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STRONG CONVERGENCE THEOREMS OF AVERAGING ITERATIONS OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

LU-CHUAN CENG*, ADRIAN PETRUŞEL** AND JEN-CHIH YAO***

*Department of Mathematics, Shanghai Normal University
Shanghai 200234, China
E-mail: zenglc@hotmail.com

**Department of Applied Mathematics, Babeş-Bolyai University Cluj-Napoca
1 Kogălniceanu street, 400084 Cluj-Napoca, Romania
E-mail: petrusel@math.ubbcluj.ro

***Department of Applied Mathematics, National Sun Yat-sen University
804 Kaohsiung, Taiwan
E-mail: yaojc@math.nsysu.edu.tw

Abstract. Let E be a uniformly convex Banach space whose norm is Gâteaux differentiable and which has a weakly continuous duality mapping; for example, every l^p $(1 < p < \infty)$ space has a weakly continuous duality map with gauge function $\varphi(t) = t^{p-1}$. Let C be a nonempty closed convex subset of E, $T : C \to E$ be a nonexpansive nonself-mapping, and x_0, x, y_0, y be elements of C. In this paper, we study the strong convergence of two sequences generated by

$$x_{n+1} = \tfrac{1}{n+1}\sum_{j=0}^{n}(\alpha_n x + (1-\alpha_n)(PT)^j x_n) \text{ for } n = 0, 1, 2, ...,$$
$$y_{n+1} = \tfrac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) \text{ for } n = 0, 1, 2, ...,$$

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where $\{\alpha_n\}$ is a real sequence such that $0 \leq \alpha_n \leq 1$, and P is a sunny and nonexpansive retraction of E onto C.

Key Words and Phrases: Fixed point, nonexpansive nonself-mapping, strong convergence, sunny and nonexpansive retraction, Banach space.

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1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be a nonexpansive mapping such that the set $F(S)$ of all fixed points of S is nonempty. Recently, Shimizu and Takahashi [15] proved the strong convergence of an iteration process to a common fixed point of a family of nonexpansive mappings in a Hilbert space H. Using Shimizu and Takahashi's idea in [15], Shioji and Takahashi [16] proved the strong convergence of an iterative sequence $\{x_n\}$ to an element of $F(S)$ which is nearest to x in the setting of a Banach space E where for a given sequence $\{\alpha_n\}$ with $0 \leq \alpha_n \leq 1$, the sequence $\{x_n\}$ is generated from any elements $x_0, x \in C$ by

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} S^j x_n \quad \text{for } n = 0, 1, 2, ... \tag{1}$$

But unfortunately this approximation method is not suitable for some nonexpansive nonself-mappings. On the other hand Matsushita and Kuroiwa [9] studied the convergence of two sequences generated by

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n x + (1-\alpha_n)PTx_n \quad \text{for } n = 1, 2, ..., \tag{2}$$

$$y_1 = y \in C, \quad y_{n+1} = P(\alpha_n y + (1-\alpha_n)Ty_n) \quad \text{for } n = 1, 2, ... \tag{3}$$

where P is the metric projection from H onto C and T is a nonexpansive nonself-mapping from C into H. They proved that $\{x_n\}$ and $\{y_n\}$ converge strongly to fixed points of T when $F(T)$ is nonempty. Furthermore Matsushita and Kuroiwa [10] studied the new iteration processes which are mixed iteration processes of (1)-(3) as follows:

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}(\alpha_n x + (1-\alpha_n)(PT)^j x_n) \quad \text{for } n = 0, 1, 2, ..., \tag{4}$$

$$y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) \quad \text{for } n = 0, 1, 2, ..., \tag{5}$$

where x_0, x, y_0, y are elements of C, P is the metric projection from H onto C and T is a nonexpansive nonself-mapping from C into H. By using the nowhere normal outward condition given earliest in [7], under the assumption that $F(T)$ is nonempty, they proved not only that the sequence $\{x_n\}$ generated by (4) converges strongly to an element of $F(T)$ but also that the sequence $\{y_n\}$ generated by (5) converges strongly to an element of $F(T)$.

The purpose of this paper is to study the strong convergence of the iteration processes (4) and (5) in the framework of a Banach space. Let E be a uniformly convex Banach space whose norm is Gâteaux differentiable and which has a weakly continuous duality mapping; for example, every l^p $(1 < p < \infty)$ space has a weakly continuous duality map with gauge function $\varphi(t) = t^{p-1}$. Let C be a nonempty closed convex subset of E, P be a sunny and nonexpansive retraction of E onto C, $T : C \to E$ be a nonexpansive nonself-mapping such that $F(T)$ is nonempty, and $\{\alpha_n\}$ be a real sequence such that $0 \leq \alpha_n \leq 1$. Firstly by using the property of the sunny and nonexpansive retraction, we consider the sequence $\{x_n\}$ generated by (4) and prove that $\{x_n\}$ converges strongly to an element of $F(T)$. Secondly by using the same property, we consider the sequence $\{y_n\}$ generated by (5) and prove that $\{y_n\}$ converges strongly to an element of $F(T)$.

