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# REMARKS ON THE STABILITY OF MONOMIAL FUNCTIONAL EQUATIONS

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**Abstract.** The aim of this paper is to prove the generalized stability in the Ulam-Hyers-Bourgin sense of the monomial equations, for functions from 2-divisible groups into complete $\beta$-normed spaces. There is used a fixed point method, previously applied by the authors to some particular functional equations. A special case, the stability of type Aoki-Rassias is emphasized in section 3 and finally we give some (counter)examples in order to clarify the role of each of the control conditions.
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## 1. Introduction

Different methods are known for demonstrating the stability of functional equations. Nevertheless, almost all proofs use the *direct method*, conceived by Hyers in [22] and revealed by Bourgin in [5] for unbounded differences (see also [23], [30], [14] and [18]). For supplementary details, we refer the reader to the expository papers [15] and [31] or to the books [24], [12], [13] and [25]. A slightly different method, by using the *fixed point alternative*, has been applied in [29], [6]-[11], [26] and [28]. It does essentially insinuate a metrical context and is seen to better clarify the ideas of stability as well as the role of each controlling condition.

For an Abelian group $X$ and a vector space $Y$ consider the difference operators defined, for each $y \in X$ and any mapping $f : X \to Y$, in the following manner:

$$\Delta_y^1 f(x) := f(x+y) - f(x), \text{ for all } x \in X,$$

and, inductively, $\Delta_y^{n+1} = \Delta_y^1 \circ \Delta_y^n$, for all $n \geq 1$.

A mapping $f : X \to Y$ is called a *monomial function of degree N* if it is a solution of the *monomial functional equation*

$$\Delta_y^N f(x) - N!f(y) = 0, \ \forall \ x, y \in X. \tag{1.1}$$

Notice that the monomial equation of degree 1 is exactly the Cauchy equation, while for N=2 the monomial equation has the form $f(x+2y) - 2f(x+y) + f(x) - 2f(y) = 0$, which is equivalent to the well-known quadratic functional equation:

$$f(x+y) + f(x-y) = 2(f(x) + f(y)), \forall x, y \in X. \tag{1.2}$$

In what follows, the positive integer $N$ will be fixed.

M.H. Albert and A. Baker, in [3], demonstrated the Ulam-Hyers stability of the monomial equation (1.1) (see also [20]). Subsequently, A. Gilanyi proved in [21] that the equation is stable in the sense of Aoki-Rassias (see also [35] for the asymptotic stability):

**Proposition 1.1.** *Let $X$ be a normed linear space, let $Y$ be a Banach space and let $p \neq N$ be a non-negative real number. If, for a function $f : X \to Y$, there exists a real number $\eta \geq 0$ such that*

$$\|\Delta_y^N f(x) - N!f(y))\| \leq \eta(\|x\|^p + \|y\|^p), \ \forall x, y \in X, \tag{1.3}$$

*then there exist a real number $c = c(N, p)$ and a unique monomial function $g : X \to Y$ of degree $N$ with the property*

$$\|f(x) - g(x)\| \leq c\eta\|x\|^p \quad , \ \forall x \in X.$$

On the other hand, by using the fixed point method, we proved in [8] the following generalized stability result for additive Cauchy equations and mappings with values into $\beta$-normed spaces:

**Proposition 1.2.** *Let $X, Y$ be two linear spaces over the same (real or complex) field, with $Y$ a complete $\beta$-normed space, and set* $r_j = \begin{cases} 2, & \text{if } j = 0 \\ \frac{1}{2}, & \text{if } j = 1 \end{cases}$.

