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FIXED POINT APPROACH TO SOME TWO-POINT BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS ON MANIFOLDS

YURI E. GLIKLIKH* AND PETER S. ZYKOV**

* Mathematics Faculty, Voronezh State University,
Universitetskaya pl. 1, 394006 Voronezh, Russia
E-mail: yeg@math.vsu.ru

** Physics and Mathematics Faculty, Kursk State University,
ul. Radishcheva 33, 305416 Kursk, Russia
petya39b@mail.ru

Abstract. We investigate the two-point boundary value problem for second order differential inclusions of the form $\frac{D}{dt}\dot{m}(t) \in F(t, m(t), \dot{m}(t))$ on a complete Riemannian manifold for a couple of points, non-conjugate along at least one geodesic of Levi-Civitá connection, where $\frac{D}{dt}$ is the covariant derivative of Levi-Civitá connection and $F(t, m, X)$ is convex-valued and satisfies the upper Carathéodory condition or is almost lower semi-continuous set-valued vector field such that $\|F(t, m, X)\| < a(t, m)\|X\|^2$ with continuous $a(t, m) > 0$. Some conditions on certain geometric characteristics, on the distance between points and on $a(t, m)$ are found, under which the problem is solvable on any time interval. The solution is constructed from a fixed point of a certain integral-type operator, acting in the space of continuous curves in the tangent space at initial point. The existence of fixed point is proved by application of Bohnenblust-Karlin and Schauder theorems.

Key Words and Phrases: Second order differential inclusion, complete Riemannian manifold, quadratic growth, two-point boundary value problem, set-valued map, fixed point.
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1. Introduction

Let M be a finite-dimensional complete Riemannian manifold and TM be its tangent bundle with the natural projection $\pi : TM \to M$. Consider a set-valued map $F : R \times TM \to TM$ such that for any point $(m, X) \in TM$

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(this means that $X \in T_mM$, i.e., X is a tangent vector to M at the point $m \in M$) the relation $\pi F(t,m,X) = \pi(m,X) = m$ holds.

We investigate the differential inclusion of the form

$$\frac{D}{dt}\dot{m}(t) \in F(t, m(t), \dot{m}(t)) \tag{1}$$

where $\frac{D}{dt}$ is the covariant derivative of Levi-Civita connection. Such inclusions arise in description of complicated mechanical systems on nonlinear configuration spaces where the set-valued right-hand side F is generated by an essentially discontinuous force field or by a force with control (see, e.g., [4, 5]). That is why everywhere below we call F a set-valued force field.

In this paper we deal with the two-point boundary value problem for inclusion (1). For differential equations and inclusions of this sort with various kinds of F this problem was investigated in many publications (see, e.g., references in [8]). An important feature of this problem on manifolds is that if two points are conjugate along all geodesics of Levi-Civita connection, joining them, there may not exist a solution of this problem even for single-valued smooth uniformly bounded force fields (as well as for force fields with linear and quadratic growth, see examples in [4, 8]). Recall that for F with quadratic growth this problem may not be solvable also for non-conjugate points (even in Euclidean spaces, see an example in [8]).

The set-valued force field $F(t, m, X)$ is said to have quadratic growth in X if for any compact $\Theta \subset M$ and any finite interval $[0, l]$ the relation $\lim\limits_{\|X\|\to\infty} \frac{\|F(t,m,X)\|}{\|X\|^2} = a(t, m)$ holds uniformly in $t \in [0, l]$ and $m \in \Theta$ where $a(t, m) \geq 0$ is a continuous real-valued function on $[0, l] \times \Theta$ that is not identically equal to zero. As usual, here $\|F(t, m, X)\| = \sup\limits_{y\in F(t,m,X)} \|y\|$ where the norms of vectors are generated by the Riemannian metric.

In [8] some conditions on F with quadratic growth, on the two points and on some geometric characteristics of the manifold were found, under which for the points, that are not conjugate along at least one geodesic, the problem is solvable on a small enough time interval.

Obviously, if the estimate $\|F(t, m, X)\| < a(t, m)(1 + \|X\|^2)$ holds for F for a certain continuous real function $a(t, m) > 0$, it is a particular case of quadratic growth and so the results of [8] are valid for such F.