2. Preliminaries and Notations

Throughout this paper, all vector spaces are real and we denote by $\mathbb{N}$ and $\mathbb{N}_+$, the set of all nonnegative integers and the set of all positive integers, respectively. Let E be a real Banach space and E^* be the topological dual of E. By 2^{E^*} we will denote the power set of E^*. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let C be a nonempty closed convex subset of E, and $T : C \to E$ be a nonself-mapping. We denote the set of all fixed points of T by $F(T)$. T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by B_r, the closed ball in E with center 0 and radius r. E is said to be uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$ for each $x, y \in B_1$ with $\|x - y\| \geq \varepsilon$.

For simplicity, the notation $\rightharpoonup$ denotes weak convergence and the notation $\to$ denotes strong convergence. By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}_+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x^*\| = \varphi(\|x\|)\} \quad \text{for all } x \in E$$

is called the duality mapping with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Browder [3] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Set for every $t \geq 0$,

$$\Phi(t) = \int_0^t \varphi(r)\mathrm{d}r.$$

Then it is known [8, p. 1350] that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x. Thus it is easy to see that the normalized duality mapping $J(x)$ can also be defined as the subdifferential of the convex functional $\Phi(\|x\|) = \|x\|^2/2$, that is,

$$\begin{aligned} J(x) = \partial\Phi(\|x\|) = \{f \in E^* : \Phi(\|y\|) - \Phi(\|x\|) \\ \geq \langle y - x, f \rangle, \forall y \in E\} \quad \text{for all } x \in E. \end{aligned}$$

We will use the following properties of duality mappings, respectively.

Proposition 1 [23, p. 193-194].

(i) $J = I$ (i.e., the identity mapping of E) if and only if E is a Hilbert space.

(ii) J is surjective if and only if E is reflexive.

(iii) $J_\varphi(\lambda x) = \text{sign}(\lambda)(\varphi(|\lambda| \cdot \|x\|)/\|x\|)J(x)$ for each $x \in E \setminus \{0\}$ and each real number λ. In particular, $J(-x) = -J(x)$ for all $x \in E$.

Recall that a Banach space E is said to satisfy Opial's condition [12] if for any sequence $\{x_n\}$ in E the condition that $\{x_n\}$ converges weakly to $x \in E$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in E,\ y \neq x$. It is known [5] that any separable Banach space can be equivalently renormed so that it satisfies Opial's condition. Recall also that E is said to have a weakly continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology. A space with a weakly continuous duality mapping is easily seen to satisfy Opial's condition; see [3] for more details. Every l^p $(1 < p < \infty)$ space has a weakly continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$.

The following proposition plays an important role in our proofs; see [18] for more details.

Proposition 2 [17, Proposition 1]. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Then for each $r > 0$, $R \geq r$ and $\varepsilon > 0$, there exist $\eta > 0$ and $M \in N_+$ such that for each $j \in N$ and for each mapping T from C into itself satisfying $\sup\{\|T^m x\| : m \in N, x \in C \cap B_r\} \leq R$ and $\|T^j x - T^j y\| \leq (1+\eta)\|x - y\|$ for each $x, y \in C$, there holds

$$\|\frac{1}{n+1}\sum_{i=0}^{n} T^i x - T^j(\frac{1}{n+1}\sum_{i=0}^{n} T^i x)\| \leq \varepsilon$$

for all $n \geq jM$ and $x \in C \cap B_r$.

Let μ be a continuous linear functional on l^∞ and let $(a_0, a_1, \cdots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. We call μ a Banach limit [1] when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, \cdots) \in l^\infty$. For a Banach limit μ, we know that

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n \quad \text{for all } (a_0, a_1, \cdots) \in l^\infty. \tag{6}$$

Proposition 3 [17, Proposition 2]. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $\{x_n\}$ be a bounded sequence of E and μ be a Banach limit. Let g be a real valued function on C defined by

$$g(y) = \mu_n(\|x_n - y\|^2) \quad \text{for each } y \in C.$$

Then g is continuous, convex and g satisfies $\lim_{\|y\|\to\infty} g(y) = \infty$. Moreover, for each $R > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$g(\frac{y+z}{2}) \leq \frac{g(y)+g(z)}{2} - \delta$$

for all $y, z \in C \cap B_R$ with $\|y - z\| \geq \varepsilon$.