*Suppose the mapping $f : X \to Y$, with $f(0) = 0$, verifies* ***the control condition***

$$\|f(x+y) - f(x) - f(y)\|_\beta \leq \varphi(x,y), \text{ for all } x, y \in X,$$

*where $\varphi : X \times X \to \mathbb{R}_+$ has the following property:*

$$\lim_{n\to\infty} \frac{\varphi\left(r_j^n x, r_j^n y\right)}{r_j^{n\beta}} = 0.$$

*If there exists a positive constant $L < 1$ such that the mapping*

$$x \to \psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

*satisfies the relation*

$$\psi(x) \leq L \cdot r_j^\beta \cdot \psi\left(\frac{x}{r_j}\right), \text{ for all } x \in X,$$

*then there exists a* ***unique additive mapping*** *$a : X \to Y$ such that*

$$\|f(x) - a(x)\|_\beta \leq \frac{L^{1-j}}{1-L}\psi(x), \text{for all } x \in X.$$

In the present paper, by developing to some extent the above results, we prove a stability property in the **Ulam-Hyers-Bourgin sense**: For every $f : G \to Y$, from a 2-divisible group $G$ into a complete $\beta$-normed space $Y$, which verifies a *slightly perturbed monomial equation* ( i.e. controlled by a mapping $\varphi : G \times G \to [0, \infty)$ with suitable properties), there exists a *unique monomial solution* fittingly approximating $f$ (see Agarwal, Xu & Zhang [1] for a general definition)

## 2. A stability theorem via the fixed point alternative

We intend to show that Proposition 1.1 and Proposition 1.2 can really be generalized by the *fixed point method*, proposed in [29] and already used in [6]-[11] and [28] to different functional equations (see also [4] and [16]). Actually, we shall prove a new stability theorem of the Ulam-Hyers-Bourgin type for the monomial functional equation (1.1). As it will be seen, the fixed point alternative is a meaningful device on the road to a better understanding of the stability property, plainly related to some fixed point of a concrete operator. Specifically, our control conditions are perceived to be responsible for three fundamental facts:

1) The *contraction property* of the operator $S$ (given by $(\mathbf{OP^j_{mon}})$ below).
2) The distance between $f$ and $Sf$, the first two approximations, *is finite.*
3) The fixed point function of $S$ is forced to be a *monomial function.*

Let $X$ be a 2-divisible group[1], let $Y$ be a (real or complex) complete $\beta$-normed space[2], and assume we are given a function $\varphi : X \times X \to [0, \infty)$ with the following property:

$$\left(\mathbf{H}^*_\mathbf{j}\right) \qquad \lim_{m\to\infty} \frac{\varphi\left(r_j^m x, r_j^m y\right)}{r_j^{mN\beta}} = 0, \forall x, y \in X, \text{ for } r_j := 2^{1-2j}, j \in \{0, 1\}.$$

**Theorem 2.1.** *Suppose the mapping $f : X \to Y$, with $f(0) = 0$, verifies* ***the control condition***

$$\|\Delta_y^N f(x) - N!f(y)\|_\beta \leq \varphi(x, y)\ ,\ \forall x, y \in X. \tag{2.4}$$

*If there exists a positive constant $L < 1$ such that the mapping*

$$x \to \psi(x) = \frac{1}{(N!)^\beta}\left(\varphi(0, x) + \sum_{i=0}^{N}\begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \varphi\left(\frac{ix}{2}, \frac{x}{2}\right)\right)\ ,\ \forall x \in X,$$

*satisfies the inequality*

$$(\mathbf{H_j}) \qquad \psi(r_j x) \leq L \cdot r_j^{N\beta} \cdot \psi(x), \forall x \in X,$$

*then there exists a* ***unique monomial mapping*** *$g : X \to Y$, of degree $N$, with the following fitting property:*

$$(\mathbf{Est_j}) \qquad \|f(x) - g(x)\| \leq \frac{L^{1-j}}{1-L}\psi(x), \forall x \in X.$$

For the proof of our theorem, we need the following two fundamental lemmas. The first one gives a crucial intermediary result and the second one, which is recalled for convenience only, is a celebrated result in fixed point theory.

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[1]That is to say an Abelian group $(X, +)$ such that for any $x \in X$ there exists a unique $a \in X$ with the property $x = 2a$; this unique element $a$ is denoted by $\frac{x}{2}$.

