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ON PERIODIC BOUNDARY VALUE PROBLEMS OF FIRST ORDER DISCONTINUOUS IMPULSIVE DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we prove existence results for extremal solutions of the first order discontinuous impulsive differential inclusions with periodic boundary conditions and impulses at the fixed times under certain monotonicity conditions of the multi-valued functions.
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1. Introduction

In this paper, we shall deal with the existence as well as the existence of the extremal solutions of a periodic nonlinear boundary value problems for first order Carathéodory impulsive ordinary differential inclusions without the convex values of multi-valued functions. Given a closed and bounded interval $J := [0, T]$ in $\mathbf{R}$, $\mathbf{R}$ the set of real numbers, and given the impulsive moments $t_0, t_1, t_2, \ldots, t_p$ with $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $J' = J - \{t_1, t_2, \ldots, t_p\}$, $J_j = (t_j, t_{j+1})$, consider the following periodic boundary value problem of impulsive differential inclusions (in short IDI)

$$x'(t) \in F(t, x(t)) \text{ a.e. } t \in J', \tag{1.1}$$

$$x(t_j^+) = x(t_j^-) + I_j(x(t_j)), \tag{1.2}$$

$$x(0) = x(T), \tag{1.3}$$

where $F : J \times \mathbf{R} \to \mathcal{P}_p(\mathbf{R})$ is an impulsive Carathéodory multi-valued function , $\mathcal{P}_p(\mathbf{R})$ denotes the class of non-empty subsets of $\mathbf{R}$ with the property p, $I_j : \mathbf{R} \to \mathbf{R}$, $j = 1, 2, \dots, p$ are the impulse functions and $x(t_j^+)$ and $x(t_j^-)$ are respectively the right and the left limit of x at $t = t_j$.

Let $C(J, \mathbf{R})$ and $L^1(J, \mathbf{R})$ denote respectively the spaces of continuous and Lebesgue integrable real-valued functions on J. We equip the space $L^1(J, \mathbf{R})$ with the norm $\| \cdot \|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\, dt.$$

Denote

$$\begin{aligned} X := \{\, x : J \to \mathbf{R} \,:\, x \in C(J', \mathbf{R})\,,\ x(t_j^+)\,,\ x(t_j^-)\ \text{exist}, \\ \text{and}\ \ x(t_j^-) = x(t_j)\,,\ j = 1, 2, \dots, p \,\} \end{aligned} \tag{1.4}$$

and the space

$$Y := \{\, x \in X \,:\, x \text{ is differentiable a.e. on } (0, T)\,,\ x' \in L^1(J, \mathbf{R}) \,\}\,.$$

By a solution of (1.1)-(1.2)-(1.3), we mean a function

$$x \in Y_T := \{\, x \in Y \,:\, x(0) = x(T) \,\}$$

that satisfies the differential inclusion (1), and the impulsive conditions given in (1.2).

Several papers have been devoted to the study of initial and boundary value problems for impulsive differential inclusions (see for example [2, 3]). For impulsive differential equations in the nonresonance case, see [15, 16, 17] and the references therein. Also, for a general theory on impulsive differential equations we refer the interested reader to the monographs [12] and [18]. Note that the IDI (1.1)-(1.3) has been studied for the existence theorems in Dhage [5] under Lipschitz and Carathéodory conditions. The multi-valued functions were assumed to be upper semi-continuous and the existence was proved via a hybrid fixed point theorem the present author [5]. Our aim in this paper is to provide monotonic sufficient conditions on the multi-valued functions F and the impulsive functions I_j for guarantying the existence as well as well existence of maximal and minimal solutions for the IDI (1.1)-(1.2)-(1.3) on J.

In this approach, we do not require the multi-valued functions in question to satisfy any continuity criteria on the domain of definitions.