The following lemma is similar to a result in Hilbert spaces which was proved by Matsushita and Kuroiwa; see [10, Lemma 1]. The method of their proof can be found in Shimizu and Takahashi [14].

Lemma 1. Let E be a uniformly convex Banach space which satisfies Opial's condition, C be a nonempty closed convex subset of E and $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\{x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} S^j x_n\}$ converges strongly to 0, and let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $x \in C$. Then x is a fixed point of S.

Proof. We claim that $\{S^l x\}$ converges strongly to x. Indeed if this is not true, then there exist a real number $\varepsilon > 0$ and a subsequence $\{S^{l_k}x\}$ of $\{S^l x\}$ such that $\|S^{l_k}x - x\| \geq \varepsilon$ for each k. Since $\{x_{n_i}\}$ converges weakly to x, for each $y \in C$ with $y \neq x$ we have

$$\liminf_{i\to\infty} \|x_{n_i} - x\| < \liminf_{i\to\infty} \|x_{n_i} - y\|.$$

Let $r = \liminf_{i\to\infty} \|x_{n_i} - x\|$. Then there exists a subsequence $\{x_{m_i}\}$ of $\{x_{n_i}\}$ such that $r = \lim_{i\to\infty} \|x_{m_i} - x\|$. Now let μ be a Banach limit and set $R = \sup\{\|S^l x\| : l \in N\}$. By Proposition 3, there exists $\hat{\delta} > 0$ such that

$$\mu_i(\|x_{m_i} - \frac{u+v}{2}\|^2) \leq \frac{1}{2}[\mu_i(\|x_{m_i} - u\|^2) + \mu_i(\|x_{m_i} - v\|^2)] - \hat{\delta} \tag{7}$$

for all $u, v \in C \cap B_R$ with $\|u - v\| \geq \varepsilon$. Choose $\delta > 0$ such that $\delta < \sqrt{r^2 + \hat{\delta}} - r$. Without loss of generality, we may assume that $\|x_{m_i} - x\| < r + \delta/6$ for every

i. On the other hand we have

$$\begin{aligned}\|x_{m_i} - S^l x\| &\leq \|x_{m_i} - \tfrac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1}\| \\ &+\|\tfrac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1} - S^l(\frac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1})\| \\ &+\|S^l(\tfrac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1}) - S^l x\| \\ &\leq 2\|x_{m_i} - \tfrac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1}\| + \|x_{m_i} - x\| \\ &+\|\tfrac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1} - S^l(\frac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1})\|.\end{aligned}$$

In particular,

$$\begin{aligned}&\|\tfrac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1} - S^l(\frac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1})\| \\ &\leq \|\tfrac{1}{m_i}\sum_{j=1}^{m_i} S^j x_{m_i-1} - S^l(\frac{1}{m_i}\sum_{j=1}^{m_i} S^j x_{m_i-1})\| + \frac{1}{m_i}\|x_{m_i-1} - S^{m_i} x_{m_i-1}\| \\ &\quad+\|S^l(\tfrac{1}{m_i}\sum_{j=1}^{m_i} S^j x_{m_i-1}) - S^l(\frac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1})\| \\ &\leq \|\tfrac{1}{m_i}\sum_{j=1}^{m_i} S^j x_{m_i-1} - S^l(\frac{1}{m_i}\sum_{j=1}^{m_i} S^j x_{m_i-1})\| + \frac{2}{m_i}\|x_{m_i-1} - S^{m_i} x_{m_i-1}\|.\end{aligned}$$

Since $\{x_{m_i-1}\}$ and $\{S^{m_i} x_{m_i-1}\}$ are bounded, there exists a positive integer i_1 such that

$$\frac{1}{m_i}\|x_{m_i-1} - S^{m_i} x_{m_i-1}\| < \frac{\delta}{6}$$

for each $i \geq i_1$. Since $\{x_{m_i-1}\}$ is bounded, by Proposition 2 there exists a positive integer L_0 such that for every $l \geq L_0$, there exists a positive integer i_l such that for each $i \geq i_l$

$$\|\frac{1}{m_i}\sum_{j=1}^{m_i} S^j x_{m_i-1} - S^l(\frac{1}{m_i}\sum_{j=1}^{m_i} S^j x_{m_i-1})\| < \frac{\delta}{6}.$$

Actually let us notice that one may assume that $\{m_i\}$ is strictly increasing to $+\infty$ and that $m_i > i$. In Proposition 2, put $r =: \sup\{\|x_{m_i-1}\| : i \in N\},\ R =: \sup\{\|S^m x\| : m \in N, x \in C \cap B_r\}$ and $L_0 = M$. For each $l \geq L_0$, take $i_l = lL_0$.