[2]Usually, a mapping $|| \cdot ||_\beta : Y \to \mathbb{R}_+$, where $\beta \in (0, 1]$, is called a $\beta-norm$ iff it has the properties $(n^I_\beta)$ : $||y||_\beta = 0 \Leftrightarrow y = 0$, $(n^{II}_\beta)$ : $||\lambda \cdot y||_\beta = |\lambda|^\beta \cdot ||y||_\beta$, and $(n^{III}_\beta)$ : $||y + z||_\beta \leq ||y||_\beta + ||z||_\beta$, for all $y, z \in Y$, and $\lambda \in \mathbb{K}$.

**Lemma 2.2.** *Let us consider an Abelian group $G$, a $\beta-$normed linear space $Y$ and a mapping $\varphi : G \times G \to [0, \infty)$. If the function $f : G \to Y$ satisfies (2.4) then, for all $x \in G$,*

$$\left\|\frac{f(2x)}{2^N} - f(x)\right\|_\beta \le \frac{1}{2^{N\beta} \cdot (N!)^\beta} \cdot \left(\varphi(0, 2x) + \sum_{i=0}^{N} \binom{N}{N-i} \cdot \varphi(ix, x)\right). \tag{2.5}$$

**Proof.** As in [21], we define the functions $F_i : G \to Y$ by

$$F_i(x) := \Delta_x^N f(ix) - (N!)f(x) \ , \quad \forall x \in G.$$

Using (2.4) we see that

$$\|F_i(x)\|_\beta \le \varphi(ix, x) \ , \ \forall x \in G \tag{2.6}$$

and

$$||F_0(2x)||_\beta \le \varphi(0, 2x) \ , \ \forall x \in G. \tag{2.7}$$

On the other hand, if we consider (as in [19], Lemma 2.2.) the $(N+1)\times(2N+1)$ matrix

$$A = \begin{pmatrix} \alpha_0^{(0)} & \alpha_0^{(1)} & \dots & \alpha_0^{(2N)} \\ \alpha_1^{(0)} & \alpha_1^{(1)} & \dots & \alpha_1^{(2N)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_N^{(0)} & \alpha_N^{(1)} & \dots & \alpha_N^{(2N)} \end{pmatrix}$$

where

$$\alpha_i^{(i+k)} = \begin{cases} (-1)^k \binom{N}{N-k} & , \text{ if } 0 \le k \le N \\ 0 & , \text{ otherwise} \end{cases} \tag{2.8}$$

for $i = 0, ..., N$ , $k = -i, ..., 2N - i$, and the $1 \times (2N + 1)$ matrix

$$B = \begin{pmatrix} \beta^{(0)} & \beta^{(1)} & \beta^{(2)} & \dots & \beta^{(2N)} \end{pmatrix},$$

with

$$\beta^{(k)} = \begin{cases} (-1)^{\frac{k}{2}} \binom{N}{N-\frac{k}{2}} & , \text{ if } 2 \mid k \\ 0 & , \text{ if } 2 \nmid k \end{cases} , \qquad \text{for } k = 0, ..., 2N, \tag{2.9}$$

then, for the positive constants $K_i = \binom{N}{N-i}$, one has

$$K_0 A_0 + K_1 A_1 + ... + K_N A_N = B \ \text{ and } \ K_0 + K_1 + ... + K_N = 2^N, \tag{2.10}$$

where $A_i$, $i = 0, ..., N$ is the i-th row of the matrix $A$.
Recall the following noteworthy formula for the difference operator:

$$\Delta_y^N f(x) = \sum_{j=0}^{N} (-1)^{N+j} \left( \begin{array}{c} N \\ N-j \end{array} \right) f(x + jy). \qquad (2.11)$$

Therefore

$$\Delta_x^N f(ix) \quad = \quad (-1)^N \sum_{k=i}^{N+i} (-1)^{k-i} \left( \begin{array}{c} N \\ N-(k-i) \end{array} \right) f(kx)$$

so that, with the notation (2.8),

$$\Delta_x^N f(ix) = (-1)^N \sum_{k=0}^{2N} \alpha_i^{(k)} \cdot f(kx) \ , \forall x \in G,$$

hence

$$F_i(x) = (-1)^N \sum_{k=0}^{2N} \alpha_i^{(k)} \cdot f(kx) - (N!) f(x) \ , \forall x \in G.$$