2. Preliminaries

Let the space X be defined above and define a norm $\|\cdot\|$ in X by

$$\|x\| = \sup_{t\in J} |x(t)|$$

and define the order relation $\leq$ in X by the cone K given by

$$K = \{x \in X \mid x(t) \geq 0 \text{ for all } t \in J\},$$

which is obviously normal cone in X. Thus we have

$$x \leq y \iff x(t) \leq y(t) \text{ for all } t \in J.$$

Clearly, X becomes a ordered Banach space with respect to the above norm and order relation in X.

Let $A, B \in \mathcal{P}_p(X)$. Denote

$$A \pm B = \{a \pm b : a \in A \quad \text{and} \quad b \in B\},$$

$$\lambda A = \{\lambda a : \lambda \in \mathbf{R} \quad \text{and} \quad a \in A\}.$$

Also denote

$$\|A\| = \{\|a\| : a \in A\}$$

and

$$\|A\|_{\mathcal{P}} = \sup\{\|a\| : a \in A\}.$$

Let the Banach space X be equipped with the order relation $\leq$ and define the order relation in $\mathcal{P}_p(X)$ as follows.

Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean " for every $a \in A$ there exists a $b \in B$ such that $a \leq b$ ". Again $A \overset{d}{\leq} B$ means for each $b \in B$ there exists a $a \in A$ such that $a \leq b$. Further we have $A \overset{id}{\leq} B \iff A \overset{i}{\leq} B$ and $A \overset{d}{\leq} B$. Finally $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$. Note that if $A \leq A$, then it follows that A is a singleton set. See Dhage [6] and references therein.

A correspondence $Q : X \to \mathcal{P}_p(X)$ is called a multi-valued operator on X and a point $u \in X$ is called a fixed point of Q if $u \in Qu$. We denote $T(S) = \bigcup_{x\in S} Tx$ for any subset S of X.

Definition 2.1. *A mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone increasing (resp. left monotone increasing) if $Qx \overset{i}{\leq} Qy$ (resp. $Qx \overset{d}{\leq} Qy$) for all $x, y \in X$ for which $x \leq y$. Similarly, Q is called monotone increasing if it is left as well as right monotone increasing on X. Finally, Q is strict monotone increasing if $Qx \leq Qy$ for all $x, y \in X$ for which $x \leq y$, $x \neq y$.*

Remark 2.1. *Note that every strict monotone increasing multi-valued mapping is right monotone increasing, but the converse may not be true.*

It is known that the monotone technique is a very useful tool for proving the existence of the extremal solutions for differential equations and inclusions. The exhaustive treatment of this method for discontinuous differential equations may be found in Heikkilä and Lakshmikantham [13]. But the use of monotone technique in the theory of differential inclusion involving discontinuous multi-valued functions is relatively new to the literature. Some recent results in this direction appear in Dhage [5, 6, 7, 8]. In this method of monotone technique, the operator in question is required to satisfy certain monotonicity condition with respect to certain order relation on the domain of the definition. The following two fixed point theorems are fundamental in the monotone theory for discontinuous differential inclusions involving the right or strict monotone increasing multi-valued functions.

Theorem 2.1 (Dhage [7])**.** *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a right monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset Q([a, b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$ has a cluster point, whenever $\{x_n\}$ is a monotone increasing sequence in $[a, b]$, then Q has a fixed point.*

Theorem 2.2 (Dhage [8])**.** *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a strict monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset Q([a, b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$ has a cluster point, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then Q has the least fixed point x_* and the greatest fixed point x^* in $[a, b]$. Moreover,*

$$x_* = \min\{y \in [a, b] \mid Qy \leq y\} \quad \text{and} \quad x^* = \max\{y \in [a, b] \mid y \leq Qy\}.$$

A single-valued mapping $T : X \to X$ is called Lipschitz if there exists a constant $\alpha > 0$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in X$. The constant α is called the Lipschitz constant of T on X. Further if $\alpha < 1$, then T is called a contraction on X with contraction constant α. A multi-valued mapping $T : X \to \mathcal{P}_{cp}(X)$ is called totally bounded if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X.