Since $\lim_{n\to\infty} \|x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} S^j x_n\| = 0$, there exists a positive integer i_0 such that for all $i \geq i_0$

$$\|x_{m_i} - \frac{1}{m_i}\sum_{j=0}^{m_i-1} S^j x_{m_i-1}\| < \frac{\delta}{6}.$$

Hence for any $l \geq L_0$ and $i \geq \max\{i_l, i_0, i_1\}$, we have

$$\|x_{m_i} - S^l x\| < 2 \times \frac{\delta}{6} + r + \frac{\delta}{6} + \frac{\delta}{6} + 2 \times \frac{\delta}{6} = r + \delta.$$

Choose $l_k \geq L_0$. Then for $i \geq \max\{i_{l_k}, i_0, i_1\}$, we have

$$\frac{1}{2}(\|x_{m_i} - S^{l_k}x\|^2 + \|x_{m_i} - x\|^2) - \hat{\delta} < \frac{1}{2}((r+\delta)^2 + (r+\delta/6)^2) - \hat{\delta} < (r+\delta)^2 - \hat{\delta} < r^2$$

which implies that

$$\limsup_{i\to\infty}[\frac{1}{2}(\|x_{m_i} - S^{l_k}x\|^2 + \|x_{m_i} - x\|^2) - \hat{\delta}] \leq r^2.$$

Since $S^{l_k}x$ and x lie in $C \cap B_R$ such that $\|S^{l_k}x - x\| \geq \varepsilon$, from (6) and (7) we obtain

$$\begin{aligned} \liminf_{i\to\infty} \|x_{m_i} - \tfrac{S^{l_k}x+x}{2}\|^2 &\leq \mu_i(\|x_{m_i} - \tfrac{S^{l_k}x+x}{2}\|^2) \\ &\leq \tfrac{1}{2}[\mu_i(\|x_{m_i} - S^{l_k}x\|^2) + \mu_i(\|x_{m_i} - x\|^2))] - \hat{\delta} \\ &= \mu_i[\tfrac{1}{2}(\|x_{m_i} - S^{l_k}x\|^2 + \|x_{m_i} - x\|^2) - \hat{\delta}] \\ &\leq \limsup_{i\to\infty}[\tfrac{1}{2}(\|x_{m_i} - S^{l_k}x\|^2 + \|x_{m_i} - x\|^2) - \hat{\delta}] \leq r^2. \end{aligned}$$

This shows that

$$\liminf_{i\to\infty} \|x_{m_i} - \frac{S^{l_k}x + x}{2}\| \leq r = \liminf_{i\to\infty} \|x_{m_i} - x\|.$$

However we see that $\frac{S^{l_k}x+x}{2} \neq x$ from $\|S^{l_k}x - x\| \geq \varepsilon$. Thus we obtain

$$\liminf_{i\to\infty} \|x_{m_i} - x\| < \liminf_{i\to\infty} \|x_{m_i} - \frac{S^{l_k}x + x}{2}\|$$

which yields a contradiction. Thus $\{S^l x\}$ converges strongly to x. Consequently for each $\varepsilon > 0$, there exists a positive integer l_0 such that

$$\|S^l x - x\| \leq \frac{\varepsilon}{2} \quad \text{for all } l \geq l_0.$$

Therefore for all $l \geq l_0 + 1$ we have

$$\|Sx - x\| \leq \|S^{l-1}x - x\| + \|S^l x - x\| \leq \varepsilon.$$

Since ε is arbitrary, we derive $Sx = x$ and the proof is complete. □

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. The norm of E is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. In this case E is said to be smooth. It is known [4] that if E is smooth then the normalized duality mapping J is single-valued and continuous from the strong topology to the weak* topology.

Lemma 2. Let E be a smooth Banach space. Then there holds

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle \quad \text{for all } x, y \in E.$$

Proof. Since E is smooth, J is single-valued. Note that J can be equivalently defined as the subdifferential of the functional $\Phi(\|x\|) = \|x\|^2/2$. Therefore the conclusion follows immediately from the definition of the subdifferential of $\Phi(\|x\|)$. □

Let C be a convex subset of E, K be a nonempty subset of C and let P be a retraction from C onto K, i.e., $Px = x$ for each $x \in K$. We say that P is sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If there is a sunny and nonexpansive retraction from C onto K, K is said to be a sunny and nonexpansive retract of C. For a sunny and nonexpansive retraction, there exists the following useful characterization:

Lemma 3 [17, p. 59, Proposition 4]. Let C be a convex subset of a smooth Banach space E, K be a nonempty subset of C and let P be a retraction from C onto K. Then P is sunny and nonexpansive if and only if for all $x \in C$ and $y \in K$,

$$\langle x - Px, J(y - Px)\rangle \leq 0.$$

Hence there is at most one sunny and nonexpansive retraction from C onto K.