On the other hand, with the notation (2.9),

$$\Delta_{2x}^N f(0) = (-1)^N \sum_{j=0}^{N} (-1)^j \left( \begin{array}{c} N \\ N-j \end{array} \right) f(2xj) = (-1)^N \sum_{k=0}^{2N} \beta^{(k)} \cdot f(kx),$$

so that

$$F_0(2x) = (-1)^N \sum_{k=0}^{2N} \beta^{(k)} \cdot f(kx) - (N!) f(2x) \ , \forall x \in G.$$

By (2.10), we can write:

$$K_0 \cdot \left( \alpha_0^{(k)} \right)_{k=\overline{0,2N}} + ... + K_N \cdot \left( \alpha_N^{(k)} \right)_{k=\overline{0,2N}} = \left( \beta^{(k)} \right)_{k=\overline{0,2N}} = B, \qquad (2.12)$$

where $K_i = \left( \begin{array}{c} N \\ N-i \end{array} \right)$, $i = 0, 1, ..., N$.
If we multiply (2.12) by $f(kx)$ and then add for $k = \overline{0, 2N}$, we obtain

$$K_0 \cdot \Delta_x^N f(0) + K_1 \cdot \Delta_x^N f(x) + ... + K_N \cdot \Delta_x^N f(Nx) = \Delta_{2x}^N f(0) \ , \ \forall x \in G,$$

hence

$$K_0 F_0(x) + K_1 F_1(x) + ... + K_N F_N(x) + 2^N (N!) f(x) = F_0(2x) + (N!) f(2x)$$

for all $x \in G$. Using (2.6) and (2.7) we get, for all $x \in G$,

$$\left\|\frac{f(2x)}{2^N} - f(x)\right\|_\beta \leq \frac{1}{2^{N\beta} \cdot (N!)^\beta} \left(\varphi(0, 2x) + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \varphi(ix, x)\right). \quad \square$$

**Remark 2.3.** It easy to see ( e.g. by taking $\varphi \equiv 0$ in Lemma 2.2) that

$$g(2x) = 2^N g(x) \text{ and } g(2^m x) = 2^{Nm} g(x) \ , \forall x \in G, \forall m \in \mathbb{N},$$

for any monomial function $g$ of degree $N$.

**Remark 2.4.** Let $G$ be a 2 - divisible group and formally replace $x$ with $\frac{x}{2}$ in (2.5). Then one has the following result:

*Let $Y$ be a $\beta-$normed linear space and $\varphi : G \times G \to [0, \infty)$ a given mapping. If $f : G \to Y$ satisfies*

$$\|\Delta_y^N f(x) - N! f(y))\|_\beta \leq \varphi(x, y) \ , \ \forall x, y \in G,$$

*then*

$$\left\|f(x) - 2^N f\left(\frac{x}{2}\right)\right\|_\beta \leq \frac{1}{(N!)^\beta} \left(\varphi(0, x) + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \varphi\left(\frac{ix}{2}, \frac{x}{2}\right)\right), \ \forall x \in G.$$

Now we recall **the fixed point alternative** (see, e.g., [27, 32]):

**Lemma 2.5.** *Suppose we are given a complete generalized metric space $(E, d)$ (i.e. one for which d may assume infinite values), and a strictly contractive mapping $S : E \to E$, with the Lipschitz constant $L < 1$. Then, for each of the elements $x \in E$, exactly one of the following assertions is true:*
$(A_1)$ $d(S^n x, S^{n+1} x) = +\infty$, *for all $n \geq 0$,*
$(A_2)$ *There exists $k$ such that $d(S^n x, S^{n+1} x) < +\infty$, for each $n \geq k$.*
*Actually, if $(A_2)$ holds, then (see [34])*
$(A_{21})$ *The sequence $(S^n x)$ is convergent to a fixed point $z^*$ of $S$;*
$(A_{22})$ *$z^*$ is the unique fixed point of $S$ in the set*

$$Z := \left\{z \in E, \ d\left(S^k x, z\right) < +\infty\right\};$$

$(A_{23})$ *The following estimation holds:*

$$d\left(z, z^*\right) \leq \frac{1}{1 - L} d\left(z, Sz\right), \ \forall z \in Z.$$