We also need the following two hybrid fixed point theorems in the sequel.

Theorem 2.3 (Dhage [7, 9])**.** *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $A : [a, b] \to X$ be a single-valued and let $B : [a, b] \to \mathcal{P}_{cp}(X)$ be a multi-valued operator satisfying*

(*a*) *A is nondecreasing and single-valued contraction,*
(*b*) *B is totally bounded and right monotone increasing, and*
(*c*) *$Ax + By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further if the cone K in X is normal, then the operator inclusion $x \in Ax+Bx$ has a solution in $[a, b]$.

Theorem 2.4 (Dhage [8])**.** *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $A : [a, b] \to X$ be a single-valued and let $B : [a, b] \to \mathcal{P}_{cp}(X)$ be a multi-valued operator satisfying*

(*a*) *A is a nondecreasing and single-valued contraction,*
(*b*) *B is totally bounded and strict monotone increasing, and*
(*c*) *$Ax + By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further if the cone K in X is normal, then the operator inclusion $x \in Ax+Bx$ has the least and the greatest solution in $[a, b]$.

Remark 2.2. *We note that the hypothesis (c) of Theorems 2.3 and 2.4 holds if there exist the elements $a, b \in X$ such that $a \leq Aa + Ba$ and $Ab + Bb \leq b$.*

In the following section we prove the main existence results of this paper.

3. Main Results

We need the following definitions in the sequel.

Definition 3.1. *A multi-valued function $F : J \to \mathcal{P}_{cp}(\mathbf{R})$ is said to be measurable if for every $y \in \mathbf{R}$, the function $t \to d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.*

Definition 3.2. *Let $F : [0,T] \subset \mathbf{R} \to P_{cl,bd}(\mathbf{R})$ be a multi-function. A function $v : [0,T] \to \mathbf{R}$ is called a measurable selector of F if v is measurable and $v(t) \in F(t)$ almost everywhere $t \in [0,T]$.*

Definition 3.3. *A multi-valued function $F(t,x)$ is called right monotone increasing in x almost everywhere for $t \in J$ if $F(t,x) \overset{i}{\leq} F(t,y)$ a. e. $t \in J$, for all $x, y \in \mathbf{R}$ for which $x \leq y$. Similarly, a multi-valued function $F(t,x)$ is called strict monotone increasing in x almost everywhere for $t \in J$ if $F(t,x) \leq F(t,y)$ a. e. $t \in J$ for all $x, y \in \mathbf{R}$ for which $x \leq y$, $x \neq y$.*

Definition 3.4. *A multi-valued function $F : J \times \mathbf{R} \to \mathcal{P}_p(\mathbf{R})$ is called Chandrabhan if*

(i) *$t \mapsto F(t,x(t))$ is measurable for each $x \in C(J,\mathbf{R})$ and*

(ii) *$F(t,x)$ is right monotone increasing in x almost everywhere for $t \in J$.*

Further a Chandrabhan multi-valued function F is called L^1-Chandrabhan, if

(iii) *for each $r > 0$, there exists a function $h_r \in L^1(J,\mathbf{R})$ such that*

$$\|F(t,x)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t,x)\} \leq h_r(t) \text{ a.e. } t \in J$$

for all $x \in \mathbf{R}$ with $|x| \leq r$.

For any $x \in C(J,\mathbf{R})$, denote

$$S_F^1(x) = \{v \in L^1(J,\mathbf{R}) \mid v(t) \in F(t,x(t)) \text{ a.e. } t \in J'\}.$$

The integral of the multi-function F is defined as

$$\int_0^t F(s,x(s))\,ds = \left\{ \int_0^t v(s)\,ds : v \in S_F^1(x) \right\}.$$

Definition 3.5. *A function $a \in C(J,\mathbf{R})$ is called a strict lower solution of the FDI (1.1) if for all $v \in S_F^1(a)$,*

$$a'(t) \leq v(t) \text{ a. e. } t \in J \setminus \{t_1, \dots, t_p\}$$
$$a(t_j^+) - a(t_j^-) \leq I_j(a(t_j))$$
$$a(0) \leq a(T).$$

Similarly, a strict upper solution b to IDI (1.1) is defined.