More properties regarding sunny and nonexpansive retractions can be found in [6, 13].

Remark 1. If E is a real Hilbert space H and C is a nonempty closed convex subset of H, then every nearest point projection of H onto C is a sunny and nonexpansive retraction of H onto C where mapping $P_C : H \to C$ is defined as follows: for each $x \in H$, $P_C x$ is the unique element of C that satisfies $\|x - P_C x\| = d(x, C) := \inf_{y \in C} \|x - y\|$. Indeed it is easy to see that P_C is a retraction of H onto C. Moreover for all $x \in H$ and $y \in C$, we have

$$\langle x - P_C x, P_C x - y \rangle \geq 0.$$

According to Lemma 3, we know that P_C is a sunny and nonexpansive retraction of H onto C.

Lemma 4 (see [2]). Let $\{\lambda_n\}$ be a sequence in $[0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 0$. Then

$$\sum_{n=1}^{\infty} \lambda_n = \infty \;\Leftrightarrow\; \prod_{n=1}^{\infty} (1 - \lambda_n) = 0.$$

Lemma 5 [11, Lemma 2.2]. Let $\{\lambda_n\}$ be a sequence in $[0, 1)$ that satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfies any one of the following conditions:

(a) For all $\varepsilon > 0$, there exists an integer $M \geq 1$ such that for all $n \geq M$,

$$a_{n+1} \leq (1 - \lambda_n) a_n + \lambda_n \varepsilon.$$

(b) $a_{n+1} \leq (1-\lambda_n)a_n + \sigma_n,\ n \geq 0$ where $\sigma_n \geq 0$ satisfies $\lim_{n\to\infty} \sigma_n/\lambda_n = 0$.

(c) $a_{n+1} \leq (1 - \lambda_n) a_n + \lambda_n c_n$ where $\limsup_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Remark 2. The proof of Lemma 5 can be found in [22].

Lemma 6. Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers with

$$\limsup_{n\to\infty} \alpha_n < \infty$$

and $\{\beta_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \beta_n \le 0$. Then

$$\limsup_{n\to\infty} \alpha_n \beta_n \le 0.$$

Proof. We prove the conclusion in two cases.

Case 1: $\sup_{j\ge n} \beta_j \ge 0$ for all $n \ge 0$. For any fixed $n \ge 0$, observe that

$$\sup_{i\ge n} \alpha_i \beta_i \le \sup_{i\ge n} (\alpha_i \cdot \sup_{j\ge n} \beta_j) = (\sup_{i\ge n} \alpha_i)(\sup_{j\ge n} \beta_j).$$

Thus taking the limit as $n \to \infty$, we obtain the conclusion.

Case 2: $\beta = \sup_{n\ge m_0} \beta_n < 0$ for some $m_0 \ge 0$. It is easy to see that $\alpha_n \beta_n \le \alpha_n \beta \le 0$ for all $n \ge m_0$. This implies the conclusion. □

Throughout the rest of the paper, we shall use the notation: for any sequence $\{x_n\}$ in E, we denote by $\omega_w(x_n)$ the weak ω-limit set of $\{x_n\}$; that is,

$$\omega_w(x_n) := \{x \in E : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

3. Strong Convergence Theorems

Now we can state and prove the main results in this paper. The method employed in [10, 19, 21] is extended to develop the new technique for proving our results.

Theorem 1. Let E be a uniformly convex Banach space whose norm is Gateaux differentiable and which has a weakly continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E, P_1 be a sunny and nonexpansive retraction of E onto C, $T : C \to E$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$, and $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_0,\ x \in C, \\ x_{n+1} = \frac{1}{n+1} \displaystyle\sum_{j=0}^{n} (\alpha_n x + (1-\alpha_n)(P_1 T)^j x_n) \text{ for } n = 0, 1, 2, ... \end{cases}$$

Assume that there hold the following conditions:

(i) $T\omega_w(x_n) \subseteq C$;

(ii) there exists a sunny and nonexpansive retraction P_2 of C onto $F(T)$ such that

$$\sup\{\|x_n - P_2x\|/\varphi(\|x_n - P_2x\|) : x_n \neq P_2x\} < \infty.$$

Then $\{x_n\}$ converges strongly to $P_2x \in F(T)$.

Proof. We divide the proof into four steps.

Step 1. We claim that the sequence$\{x_n\}$ is bounded. Indeed let $z \in F(T)$ and $D = \max\{\|x - z\|, \|x_0 - z\|\}$. Then we have

$$\|x_1 - z\| = \|\alpha_0 x + (1 - \alpha_0)x_0 - z\| \leq \alpha_0\|x - z\| + (1 - \alpha_0)\ |x_0 - z\| \leq D.$$

If $\|x_n - z\| \leq D$ for some $n \in N$, then we can show that $\|x_{n+1} - z\| \leq D$ similarly. Therefore, by induction we obtain $\|x_n - z\| \leq D$ for all $n \in N$ and hence $\{x_n\}$ is bounded.