**The proof of Theorem 2.1**.
We consider the set $E := \{g : X \to Y, g(0) = 0\}$ and introduce a *generalized metric* $d = d_\psi$ on $E$, where

$$(\mathbf{GM}_\psi) \qquad d_\psi(g,h) = \inf\left\{K \in \mathbb{R}_+, \|g(x) - h(x)\|_\beta \le K\psi(x), \forall x \in X\right\}.$$

It is easy to see that $(E, d)$ *is complete.* Now, consider the mapping

$$(\mathbf{OP^j_{mon}}) \qquad S : E \to E, Sg(x) := \frac{g(r_j x)}{r_j^N}.$$

and notice that $r_j = 2^{1-2j}$, for $j$ fixed in $\{0, 1\}$.
**Step I.** Using the hypothesis $(\mathbf{H}_j)$, one can see that $S$ *is strictly contractive* on $E$. Namely, we can write, for any $g, h \in E$ :

$$d(g,h) < K \Longrightarrow \|g(x) - h(x)\|_\beta \le K\psi(x), \forall x \in X \Longrightarrow$$

$$\left\|\frac{1}{r_j^N} g(r_j x) - \frac{1}{r_j^N} h(r_j x)\right\|_\beta \le \frac{1}{r_j^{N\beta}} K\psi(r_j x), \forall x \in X \Longrightarrow$$

$$\left\|\frac{1}{r_j^N} g(r_j x) - \frac{1}{r_j^{N\beta}} h(r_j x)\right\|_\beta \le LK\psi(x), \forall x \in X \Longrightarrow$$

$$d(Sg, Sh) \le LK.$$

Therefore

$$(\mathbf{CC_L}) \qquad d(Sg, Sh) \le L d(g,h), \forall g, h \in E,$$

that is $S$ is a *strictly contractive* self-mapping of $E$ relative to $d$, with the Lipschitz constant $L < 1$.
**Step II.** We show that $d(f, Sf) < \infty$.
For j=0, by Lemma 2.2, we have

$$\left\|\frac{f(2x)}{2^N} - f(x)\right\|_\beta \le \frac{1}{2^{N\beta} \cdot (N!)^\beta} \left(\varphi(0, 2x) + \sum_{i=0}^{N} \binom{N}{N-i} \varphi(ix, x)\right), \forall x \in X.$$

Therefore, using $(\mathbf{H_0})$,

$$\left\|\frac{f(2x)}{2^N} - f(x)\right\|_\beta \le \frac{\psi(2x)}{2^{N\beta}} \le L\psi(x), \forall x \in X,$$

that is $d(f, Sf) \le L < \infty$.

For j=1, by Remark 2.4, we see that

$$\left\|f(x)-2^N f\left(\frac{x}{2}\right)\right\|_\beta \leq \frac{1}{(N!)^\beta}\left(\varphi(0,x)+\sum_{i=0}^{N}\begin{pmatrix} N \\ N-i \end{pmatrix}\cdot\varphi\left(\frac{ix}{2},\frac{x}{2}\right)\right)$$

$$= \psi(x)\ ,\ \forall x \in X.$$

Therefore $d\left(f, Sf\right) \leq 1 = L^0 < \infty$.

**Step III.** In both cases we can apply the fixed point alternative (see Lemma 2.5), which tells us that there is a mapping $g : X \to Y$ such that:

- $g$ is a fixed point of $S$, that is

$$g\left(2x\right) = 2^N g\left(x\right), \forall x \in X. \tag{2.13}$$

The mapping $g$ is the unique fixed point of $S$ in the set

$$F = \left\{h \in E,\ d\left(f, h\right) < \infty\right\}.$$

This says that $g$ is the unique mapping with *both* the properties (2.13) and (2.14), where

$$\exists K < \infty \text{ such that } \|f\left(x\right) - g\left(x\right)\| \leq K\psi(x), \forall x \in X. \tag{2.14}$$

- $d(S^m f, g) \longrightarrow 0$, for $m \to \infty$, which implies the equality

$$\lim_{m\to\infty}\frac{f\left(r_j^m x\right)}{r_j^{mN}} = \lim_{m\to\infty} g_m(x) = g(x), \forall x \in X.$$

- $d(f,g) \leq \dfrac{1}{1-L} d\left(f, Sf\right)$, which implies the inequality

$$d(f,g) \leq \frac{L^{1-j}}{1-L}\ ,$$

from which ($\mathbf{Est_j}$) is seen to be true.