We consider the following set of hypotheses in the sequel.

(A_1) The impulsive function I_j is continuous and there exist constants $c_j > 0$ such that $|I_j(x)| \leq c_j$, $j = 1, 2, ..., p$; for all $x \in \mathbf{R}$.

(A_2) The mapping $x \mapsto I_j(x)$ is nondecreasing in $x \in \mathbf{R}$ for each $j = 1, \dots, p$.

(B_1) $F(t, x)$ is closed and bounded for each $(t, x) \in J \times \mathbf{R}$.

(B_2) There exists a function $h \in L^1(J, \mathbf{R})$ such that the multi-valued function
$F_1(t, x) = F(t, x) + h(t)x$ is Chandrabhan on $J \times \mathbf{R}$.

(B_3) $S^1_{F+h}(x) \neq \emptyset$ for all $x \in C(J, \mathbf{R})$.

(B_4) The multi-valued map $x \mapsto S^1_{F+h}(x)$ is right monotone increasing in $C(J, \mathbf{R})$.

(B_5) FDI (1.1) has a strict lower solution a and a strict upper solution b with $a \leq b$.

(B_6) The function $\ell : J \to \mathbf{R}$ defined by

$$\ell(t) = \|F_1(t, a(t))\|_{\mathcal{P}} + \|F_1(t, b(t))\|_{\mathcal{P}}$$

is Lebesgue integrable.

Remark 3.1. *Note that if* (B_2), (B_5)-(B_6) *hold, then we have*

$$\|F_1(t, x(t))\|_{\mathcal{P}} \leq \ell(t) \quad a.\ e. \quad t \in J$$

for all $x \in [a, b]$.

Hypotheses (B_1) and (B_3) are much common in the literature. Some nice sufficient conditions for guarantying (B_3) appear in Deimling [4], and Lasota and Opial [14]. A mild hypothesis of (B_5) has been used in Halidias and Papageorgiou [11]. Hypothesis (B_5) holds in particular if F is bounded on $J \times \mathbf{R}$. Hypotheses (B_2), (B_3) and (B_6) are relatively new to the literature, but these are assumed for (B_4) to make sense and the special forms of these hypotheses have been appeared in the works of several authors. See Dhage [5, 6] and references therein.

Now consider the following impulsive differential inclusion

$$\left.\begin{aligned} x'(t) + h(t)x(t) \in F_1(t, x(t)) \ \text{ a. e. } t \in J \setminus \{t_1, \dots, t_p\} \\ x(t_j^+) - x(t_j^-) = I_j(x(t_j)) \\ x(0) = x(T) \end{aligned}\right\} \tag{3.1}$$

where, $F_1 : J \times \mathbf{R} \to \mathcal{P}_p(\mathbf{R})$ defined by $F_1(t, x(t)) = F(t, x(t)) + h(t)x(t)$ and $x(t_j^+)$, $x(t_j^+)$, $I_j(x(t_j))$ have the usual meanings as in IDI (1.1).

Note that, if a function a is a lower solution for the IDI (1.1), then it is also a lower solution for the IDI (3.1) on J. The same fact is also true for upper solution. Thus x is a solution for the IDI (1.1) if and only if it is a solution for the IDI (3.1). We need the following result in the sequel.