The main result of this paper is that if the following more restrictive estimate

$$\|F(t, m, X)\| < a(t, m)\|X\|^2$$

holds and the conditions of [8] are satisfied, a solution of two-point boundary value problem for (1) exists on arbitrary finite time interval.

It should be pointed out that the effect of existence of the above-mentioned solution on arbitrary finite time interval was previously known for uniformly bounded F in Euclidean spaces and for single-valued quadratic fields F on manifolds that correspond to vector fields of geodesic sprays of connections on the tangent bundles. In the latter case, applying linear change of time along the solution on a given time interval, one obtains a solution on another time interval and by this method a solution on arbitrary finite interval can be constructed. Notice that this approach is absolutely not applicable for our general set-valued case.

We construct the above mentioned solution from a fixed point of a special integral-type operator that acts in the space of continuous curves of the tangent space at initial point.

Preliminaries from set-valued analysis can be found in [2, 3, 10], and those from geometry of manifolds in [1, 4, 5, 9, 11].

2. Technical statements

In this section we modify some constructions from [4, 5] for the problem under consideration.

Let M be a complete Riemannian manifold. Consider $m_0 \in M$, $[0, 1] \subset R$ and let $v\colon [0, 1] \to T_{m_0}M$ be a continuous curve. It is shown in [4, 5] that there exists unique C^1-curve $m : [0, 1] \to M$ such that $m(0) = m_0$ and the vector $\dot{m}(t)$ is parallel along $m(\cdot)$ to the vector $v(t) \in T_{m_0}M$ at any $t \in [0, 1]$.

Denote the curve $m(t)$, constructed above from the curve $v(t)$, by the symbol $\mathcal{S}v(t)$. Thus, we have defined a continuous operator $\mathcal{S}$ that sends the Banach space $C^0([0, 1], T_{m_0}M)$ of continuous maps (curves) from $[0, 1]$ to $T_{m_0}M$ into the Banach manifold $C^1([0, 1], M)$ of C^1-maps from $[0, 1]$ to M.

By $U_k \subset C^0([0, 1], T_{m_0}M)$ we denote the ball of radius k centered at the origin in $C^0([0, 1], T_{m_0}M)$.

Let a point $m_1 \in M$ be non-conjugate to the point $m_0 \in M$ along a geodesic $g(t)$ of the Levi-Civitá connection. Without loss of generality we postulate that the parameter t on $g(t)$ is taken so that $g(0) = m_0$ and $g(1) = m_1$.

Lemma 1. *There exists a ball $U_\varepsilon \subset C^0([0,1], T_{m_0}M)$ with a radius $\varepsilon > 0$ and centered at the origin such that for any curve $\hat{u}(t) \in U_\varepsilon \subset C^0([0,1], T_{m_0}M)$ there exists a unique vector $\mathbf{C}_{\hat{u}}$, belonging to a certain bounded neighborhood V of the vector $\dot{g}(0)$ in $T_{m_0}M$, that is continuous in $\hat{u}$ and such that $\mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(1) = m_1$.*

Proof. By the construction of operator $\mathcal{S}$ its value $\mathcal{S}v_g(1)$ on the constant curve $v_g(t) = \dot{g}(0)$ coincides with $exp_{m_0}\dot{g}(0) = m_1$. Since m_0 are m_1 are not conjugate along g, exp_{m_0} is a diffeomorphism of a certain neighborhood of $\dot{g}(0) \in T_{m_0}M$ onto a neighborhood of the point m_1 in M. Applying the implicit function theorem, one can easily show that the perturbation of exponential map, that sends $X \in T_{m_0}M$ to $\mathcal{S}(X + \hat{u})(1)$, is also a diffeomorphism of a certain neighborhood V of $\dot{g}(0)$ onto a neighborhood of m_1 in M for any curve $\hat{u}(t)$ from a small enough ε-neighborhood of the origin in $C^0([0,1], T_{m_0}M)$. $\square$

Introduce the notation $\sup\limits_{\mathbf{C} \in V} \|\mathbf{C}\| = C$ where V is from Lemma 1.