**Step IV.** We show that $g$ is a *monomial function of degree* $N$. To this end, we replace $x$ by $r_j^m x$ and $y$ by $r_j^m y$ in relation (2.4), then divide the obtained relation by $r_j^{mN}$ and we obtain

$$\left\|\frac{\Delta^N_{r_j^m y} f(r_j^m x)}{r_j^{mN}} - N!\frac{f(r_j^m y))}{r_j^{mN}}\right\|_\beta \leq \frac{\varphi(r_j^m x, r_j^m y)}{r_j^{mN\beta}}\ ,\ \forall x, y \in X.$$

On the other hand, by (2.11),

$$\begin{aligned}\frac{\Delta^N_{r_j^m y} f(r_j^m x)}{r_j^{mN}} &= \sum_{k=0}^{N}(-1)^{N-k}\begin{pmatrix} N \\ k \end{pmatrix}\frac{f(r_j^m x + k r_j^m y)}{r_j^{mN}} = \\ &= \sum_{k=0}^{N}(-1)^{N-k}\begin{pmatrix} N \\ k \end{pmatrix} g_m(x+ky) = \\ &= \Delta^N_y g_m(x), \forall x,y \in X.\end{aligned}$$

And we get

$$\left|\left|\Delta^N_y g_m(x) - N!\cdot g_m(y)\right|\right|_\beta \le \frac{\varphi(r_j^m x, r_j^m y)}{r_j^{mN\beta}}\ ,\ \forall x,y \in X.$$

By letting $m \to \infty$ in the above relation and using $\left(\mathbf{H_j^*}\right)$, we obtain

$$\Delta^N_y g(x) - N!\cdot g(y) = 0,\ \forall x,y \in G. \qquad \square$$

**Remark 2.6.** For $N = 1$ in the above theorem, we obtain a generalized stability result for the additive Cauchy equation for functions with values in complete $\beta$-normed spaces, with

$$\psi(x) = \varphi\left(0,x\right) + \varphi\left(0,\frac{x}{2}\right) + \varphi\left(\frac{x}{2},\frac{x}{2}\right), \forall x \in X.$$

(compare with Proposition 1.2).

If $N = 2$ in Theorem 2.1 it results (as in [7] for $\beta = 1$) that the quadratic functional equation (again for functions with values in complete $\beta$-normed spaces) is stable in the Ulam-Hyers-Bourgin sense, with

$$\psi(x) = \frac{1}{2^\beta}\left(\varphi\left(0,x\right) + \varphi\left(0,\frac{x}{2}\right) + 2\varphi\left(\frac{x}{2},\frac{x}{2}\right) + \varphi\left(x,\frac{x}{2}\right)\right), \forall x \in X.$$

It is worth noting that the estimations obtained directly for particular values of $N$ (as in [6], [7], [8] or [11]) are generally better than those resulting from ($\mathbf{Est_j}$), which in its turn **is applicable for all** $N$.

## 3. Stability of the Aoki-Rassias type

Let $\alpha \in \mathbb{R}_+$. A mapping $||.||_\alpha : X \times X \to \mathbb{R}_+$ is called an *sh-functional of order* $\alpha$ iff it is $\alpha-$sub-homogeneous:

$$(h_\alpha) : ||\lambda \cdot z||_\alpha \le |\lambda|^\alpha \cdot ||z||_\alpha,\ \forall \lambda \in \mathbb{K},\ \forall z \in X \times X.$$

As usual, $X$ is identified with $X \times \{0\}$ in $X \times X$, so that $||x||_\alpha = ||(x,0)||_\alpha$ defines an sh-functional of order $\alpha$ on $X$.