Lemma 3.1. *Let $\sigma \in L^1(J, \mathbf{R})$. Then for any $h \in L^1(J, \mathbf{R}^+)$, the function $x : J \to \mathbf{R}$ is a solution of the differential equation*

$$x'(t) + h(t)x(t) = \sigma(t) \ \ a.\ e.\ t \in J \setminus \{t_1, \ldots, t_p\} \tag{3.2}$$

$$x(t_j^+) - x(t_j^-) = I_j(x(t_j)) \tag{3.3}$$

$$x(0) = x(T) \tag{3.4}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T g_h(t, s)\sigma(s)\ ds + \sum_{j=1}^{p} g_h(t, t_j)I_j(x(t_j)) \tag{3.5}$$

for $t \in J$, where the Green's function g_h is given by

$$g_h(t, s) = \begin{cases} \dfrac{e^{-H(t)-H(s)}}{1 - e^{-H(T)}}, & 0 \le s \le t \le T \\ \\ \dfrac{e^{-H(T)+H(t)-H(s)}}{1 - e^{-H(T)}}, & 0 \le t < s \le T\,. \end{cases}$$

and $H(t) = \int_0^t h(s)\, ds$.

Notice that the Green's function $g_h(t, s)$ is nonnegative on $J \times J$ and the number $M_h := \max\{\, |g_h(t, s)| \, : \, t, s \in [0, T]\,\}$ exists for all $h \in L^1(J, \mathbf{R})$. Note that $x \in Y_T$ is a solution of (3.1)-(3.2)-(3.3) if and only if $x \in Qx$, where the multi-valued operator Q is defined by

$$Qx(t) := \int_0^T g_h(t, s)\,[h(s)x(s) + F(s, x(s))]\, ds + \sum_{j=1}^{p} g_h(t, t_j)I_j(x(t_j))\,. \tag{3.6}$$

See Nieto [15, 16] and the references given therein.

Theorem 3.1. *Assume that $(A_1) - (A_2)$ and $(B_1) - (B_6)$ hold. Then the FDI (1.1) has a solution in $[a, b]$ defined on J.*

Proof. Define an order interval $[a, b]$ in X which is well defined in view of hypothesis (B_5). Now the IDI (1.1) is equivalent to the integral inclusion

$$x(t) \in \int_0^T g_h(t,s)F_1(s,x(s))\,ds + \sum_{j=1}^{p} g_h(t,t_j)I_j(x(t_j)). \tag{3.7}$$

Define a multi-valued operator $Q : [a, b] \to \mathcal{P}_p(X)$ by

$$\begin{aligned} Qx &= \Big\{u \in X : u(t) = \int_0^T g_h(t,s)F_1(s,x(s))\,ds + \sum_{j=1}^{p} g_h(t,t_j)I_j(x(t_j))\Big\} \\ &= (\mathcal{L} \circ S^1_{F+h})(x), \end{aligned} \tag{3.8}$$

where $\mathcal{L} : L^1(J, \mathbf{R}) \to C(J, \mathbf{R})$ is a continuous operator defined by

$$\mathcal{L}v(t) = \int_0^T g_h(t,s)v(s)\,ds + \sum_{j=1}^{p} g_h(t,t_j)I_j(x(t_j)). \tag{3.9}$$

Clearly the operator Q is well defined in view of hypothesis (B_3). We shall show that Q satisfies all the conditions of Theorem 2.1.

Step I : First, we show that Q has compact values on $[a, b]$. Observe that if $t \in J$, then operator Q is equivalent to the composition $\mathcal{L} \circ S^1_{F+h}$ of two operators on $L^1(J, \mathbf{R})$, where $\mathcal{L} : L^1(J, \mathbf{R}) \to X$ is the continuous operator defined by (3.9). To show Q has compact values, it then suffices to prove that the composition operator $\mathcal{L} \circ S^1_F$ has compact values on $[a, b]$. Let $x \in [a, b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_{F+h}(x)$. Then, by the definition of S^1_{F+h}, $v_n(t) \in F_1(t, x(t))$ a. e. for $t \in J$. Since $F_1(t, x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in F_1(t, x(t))$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we first show that