Remark 2. *One can easily show that $\varepsilon < C$.*

Lemma 3. *In conditions and notations of Lemma 1 let $R > 0$ and $t_1 > 0$ be such that $t_1^{-1}\varepsilon > R$. Then for any curve $u(t) \in U_R \subset C^0([0,t_1], T_{m_0}M)$ there exists an unique vector C_u in a neighborhood $t_1^{-1}V$ of the vector $t_1^{-1}\dot{g}(0)$ in $T_{m_0}M$, continuously depending on u and such that $S(u + C_u)(t_1) = m_1$.*

Proof. For $u(t) \in U_R \subset C^0([0,t_1], T_{m_0}M)$ introduce $\hat{u}(t) = t_1 u(t_1 \cdot t) \in U_\varepsilon \subset C^0([0,1], T_{m_0}M)$ and $C_u = t_1^{-1}\mathbf{C}_{\hat{u}}$. From Lemma 1 we get $\mathcal{S}(\hat{u}+\mathbf{C}_{\hat{u}})(1) = m_1$ and $\frac{d}{dt}\mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(t)$ is parallel to $\hat{u}(t) + \mathbf{C}_{\hat{u}}$. For the curve $g(t) = \mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(t \cdot t_1)$ we have $\frac{d}{dt}g(t) = t_1^{-1}\frac{d}{dt}\mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(t \cdot t_1)$ and this vector is parallel along the same curve to the vector $t_1^{-1}(\hat{u}(t) + \mathbf{C}_{\hat{u}}) = u(t) + C_u$. Thus $g(t) = \mathcal{S}(u + C_u)(t) = \mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(t \cdot t_1^{-1})$ for $t \in [0, t_1]$. Hence $\mathcal{S}(u + C_u)(t_1) = \mathcal{S}(\hat{u} + \mathbf{C}_{\hat{u}})(1) = m_1$. $\square$

Lemmas 1 and 3 give a modification of theorem 3.3 from [4].

Lemma 4. *For specified $t_1 > 0$ and $R > 0$ as above all curves $S(v(t) + C_v)(t)$ with $v \in U_R \subset C^0([0, t_1], T_{m_0}M)$ lie in a compact set $\Xi \subset M$ where Ξ depends on ε and C introduced above.*

Indeed, since the parallel translation preserves the norm of a vector, for any $v(t)$ as above the length of $S(v(t)+C_v)(t)$ is not greater than $\int_0^{t_1}(R+\|C_v\|)dt \leq \int_0^{t_1} t_1^{-1}(\varepsilon + C)dt = \int_0^1(\varepsilon + C)dt = \varepsilon + C$. Since M is complete, by Hopf-Rinow theorem any metric ball of finite radius $\varepsilon + C$ is compact.

Lemma 5. *For $0 < \delta < \frac{\varepsilon}{(\varepsilon+C)^2}$ and for any $t > 0$ both roots $K_{1,2}$ of the equation*

$$\delta(Kt + Ct^{-1})^2 = K$$

are positive.

Prof. Transform the equation $\delta(Kt + Ct^{-1})^2 = K$ into the form $(\delta t^2)K + (2C\delta - 1)K + C^2t^{-2}\delta = 0$. Its discriminant is equal to $D = 1 - 4C\delta$. This means that for $\delta < \frac{1}{4C}$ the roots are real and take the form $K_{1,2} = \frac{1-2C\delta \pm \sqrt{1-4C\delta}}{2\delta t^2}$. Since $(1 - 2C\delta) > \sqrt{1 - 4C\delta}2\delta t^2$, we have $K_{1,2} > 0$. But as it is pointed out in Remark 2, $\varepsilon < C$ and so $\frac{\varepsilon}{(\varepsilon+C)^2} < \frac{1}{4C}$. □

Lemma 6. *For $0 < \delta < \frac{\varepsilon}{(\varepsilon+C)^2}$ and for all $t > 0$ the inequvality $t^{-1}\varepsilon > \frac{1-2C\delta-\sqrt{1-4C\delta}}{2\delta t}$ holds.*

Proof. In order to prove this statement, consider the following system

$$\begin{cases} \delta < \frac{1}{4C} \\ \frac{1-2C\delta-\sqrt{1-4C\delta}}{2\delta} < \varepsilon. \end{cases}$$