As a consequence of Theorem 2.1, we have the following result, which directly extends many stability theorems of the Aoki-Rassias type:

**Proposition 3.1.** *Let $X, Y$ be two linear spaces over the same (real or complex) field. Suppose we are given a complete $\beta-$norm on $Y$ and an sh-functional of order $\alpha$ on $X \times X$, with $\alpha \neq N\beta$. Under these conditions we have the following* ***stability property****:*
*For each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for every mapping $f : X \to Y$ which satisfies*

$$(\mathbf{C}_{\alpha\beta}) \qquad \|\Delta_y^N f(x) - N!f(y))\|_\beta \leq \delta(\varepsilon) \cdot ||(x,y)||_\alpha, \ \forall x, y \in X,$$

*there exists a unique monomial mapping $g : X \to Y$, of degree $N$, such that, $\forall x \in X$,*

$$(\mathbf{Est}_{\alpha\beta}) \ \ ||f(x) - g(x)||_\beta \leq \frac{\varepsilon}{(N!)^\beta} \left( \|(0,x)\|_\alpha + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \left\| \frac{ix}{2}, \frac{x}{2} \right\|_\alpha \right).$$

**Proof.** Having in mind Theorem 2.1, we take the following control function, appearing in the hypothesis $(\mathbf{C}_{\alpha\beta})$:

$$\varphi(x,y) := \delta(\varepsilon) \cdot \|(x,y)\|_\alpha, \text{ for all } x, y \in X.$$

We shall consider two cases.

**Case 1.** For $\alpha - N\beta < 0$, we work with $j = 0$, that is $r_0 = 2$. We then have:

$$\frac{\varphi(2^m x, 2^m y)}{2^{mN\beta}} \leq \delta(\varepsilon) \cdot 2^{m(\alpha - N\beta)} \cdot \|(x,y)\|_\alpha \to 0, \ \forall x, y \in X$$

and

$$\begin{aligned} \psi(2x) &= \frac{\delta(\varepsilon)}{(N!)^\beta} \left( \|(0,2x)\|_\alpha + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \|(ix,x)\|_\alpha \right) \leq \\ &\leq 2^\alpha \cdot \psi(x) = L \cdot 2^{N\beta} \cdot \psi(x), \ \forall x \in X, \end{aligned}$$

with $L = 2^{\alpha - N\beta} < 1$ and so $(\mathbf{H_0^*})$ and $(\mathbf{H_0})$ hold.

**Case 2.** For $\alpha - N\beta > 0$, we take $j = 1$, that is $r_1 = \frac{1}{2}$. We then have:

$$2^{N\beta m} \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \leq \delta(\varepsilon) \cdot 2^{m(N\beta - \alpha)} \cdot \|(x,y)\|_\alpha \to 0, \forall x, y \in X$$

and

$$\psi(x) \leq \frac{1}{2^{\alpha}} \cdot \psi(2x) = L \cdot \left(\frac{1}{2}\right)^{N\beta} \psi(2x),\ \forall x \in X,$$

with $L = 2^{N\beta-\alpha} < 1$, so that $(\mathbf{H_1^*})$ and $(\mathbf{H_1})$ are verified in this case.

Theorem 2.1 tells us that there is a *unique monomial* mapping $g : X \to Y$ such that either

$$\|f(x) - g(x)\|_{\beta} \leq \frac{L}{1-L}\psi(x),\ \text{for all } x \in X,$$

holds, with $L = 2^{\alpha-N\beta}$, or

$$\|f(x) - g(x)\|_{\beta} \leq \frac{1}{1-L}\psi(x),\ \text{for all } x \in X,$$

holds, with $L = 2^{N\beta-\alpha}$.