$\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let $t,\,\tau\in J$; then

$$\begin{aligned}
|(\mathcal{L}v_n)'(t)| &\leq \Big|\int_0^T \frac{\partial}{\partial t} g_h(t,s)v_n(s)\,ds + \sum_{j=1}^{p}\frac{\partial}{\partial t} g_h(t,t_j)I_j(y_j(t_j))\Big| \\
&= \Big|\int_0^T (-h(t))g_h(t,s)v_n(s)\,ds + \sum_{j=1}^{p}(-h(t))g_h(t,t_j)I_j(y_j(t_j))\Big| \\
&\leq H\,M_h\int_0^T v_n(s)\,ds + H\sum_{j=1}^{p} c_j \\
&= c
\end{aligned}$$

where, $H=\max_{t\in J} h(t)$. Hence for any $t,\tau\in[0,T]$ one has

$$|y_n(t)-y_n(\tau)|\leq c\;|t-\tau|\to 0 \quad \text{as} \quad t\to\tau.$$

Since $v_n\in L^1(J,\mathbf{R})$, the right hand side of above inequality tends to 0 as $t\to\tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Arzelá-Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j}\to\mathcal{L}v\in(\mathcal{L}\circ S^1_{F+h})(x)$ as $j\to\infty$, and so $(\mathcal{L}\circ S^1_{F+h})(x)$ is compact. Therefore, Q is a compact-valued multi-valued operator on $[a,b]$.

Step II : Secondly we show that Q is right monotone increasing and maps $[a,b]$ into itself. Let $x,y\in[a,b]$ be such that $x\leq y$. Since $S^1_{F+h}(x)\overset{i}{\leq} S^1_{F+h}(y)$, we have that $Q(x)\overset{i}{\leq}Q(y)$. From (B_4), it follows that $a\leq Qa$ and $Qb\leq b$. Now Q is right monotone increasing, so we have

$$a\leq Qa\overset{i}{\leq}Qx\overset{i}{\leq}Qb\leq b$$

for all $x\in[a,b]$. Hence Q defines a right monotone increasing multi-valued operator $Q:[a,b]\to\mathcal{P}_{cp}([a,b])$.

Step III : Finally let $\{x_n\}$ be a monotone increasing sequence in $[a,b]$ and let $\{y_n\}$ be a sequence in $Q([a,b])$ defined by $y_n\in Qx_n$, $n\in\mathbb{N}$. We shall show that $\{y_n\}$ has a cluster point. This is achieved by showing that $\{y_n\}$ is uniformly bounded and equi-continuous sequence.

Case I : First we show that $\{y_n\}$ is uniformly bounded sequence. By the definition of $\{y_n\}$, there is a $v_n \in S^1_{F+h}(x_n)$ such that

$$y_n(t) = \int_0^T g_h(t,s)v_n(s)\,ds + \sum_{j=1}^{p} g_h(t,t_j)I_j(x(t_j)).$$

Therefore,

$$\begin{aligned}
|y_n(t)| &\leq \Big|\int_0^T g_h(t,s)v_n(s)\,ds\Big| + \Big|\sum_{j=1}^{p} g_h(t,t_j)I_j(x(t_j))\Big| \\
&\leq M_h \int_0^T \ell(s)\,ds + M_h \sum_{j=1}^{p} c_j \\
&\leq M_h \|\ell\|_{L^1} + d
\end{aligned}$$

for all $t \in J$, where $d = M_h \sum_{j=1}^{p} c_j$. Taking the supremum over t,

$$\|y_n\| = M_h \|\ell\|_{L^1} + d$$

which shows that $\{y_n\}$ is a uniformly bounded sequence in $Q([a,b])$.