Step 2. We claim that $\|x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j x_n\| \to 0$ as $n \to \infty$. Indeed observe that

$$\begin{aligned}
&\|x_{n+1} - \tfrac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j x_n\| \\
&= \|\tfrac{1}{n+1}\sum_{j=0}^{n}(\alpha_n x + (1 - \alpha_n)(P_1T)^j x_n) - \frac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j x_n\| \\
&\leq \alpha_n\|x - \tfrac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j x_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, the sequence $\{x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j x_n\}$ converges strongly to 0 as claimed.

Step 3. We claim that $\limsup_{n\to\infty}\langle x - P_2x, J(x_{n+1} - P_2x)\rangle \leq 0$. Indeed let $\{x_{n_j+1}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x - P_2x, J(x_{n+1} - P_2x)\rangle = \lim_{j\to\infty}\langle x - P_2x, J(x_{n_j+1} - P_2x)\rangle.$$

Since E is uniformly convex, E is reflexive. Hence without loss of generality, we may further assume that $\{x_{n_j+1}\}$ converges weakly to some $u \in C$ due to the weak closedness of C. From Lemma 1 and Step 2, we obtain $u \in F(P_1T)$. Note that $T\omega_w(x_n) \subseteq C$. This implies that $Tu = P_1Tu = u$ and hence

$u \in F(T)$. On the other hand by using gauge function φ, we define for every $n \geq 0$

$$\eta_n := \begin{cases} \frac{\|x_n - P_2x\|}{\varphi(\|x_n - P_2x\|)}, & \text{if } x_n \neq P_2x, \\ 0, & \text{if } x_n = P_2x. \end{cases}$$

From $\sup\{\|x_n - P_2x\|/\varphi(\|x_n - P_2x\|) : x_n \neq P_2x\} < \infty$, we obtain $\limsup_{n\to\infty} \eta_n < \infty$. Also from Proposition 1 (iii), we obtain

$$J(x_n - P_2x) = \eta_n J_\varphi(x_n - P_2x) \quad \text{for all } n \geq 0.$$

Since J_φ is continuous from the weak topology to the weak* topology, we conclude that

$$\lim_{j\to\infty} \langle x - P_2x, J_\varphi(x_{n_j+1} - P_2x)\rangle = \langle x - P_2x, J_\varphi(u - P_2x)\rangle.$$

It is clear that Proposition 1 (iii) yields

$$J_\varphi(u - P_2x) = \begin{cases} \frac{\varphi(\|u-P_2x\|)}{\|u-P_2x\|} J(u - P_2x), & \text{if } u \neq P_2x, \\ 0 & \text{if } u = P_2x, \end{cases}$$

which implies that

$$\langle x - P_2x, J_\varphi(u - P_2x)\rangle = \begin{cases} \frac{\varphi(\|u-P_2x\|)}{\|u-P_2x\|} \langle x - P_2x, J(u - P_2x)\rangle, & \text{if } u \neq P_2x, \\ 0, & \text{if } u = P_2x. \end{cases}$$

Since P_2 is the sunny and nonexpansive retraction of C onto $F(T)$, from Lemma 3 we obtain

$$\lim_{j\to\infty} \langle x - P_2x, J_\varphi(x_{n_j+1} - P_2x)\rangle = \langle x - P_2x, J_\varphi(u - P_2x)\rangle \leq 0,$$

and hence we infer by Lemma 6 that

$$\begin{aligned} \limsup_{n\to\infty} \langle x - P_2x, J(x_{n+1} - P_2x)\rangle &= \lim_{j\to\infty} \langle x - P_2x, J(x_{n_j+1} - P_2x)\rangle \\ &= \lim_{j\to\infty} \eta_{n_j+1} \langle x - P_2x, J_\varphi(x_{n_j+1} - P_2x)\rangle \leq 0. \end{aligned}$$

Step 4. We claim that $x_n \to P_2x$. Indeed by Step 3, we have that for any $\varepsilon > 0$, there exists $m \in N$ such that for all $n \geq m$

$$\langle x - P_2x, J(x_{n+1} - P_2x)\rangle < \frac{\varepsilon}{2}.$$

Also observe that

$$\begin{aligned} x_{n+1} - P_2x + \alpha_n(P_2x - x) = \ & \tfrac{1}{n+1} \textstyle\sum_{j=0}^{n} (\alpha_n x + (1 - \alpha_n)(P_1T)^j x_n) \\ & -(\alpha_n x + (1 - \alpha_n)P_2x). \end{aligned}$$