By means of elementary transformations, taking into account Remark 2, this system can be transformed into the following form

$$\begin{cases} \left[\begin{array}{l} \delta < \frac{\varepsilon}{\varepsilon^2+2C\varepsilon+C^2} \\ \delta \geq \frac{1}{2(\varepsilon+C)} \end{array} \right. \\ \delta < \frac{1}{4C}. \end{cases}$$

Since by Remark 2 $\varepsilon < C$, from the last system it follows that $\delta < \frac{\varepsilon}{(\varepsilon+C)^2}$. □

3. The main results

Everywhere below M is a complete Riemannian manifold. Without loss of generality we consider $t \in [0,l] = I$ where $[0,l] = I \subset R$ is an arbitrary specified finite interval.

Definition 7. *By a solution of (1) we mean a C^1-curve $m(t)$ with absolutely continuous derivative $\dot{m}(t)$ such that inclusion (1) holds for almost all t.*

We suppose that the estimate $\|F(t,m,X)\| < a(t,m)\|X\|^2$ is satisfied where $a(t,m) > 0$ is a continuous real-valued function on $I \times M$.

Let m_0 and m_1 be two different points in M that are not conjugate along a geodesic $g(\cdot)$ of the Levi-Civita connection. So, all constructions of previous Section are valid. In particular, numbers ε and C from Lemma 1 are well-posed for m_0 and m_1. Denote by Ξ the compact set from Lemma 4. Notice that on the compact set $I \times \Xi$ for the continuous function $a(t,m)$ there exists a certain constant δ such that $0 < a(t,m) < \delta$ for all $(t,m) \in I \times \Xi$.

The main assumption, under which the existence theorems will be proven, is that $\delta < \frac{\varepsilon}{(\varepsilon+C)^2}$ on $I \times \Xi$ where ε and C are from Lemma 1.

We deal with F of the following two types:

Definition 8. *We say that $F(t,m,X)$ satisfies upper Carathéodory conditions if:*

1) for every $(m,X) \in TM$ the map $F(\cdot,m,X) : I \to T_mM$ is measurable,

2) for almost all $t \in I$ the map $F(t,\cdot,\cdot) : TM \to TM$ is upper semi-continuous.

Definition 9. *Let $I = [0,l] \subset R$. The set-valued force field $F : I \times TM \to TM$ is called almost lower semi-continuous if there exists a countable sequence of disjoint compact sets $\{I_n\}$, $I_n \subset I$ such that: (i) the measure of $I \backslash \cup_n I_n$ is equal to zero; (ii) the restriction of F on each $I_n \times TM$ is lower semi-continuous.*

Theorem 10. *Let $F(t,m,X)$ have convex closed bounded values, satisfy the upper Carathéodory condition and $\|F(t,m,X)\| < a(t,m)\|X\|^2$ with a continuous function $a(t,m) > 0$. Let the points m_1 and m_0 be non-conjugate along a certain geodesic $g(\cdot)$ of the Levi-Civitá connection and let the estimate $a(t,m) < \delta$ hold on $I \times \Xi$, where the compact set Ξ is from Lemma 4 and $\delta > 0$ satisfies the inequality $\delta < \frac{\varepsilon}{(\varepsilon+C)^2}$. Then for any $t_1 > 0$, $t_1 \in I$ there exists a solution $m(t)$ of inclusion (1), for which $m(0) = m_0$ and $m(t_1) = m_1$.*

Proof. For a C^1-curve $\gamma(t) = \mathcal{S}v(t)$, $v(\cdot) \in C^0(I, T_{m_0}M)$, consider the set-valued vector field $F(t, \gamma(t), \dot\gamma(t))$. Denote by Γ the operator of parallel translation of vectors along $\gamma(\cdot)$ at the point $\gamma(0) = m_0$. Apply operator Γ to all sets $F(t, \gamma(t), \dot\gamma(t))$ along $\gamma(\cdot)$. As a result for any $v \in C^0(I, T_{m_0}M)$ we obtain a set-valued map $\Gamma F\mathcal{S}v : [0,l] \to T_{m_0}M$ that has convex values. It is shown in [7] that the map $\Gamma F\mathcal{S} : C^0([0,l], T_{m_0}M) \times [0,l] \to T_{m_0}M$ satisfies upper Carathéodory conditions. Denote by $\mathcal{P}\Gamma F\mathcal{S}v$ the set of all measurable selections of $\Gamma F\mathcal{S}v : [0,l] \to T_{m_0}M$ (such selections exist, see e.g., [2]). Define on $C^0([0,t_1], T_{m_0}M)$ the set-valued operator $\int \mathcal{P}\Gamma F\mathcal{S}$ by the formula