Next we show that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$. Then there is a $v_n \in S_{k+F}(x)$ such that

$$y_n(t) = \int_0^T g_h(t,s)\,v_n(s)\,ds \quad + \sum_{j=1}^{p} g_h(t,t_j)I_j(x(t_j)), \quad t \in J.$$

To finish, it is enough to show that y_n' is bounded on $[0,T]$. Now for any $t \in [0,T]$,

$$\begin{aligned}
|y_n'(t)| &\leq \Big|\int_0^T \frac{\partial}{\partial t} g_h(t,s)v_n(s)\,ds + \sum_{j=1}^{p} \frac{\partial}{\partial t} g_h(t,t_j)I_j(y_j(t_j))\Big| \\
&= \Big|\int_0^T (-h(t))g_h(t,s)v_n(s)\,ds + \sum_{j=1}^{p} (-h(t))g_h(t,t_j)I_j(y_j(t_j))\Big| \\
&\leq H\,M_h \int_0^T \ell(s)\,ds + H\,M_h \sum_{j=1}^{p} c_j \\
&= c
\end{aligned}$$

where, $H = \max_{t\in J} h(t)$. Hence for any $t, \tau \in [0,T]$ one has

$$|y_n(t) - y_n(\tau)| \leq c\,|t-\tau| \to 0 \quad \text{as} \quad t \to \tau.$$

This shows that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$. Now $\{y_n\}$ is uniformly bounded and equi-continuous, so it has a cluster point in view of Arzelà-Ascoli theorem. Now the desired conclusion follows by an application of Theorem 2.1. $\square$

Below we relax the boundedness assumption on the impulsive moments I_j on $\mathbf{R}$ for each $j = 1, ..., p$; instead we assume the Lipschitz condition on I_j and prove the existence of solution for (1.1). We need the following hypothesis in the sequel.

(A_3) There exist constants $\alpha_j > 0$ such that

$$|I_j(x) - I_j(y)| \leq \alpha_j |x - y|, \ \ j = 1, 2, ..., p;$$

for all $x, y \in \mathbf{R}$.

Theorem 3.2. *Assume that the hypotheses* $(A_2) - (A_3)$ *and* $(B_1) - (B_5)$ *hold. Furthermore if* $M_h \sum_{j=1}^{p} \alpha_j < 1$, *then the IDI (1.1) has a solution in* $[a,b]$ *defined on* J.

Proof. Consider the order interval $[a,b]$ in X which is well defined in view of hypothesis (B_4). Define two operators $A : [a,b] \to X$ and $B : [a,b] \to \mathcal{P}_{cp}(X)$ by

$$Ax(t) = \sum_{j=1}^{p} g_h(t, t_j) I_j(x(t_j)) \tag{3.10}$$

and

$$Bx(t) = \int_0^T g_h(t,s) F_1(s, x(s))\, ds. \tag{3.11}$$

We show that A is a contraction on $[a, b]$. Let $x, y \in [a, b]$. By (A_3),

$$\begin{aligned}
|Ax(t) - Ay(t)| &\leq \left| \sum_{j=1}^{p} g_h(t, t_j) I_j(x(t_j)) - \sum_{j=1}^{p} g_h(t, t_j) I_j(y(t_j)) \right| \\
&\leq \sum_{j=1}^{p} |g_h(t, t_j)| \, |I_j(x(t_j)) - I_j(y(t_j))| \\
&\leq M_h \sum_{j=1}^{p} \alpha_j |I_j(x(t_j)) - I_j(y(t_j))| \\
&\leq M_h \Big(\sum_{j=1}^{p} \alpha_j \Big) \|x - y\|
\end{aligned}$$

for all $t \in J$. This further yields

$$\|Ax - Ay\| \leq \alpha \|x - y\|$$

for all $x, y \in [a, b]$, where $\alpha = M_h \sum_{j=1}^{p} \alpha_j < 1$. Hence A is a contraction operator on $[a, b]$ with contraction constant $\alpha < 1$.