This together with Lemma 2, implies that for all $n \geq m$,

$$\begin{aligned}\|x_{n+1} - P_2x\|^2 &\leq \|\frac{1}{n+1}\sum_{j=0}^{n}(\alpha_n x + (1-\alpha_n)(P_1T)^j x_n) \\ &- (\alpha_n x + (1-\alpha_n)P_2x)\|^2 \\ &+ 2\alpha_n\langle x - P_2x, J(x_{n+1} - P_2x)\rangle \\ &\leq \{(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|(P_1T)^j x_n - P_2x\ |\}^2 \\ &+ 2\alpha_n\langle x - P_2x, J(x_{n+1} - P_2x)\rangle \\ &\leq (1-\alpha_n)^2\|x_n - P_2x\|^2 + 2\alpha_n\langle x - P_2x, J(x_{n+1} - P_2x)\rangle \\ &\leq (1-\alpha_n)\|x_n - P_2x\|^2 + \alpha_n\varepsilon.\end{aligned}$$

Therefore by Lemma 5, we conclude that $x_n \to P_2x$. The proof is now complete. □

Theorem 2. Let E be a uniformly convex Banach space whose norm is Gâteaux differentiable and which has a weakly continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E, P_1 be a sunny and nonexpansive retraction of E onto C, $T : C \to E$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$, and $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Let $\{y_n\}$ be the sequence generated by

$$\begin{cases} y_0,\ y \in C, \\ y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P_1(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n) \text{ for } n = 0,1,2,.... \end{cases}$$

Assume that there hold the following conditions:

(i) $T\omega_w(y_n) \subseteq C$;

(ii) there exists a sunny and nonexpansive retraction P_2 of C onto $F(T)$ such that

$$\sup\{\|y_n - P_2y\|/\varphi(\|y_n - P_2y\|) : y_n \neq P_2y\} < \infty.$$

Then $\{y_n\}$ converges strongly to $P_2y \in F(T)$.

Proof. We divide the proof into four steps.

Step 1. We claim that the sequence $\{y_n\}$ is bounded. Indeed let $z \in F(T)$ and $D = \max\{\|y - z\|, \|y_0 - z\|\}$. Then we have

$$\begin{aligned} \|y_1 - z\| &= \|P_1(\alpha_0 y + (1-\alpha_0)y_0) - z\| \le \|\alpha_0 y + (1-\alpha_0)y_0 - z\| \\ &\le \alpha_0\|y - z\| + (1-\alpha_0)\|y_0 - z\| \le D. \end{aligned}$$

If $\|y_n - z\| \le D$ for some $n \in N$, then we can show that $\|y_{n+1} - z\| \le D$ similarly. Therefore by induction we obtain $\|y_n - z\| \le D$ for all $n \in N$ and hence $\{y_n\}$ is bounded.

Step 2. We claim that $\|y_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j y_n\| \to 0$ as $n \to \infty$. Indeed observe that

$$\begin{aligned} &\|y_{n+1} - \tfrac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j y_n\| \\ &= \|\tfrac{1}{n+1}\textstyle\sum_{j=0}^{n} P_1(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n) - \tfrac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j y_n\| \\ &\le \tfrac{1}{n+1}\textstyle\sum_{j=0}^{n} \|P_1(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n) - (P_1T)^j y_n\| \\ &= \tfrac{1}{n+1}\textstyle\sum_{j=0}^{n} \|P_1(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n) - P_1(TP_1)^j y_n\| \\ &\le \tfrac{1}{n+1}\sum_{j=0}^{n} \|\alpha_n y + (1-\alpha_n)(TP_1)^j y_n - (TP_1)^j y_n\| \\ &\le \alpha_n \tfrac{1}{n+1}\sum_{j=0}^{n} \|y - (TP_1)^j y_n\|. \end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, it follows that $\{y_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(P_1T)^j y_n\}$ converges strongly to 0.

Step 3. We claim that $\limsup_{n\to\infty}\langle y - P_2 y, J(y_{n+1} - P_2 y)\rangle \le 0$. Indeed let $\{y_{n_j+1}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle y - P_2 y, J(y_{n+1} - P_2 y)\rangle = \lim_{j\to\infty}\langle y - P_2 y, J(y_{n_j+1} - P_2 y)\rangle.$$

