said to be asymptotable if there exists a positively homogeneous map $T_\infty : C_\infty \to X$ such that, for any $u \in C_\infty$, one has $t^{-1}T(tv) \to T_\infty(u)$ as $(t, v) \to (\infty, u)$ with $tv \in C$ (see [20]).

For asymptotable maps, the following criterion was established in [20].

**Lemma 17([20]).** If $T : C \to X$ is asymptotable and if $C$ is asymptotically compact, then $T_\infty$ is a firm asymptotic semi-derivative of $T$.

We now state and prove the announced criterion for asymptotic $I$-contractiveness of the mapping $T$.

**Proposition 18.** Let $K$ be a firm asymptotic cone of a subset $C$ of $X$. Suppose $T : C \to X$ has a firm asymptotic semi-derivative $T_\infty : K \to X$ with respect to $I : C \to X$, which is asymptotically $I$-contractive on $K$. Then $T$ is asymptotic $I$-contractive on $C$.

**Proof.** From the observation following Definition 2, we can take $q = 0$ in that definition applied to $T_\infty$ and $K$, so that there exists some $c \in (0,1)$ such that

$$\|T_\infty(v)\| \leq c\|I(v)\|$$

for $v \in K$ with sufficiently large norm. Since $K$ is a cone and $T_\infty$ is posivetively homogeneous, this relation is satisfied for any $v \in K$. Let $c' \in (c,1)$ and let $\epsilon > 0$ be such that $c + 3\epsilon < c'$. Then, taking $\rho > 0$ associated with the $\epsilon$ in Definition 14, for any $x \in C\backslash\rho B_X$ we can pick a $v \in K$ satisfying $\|x - v\| < \epsilon\|x\|$, $\|T(x) - T_\infty(v)\| < \epsilon\|I(x)\|$, so that we get

$$\begin{aligned}
\|T(x) - T(q)\| &= \|T(x) - T_\infty(v) + T_\infty(v) - T(q)\| \\
&\leq \|T(x) - T_\infty(v)\| + \|T_\infty(v)\| + \|T(q)\| \\
&\leq \epsilon\|I(x)\| + \|T_\infty(v)\| + \|T(q)\| \\
&\leq \epsilon\|I(x)\| + c\|I(v)\| + \|T(q)\| \\
&\leq 2\epsilon\|I(x)\| + c\|I(x)\| + \|T(q)\| \\
&\leq (c + 2\epsilon)\|I(x) - I(q)\| + \|T(q)\| + (c + 2\epsilon)\|I(q)\| \\
&\leq (c + 3\epsilon)\|I(x) - I(q)\| \\
&\leq c'\|I(x) - I(q)\|
\end{aligned}$$

provided

$$\epsilon\|I(x) - I(q)\| \geq \|T(q)\| + (c + 2\epsilon)\|I(q)\|,$$

which occurs when $\|I(x)\| \geq \epsilon^{-1}(\|T(q)\| + (c + 3\epsilon)\|I(q)\|)$. □

Finally, combining Propositions 7 and 18 yields.

**Theorem 19.** Let $X$ be a uniformly convex Banach space, $C$ a nonempty subset of $X$ and $T, I : C \to X$. Let $C$ be an $(I,T)$-star shaped subset of $X$ and let $K$ be a firm asymptotic cone of $C$. Suppose that $I(C)$ is bounded, closed and convex and $I$ is continuous. Let $T$ be an $I$-nonexpansive map which has a firm asymptotic semi-derivative $T_\infty : K \to X$ which is asymptotically $I$-contractive on $K$. Then $T$ and $I$ have a coincidence point $v$ in $C$. Further, if $I^2v = Iv$, and $T$ and $I$ are weakly compatible in $C$, then $T$ and $I$ have a unique common fixed point.

If $I$ is the identity map on $C$, we obtain the following result.

**Corollary 20 ([20]).** Let $X$ be a uniformly convex Banach space and let $C$ be a closed convex subset of $X$. Let $K$ be a firm asymptotic cone of $C$.

Let $T : C \to X$ be a nonexpansive map which has a firm asymptotic semi-derivative $T_\infty : K \to X$ which is asymptotically contractive on $K$. Then $T$ has a fixed point.

**Retrospect.** The result in Theorem 19 does not involve any compactness assumptions. However, such compactness assumption can be used as criteria ensuring its hypothesis, according to Lemmas 11 and 16. These criteria clearly shows the links with the results by Luc [14].

The result of Theorem 19 can be extended to real worlds nonconvex situations or to more general spaces as in [3], [8]-[10], [16], [24]. We also refer the interested reader to [4],[14],[15], [17]-[23] for the use of asymptotic compactness in various fields.

**Open Question.** To what extent can one weaken the condition of asymptotic $I$-contractive assumption in Theorems 9 and 19?

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