$$\int \mathcal{P}\Gamma F\mathcal{S}v = \{\int_0^t f(\tau)d\tau | f(\cdot) \in \mathcal{P}\Gamma F\mathcal{S}v\}.$$

It is shown in [7] that $\int \mathcal{P}\Gamma F\mathcal{S}$ is upper semicontinuous, has convex values and sends bounded sets from $C^0([0,t_1], T_{m_0}M)$ into compacts ones.

Specify a time instant t_1, $0 < t_1 < l$. Taking into account the hypothesis of Theorem, we obtain from Lemma 5 that $K = \frac{1-2C\delta-\sqrt{1-4C\delta}}{2\delta t^2}$ is positive and from Lemma 6 that $t_1^{-1}\varepsilon > Kt_1$. Consider the ball U_{Kt_1} of radius Kt_1 with center at the origin in Banach space $C^0([0,t_1], T_{m_0}M)$. Since $t_1^{-1}\varepsilon > Kt_1$, by Lemma 3 for any $v(\cdot) \in U_{Kt_1}$ the vector C_v is well-posed. Thus we can introduce the operator $\mathcal{Z} : U_{Kt_1} \to C^0([0,t_1], T_{m_0}M)$ by the formula:

$$\mathcal{Z}(v) = \int \mathcal{P}\Gamma F\mathcal{S}(v + C_v).$$

As well as $\int \mathcal{P}\Gamma F\mathcal{S}$, this operator is upper semi-continuous, convex-valued and sends bounded sets from $C^0([0,t_1], T_{m_0}M)$ into compacts ones (see [8]).

Since parallel translation preserves the norms of vectors, from the construction of $\mathcal{S}$ and from the hypothesis we derive that for any $v \in U_{Kt_1}$ and $t \in [0,t_1]$ the estimate

$$\|F(t, S(v(t) + C_v), \frac{d}{d\tau}S(v(t) + C_v))\| < \delta\|v(t) + C_v\|^2$$

holds. By construction $\delta\|v(t) + C_u\|^2 \le \delta(Kt_1 + Ct_1^{-1})^2 = K$. Since parallel translation preserves the norms of vectors, for any curve $u(t) \in \mathcal{Z}v(t)$ and for any $t \in [0,t_1]$ the inequality $\|u(t)\| \le Kt \le Kt_1$ holds. Thus $\mathcal{Z}$ sends the ball U_{Kt_1} into itself and from the Bohnenblust-Karlin fixed point theorem (see, e.g., [2, 10]) it follows that it has a fixed point $u^* \in U_{Kt_1}$, i.e. $u^* \in \mathcal{Z}u^*$. Let us show that $m(t) = \mathcal{S}(u^*(t) + C_{u^*})$ is the desired solution. By the construction we have $m(0) = m_0$ and $m(t_1) = m_1$, $m(t)$ is a C^1-curve and $\dot{m}(t)$ is absolutely continuous. Note that $\dot{u}^*$ is a selection of $\Gamma F(t, \mathcal{S}(u^* + C_{u^*}), \frac{d}{dt}\mathcal{S}(u^* + C_{u^*}))$ because u^* is a fixed point of $\mathcal{Z}$. In other words, the inclusion $\dot{u}^*(t) \in \Gamma F(t, \mathcal{S}(u^* + C_{u^*}), \frac{d}{dt}\mathcal{S}(u^* + C_{u^*}))$ holds for all points t at which the derivative exists. Using the properties of the covariant derivative and the definition of u^*, one can show that $\dot{u}^*(t)$ is parallel to $\frac{D}{dt}\dot{m}(t)$ along $m(\cdot)$ and $\Gamma F(t, \mathcal{S}(u^* + C_{u^*}), \frac{d}{dt}\mathcal{S}(u^* + C_{u^*}))$ is parallel to $F(t, m(t), \dot{m}(t))$. Hence, $\frac{D}{dt}\dot{m}(t) \in F(t, m(t), \dot{m}(t))$. $\square$