Since E is uniformly convex, E is reflexive. Hence without loss of generality, we may further assume that $\{y_{n_j+1}\}$ converges weakly to some $u \in C$. From Lemma 1 and Step 2, we obtain $u \in F(P_1T)$. Note that $T\omega_w(y_n) \subseteq C$. This implies that $Tu = P_1Tu = u$ and hence $u \in F(T)$. On the other hand by using gauge function φ, we define for every $n \ge 0$

$$\eta_n := \begin{cases} \frac{\|y_n - P_2 y\|}{\varphi(\|y_n - P_2 y\|)}, & \text{if } y_n \ne P_2 y, \\ 0, & \text{if } y_n = P_2 y. \end{cases}$$

From $\sup\{\|y_n - P_2y\|/\varphi(\|y_n - P_2y\|) : y_n \neq P_2y\} < \infty$, we obtain $\limsup_{n\to\infty} \eta_n < \infty$. Also from Proposition 1 (iii), we obtain

$$J(y_n - P_2y) = \eta_n J_\varphi(y_n - P_2y) \quad \text{for all } n \geq 0.$$

Since J_φ is continuous from the weak topology to the weak* topology, we conclude that

$$\lim_{j\to\infty} \langle y - P_2y, J_\varphi(y_{n_j+1} - P_2y)\rangle = \langle y - P_2y, J_\varphi(u - P_2y)\rangle.$$

It is clear that Proposition 1 (iii) yields

$$J_\varphi(u - P_2y) = \begin{cases} \frac{\varphi(\|u-P_2y\|)}{\|u-P_2y\|} J(u - P_2y), & \text{if } u \neq P_2y, \\ 0, & \text{if } u = P_2y \end{cases}$$

which implies that

$$\langle y - P_2y, J_\varphi(u - P_2y)\rangle = \begin{cases} \frac{\varphi(\|u-P_2y\|)}{\|u-P_2y\|} \langle y - P_2y, J(u - P_2y)\rangle, & \text{if } u \neq P_2y, \\ 0, & \text{if } u = P_2y. \end{cases}$$

Since P_2 is the sunny and nonexpansive retraction of C onto $F(T)$, from Lemma 3, we obtain

$$\lim_{j\to\infty} \langle y - P_2y, J_\varphi(y_{n_j+1} - P_2y)\rangle = \langle y - P_2y, J_\varphi(u - P_2y)\rangle \leq 0$$

and hence we infer by Lemma 6 that

$$\limsup_{n\to\infty} \langle y - P_2y, J(y_{n+1} - P_2y)\rangle = \lim_{j\to\infty} \langle y - P_2y, J(y_{n_j+1} - P_2y)\rangle$$

$$= \lim_{j\to\infty} \eta_{n_j+1} \langle y - P_2y, J_\varphi(y_{n_j+1} - P_2y)\rangle \leq 0.$$

Step 4. We claim that $y_n \to P_2y$. Indeed by Step 3 we have that for any $\varepsilon > 0$, there exists $m \in N$ such that for all $n \geq m$

$$\langle y - P_2y, J(y_{n+1} - P_2y)\rangle < \frac{\varepsilon}{2}.$$

Also observe that

$$\begin{aligned} y_{n+1} - P_2y + \alpha_n(P_2y - y) = \; & \tfrac{1}{n+1} \textstyle\sum_{j=0}^n P_1(\alpha_n y + (1 - \alpha_n)(TP_1)^j y_n) \\ & - P_1(\alpha_n y + (1 - \alpha_n)P_2y). \end{aligned}$$

This together with Lemma 2 implies that for all $n \geq m$,

$$\begin{aligned}
\|y_{n+1} - P_2y\|^2 &\leq \|\tfrac{1}{n+1}\sum_{j=0}^{n} P_1(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n) \\
&\quad -P_1(\alpha_n y + (1-\alpha_n)P_2y)\|^2 + 2\alpha_n\langle y - P_2y, J(y_{n+1} - P_2y)\rangle \\
&\leq \{\tfrac{1}{n+1}\sum_{j=0}^{n} \|P_1(\alpha_n y + (1-\alpha_n)(TP_1)^j y_n) \\
&\quad -P_1(\alpha_n y + (1-\alpha_n)P_2y)\|\}^2 + 2\alpha_n\langle y - P_2y, J(y_{n+1} - P_2y)\rangle \\
&\leq \{(1-\alpha_n)\tfrac{1}{n+1}\sum_{j=0}^{n} \|(TP_1)^j y_n - P_2y\|\}^2 \\
&\quad +2\alpha_n\langle y - P_2y, J(y_{n+1} - P_2y)\rangle \\
&\leq (1-\alpha_n)^2\|y_n - P_2y\|^2 + 2\alpha_n\langle y - P_2y, J(y_{n+1} - P_2y)\rangle \\
&\leq (1-\alpha_n)\|y_n - P_2y\|^2 + \alpha_n\varepsilon.
\end{aligned}$$

Therefore by Lemma 5, we obtain $y_n \to P_2y$ and the proof is complete. □

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