It can be shown as in the proof of Theorem 3.1 that is a totally bounded operator on $[a, b]$. Again it is easy to verify that A is nondecreasing and B is right monotone increasing on $[a, b]$ satisfying $Ax + By \in [a, b]$ for $x, y \in [a, b]$. Now the desired conclusion follows by an application of Theorem 2.3. □

Next we prove a result concerning the extremal solutions for the IDI (1.1) on J. We need the following definition in the sequel.

Definition 3.6. *A multi-valued function $F : J \times \mathbf{R} \to \mathcal{P}_p(\mathbf{R})$ is called strict L^1-Chandrabhan if*

(i) *$t \mapsto F(t, x(t))$ is measurable for each $x \in C(J, \mathbf{R})$,*
(ii) *$F(t, x)$ is strict monotone increasing in x almost everywhere for $t \in J$, and*
(iii) *for each $r > 0$, there exists a function $q_r \in L^1(J, \mathbf{R})$ such that*

$$\|F(t, x)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t, x)\} \leq q_r(t) \ \text{a.e. } t \in J$$

for all $x \in \mathbf{R}$ with $|x| \leq r$.

Remark 3.2. Note that if the multi-valued function $F(t,x)$ is strict L^1-Chandrabhan and the hypothesis (H_4) holds, then it is measurable in t and integrably bounded on $J \times \mathbf{R}$, and so, by a selection theorem, S_F^1 has non-empty and closed values on $C(J,\mathbf{R})$, that is,

$$S_F^1(x) = \{u \in L^1(J,\mathbf{R}) \mid u(t) \in F(t,x(t)) \text{ a.e. } t \in J\} \neq \emptyset$$

for all $x \in C(J,\mathbf{R})$. See Deimling [4] and the references therein. Note also that if F is L^1-Chandrabhan on $J \times \mathbf{R}$, then the multi-valued function F_1 defined by $F_1(t,x(t)) = F(t,x(t)) + h(t)x(t)$ is also a strict L^1-Chandrabhan on $J \times \mathbf{R}$, but the converse may not be true.

We consider the following hypothesis in the sequel.

(B_7) The multi-function F_1 is strict L^1-Chandrabhan on $J \times \mathbf{R}$.

Theorem 3.3. *Assume that the hypotheses* $(A_1) - (A_2)$, $(B_1), (B_5)$ *and* (B_7) *hold. Then the IDI (1.1) has a minimal and a maximal solution in* $[a,b]$ *defined on* J.

Proof. The proof is similar to Theorem 3.1. Here, $S^1_{F+h}(x) \neq \emptyset$ and the multi-valued map $x \mapsto S^1_{F+h}(x)$ is strictly monotone increasing on $[a,b]$. Consequently the multi-valued operator Q defined by (3.2) is strictly monotone increasing on $[a,b]$. Hence the desired result follows by an application of Theorem 2.2. □

Theorem 3.4. *Assume that the hypotheses* (A_2), (A_3), (B_1), (B_5) *and* (B_7) *hold. If* $\sum_{j=1}^{p} \alpha_j < 1$, *then the IDI (1.1) has a minimal and a maximal solution in* $[a,b]$ *defined on* J.

Proof. The proof is similar to Theorem 3.3. Consider the order interval $[a,b]$ in X which is well defined in view of hypothesis (B_4). Define two operators $A : [a,b] \to X$ and $B : [a,b] \to \mathcal{P}_{cp}(X)$ by (3.10) and (3.11) respectively. It can be shown as in the proofs of Theorems 3.2 that the operator A is contraction and B is totally bounded on $[a,b]$. Here, the operator A is nondecreasing on $[a,b]$. Also $S^1_{F+h}(x) \neq \emptyset$ and the multi-valued map $x \mapsto S^1_{F+h}(x)$ is strictly monotone increasing on $[a,b]$, so the multi-valued operator B is strictly monotone increasing on $[a,b]$. Now the desired conclusion follows by an application of Theorem 2.4. □

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