Theorem 11. *Let $F(t, m, X)$ be almost lower semicontinuous, have closed bounded values and $\|F(t, m, X)\| < a(t, m)\|X\|^2$ with a continuous function $a(t, m) > 0$. Let the points m_1 and m_0 be non-conjugate along a certain geodesic $g(\cdot)$ of the Levi-Civitá connection and let the estimate $a(t, m) < \delta$ holds on $I \times \Xi$, where the compact set Ξ is from Lemma 4 and $\delta > 0$ satisfies the inequality $\delta < \frac{\varepsilon}{(\varepsilon + C)^2}$. Then for any $t_1 > 0$, $t_1 \in I$ there exists a solution $m(t)$ of inclusion (1), for which $m(0) = m_0$ and $m(t_1) = m_1$.*

Proof. Here we use the same notation as in the proof of Theorem 10. Notice that from the hypothesis it follows that for all $v \in C^0([0, l], T_{m_0}M)$ the curves from $\mathcal{P}\Gamma F\mathcal{S}v$ are integrable. Hence the set-valued map $\mathcal{P}\Gamma F\mathcal{S}$ sends $C^0([0, l], T_{m_0}M)$ into $L^1(([0, l], \mathcal{A}, \mu), T_{m_0}M)$, where $\mathcal{A}$ is Borel σ-algebra and μ is the normalized Lebesgue's measure. Since F is almost lower semicontinuous, in complete analogy with [10] one can easily show that the operator $\mathcal{P}\Gamma F\mathcal{S} : C^0([0, l], T_{m_0}M) \to L^1(([0, l], \mathcal{A}, \mu), T_{m_0}M)$ is lower semicontinuous and has decomposable images (for the definition see, e.g., [2, 3, 10]). Then by Fryszkowski-Bressan-Colombo theorem (see, e.g., [2, 3]) it has a continuous selection that we denote by $p\Gamma F\mathcal{S}$.

Choose the numbers t_1 and K as in the proof of Theorem 10. Then on the ball $U_{Kt_1} \subset C^0([0, t_1], T_{m_0}M)$ the operator

$$\mathcal{G}v = \int_0^t p\Gamma F\mathcal{S}(v(s) + C_v), \frac{d}{dt}\mathcal{S}(v(s) + C_v))ds : U_{Kt_1} \to C^0([0, t_1], T_{m_0}M)$$

is well posed. As a corollary to Lemma 19 of [6] we obtain that $\mathcal{G}$ is completely continuous. Since parallel translation preserves the norm of a vector, from the construction of $\mathcal{S}$, from Lemma 5 and from Lemma 6 for any $u \in U_{Kt_1}$ with given F we get

$$\|\mathcal{G}v\| = \| \int_0^t p\Gamma F(s, \mathcal{S}(v(s)+C_v), \frac{d}{dt}\mathcal{S}(v(s)+C_v))ds\|_{C^0([0,t_1],T_{m_0}M)} \leq$$

$$\delta(Kt_1 + Ct_1^{-1})^2 t_1 = Kt_1.$$

Hence the completely continuous operator $\mathcal{G}$ sends U_{Kt_1} into itself and by classical Schauder's principle it has a fixed point $u^* \in U_{Kt_1}$. Using the same arguments, as in the proof of Theorem 10, one can easily prove that $m(t) = \mathcal{S}(u^* + C_u^*)(t)$ is a solution of (1) such that $m(0) = m_0$ and $m(t_1) = m_1$. $\square$

Remark 12. *Notice that if a geodesic, along which m_0 and m_1 are not conjugate, is a length minimizing one, the number C characterizes the Riemannian distance between these points. The numbers C and ε together provide a certain characteristics of the Riemannian geometry on M in a neighborhood of m_0.*

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