Thus, the inequality $(\mathbf{Est}_{\alpha\beta})$ holds true for $\delta(\varepsilon) = \varepsilon \cdot \frac{|2^{N\beta}-2^{\alpha}|}{2^{\alpha}}$. □

**Remark 3.2.** As a direct consequence of our proposition, we obtain the result of (Gilanyi, [21]) formulated in Proposition 1.1 above. Indeed, we apply Proposition 3.1 for the complete 1−normed space $Y$ and for the sh-functional of order $p$ on $X \times X$, given by $||x, y)||_p = ||x||^p + ||y||^p$, with $p \geq 0, p \neq N$. In this case the generalized metric is of the form

$$d_p\left(g, h\right) = \inf\left\{K \in \mathbb{R}_+, \|g\left(x\right) - h\left(x\right)\|_{\beta} \leq K \cdot \gamma(N,p) \cdot ||x||^p, \forall x \in X\right\},$$

and we obtain

$$c = c(N,p) = \frac{2^p + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot (i^p + 1)}{|2^N - 2^p| \cdot N!}.$$

**Remark 3.3.** Obviously, Proposition 3.1 can be proved by using directly the alternative of fixed point.

## 4. Comments, examples and counterexamples

As we shall see by examples, our hypotheses in Theorem 2.1 are essential for the stability result. Notice that in the proof we used the hypothesis $(\mathbf{H_j})$ to show that the operator $S$ is contractive and, subsequently, to obtain the estimation $(\mathbf{Est_j})$, while the hypothesis $\left(\mathbf{H_j^*}\right)$ was fundamentally used to show that the fixed point function $g$ does satisfy the monomial equation (1.1). In what follows we shall use the notations in Theorem 2.1.

**Assertion 4.1.** *There exist* $f : \mathbb{R} \to \mathbb{R}$ *and* $\varphi : \mathbb{R} \to [0, \infty)$ *such that (a) the relation (2.4) holds, (b)* ***none*** *of the conditions* $(\mathbf{H_j})$ *and* $(\mathbf{H_j^*})$ *is satisfied and (c) there exist* ***infinitely many monomial functions*** $g$ *which satisfy the relation* $(\mathbf{Est_j})$.

**Proof.** Indeed, if we take $X = Y = \mathbb{R}, ||\cdot||_\beta = |\cdot|$, $f(x) := x^N$ and $\varphi(x, y) = |x|^N + |y|^N$, then the inequality (2.4) clearly holds. Obviously,

$$\psi(x) = \frac{|x|^N}{N!}\left(1 + \frac{1}{2^N} \cdot \sum_{i=0}^{N} \left(\begin{array}{c} N \\ N-i \end{array}\right) \cdot (i^N + 1)\right) = \gamma(N) \cdot |x|^N,$$

and we easily see that none of the conditions $(\mathbf{H_j})$ and $(\mathbf{H_j^*})$ is satisfied. However, there exist infinitely many fixed points of $S$, $g(x) = \lambda x^N$, which satisfy

$$|f(x) - g(x)| = |x^N - \lambda x^N| \le (1 + |\lambda|) \cdot |x|^N = \frac{1 + |\lambda|}{\gamma(N)} \cdot \psi(x)$$

Actually, every mapping $g$ with $g(2x) = 2^N g(x)$ and $|g(x)| \le k \cdot |x|^N$ is seen to satisfy $|f(x) - g(x)| \le (1 + k) \cdot |x|^N$. □

**Assertion 4.2.** *Generally, the monomial functional equation (1.1) is not stable for* $\alpha = N\beta$. *More exactly, there exists* $f : \mathbb{R} \to \mathbb{R}$ *for which (1.3) holds for* $p = N$ *and there exists* ***no*** *monomial mapping* $g$ *of degree* $N$ *which could verify the relation* $(\mathbf{Est_j})$.

**Proof.** In [21] (see also [17], [25]-pp. 23-24, [24]-pp. 59-60) the following example is given: *Let* $N$ *be a positive integer,* $\varepsilon$ *be a positive real number and let*

$$\varepsilon^* = \frac{\varepsilon}{2^N(2^N + N!)N^N}.$$

*We consider the mapping* $\varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi(x) = \begin{cases} N^N \varepsilon^* , & \text{if } x \ge N \\ \varepsilon^* x^N , & \text{if } -N < x < N \\ (-1)^N N^N \varepsilon^*, & \text{if } x \le N \end{cases}$$

*and we define a function* $f$: $\mathbb{R} \to \mathbb{R}$ *by*

$$f(x) = \sum_{m=0}^{\infty} \frac{\varphi(2^m x)}{2^{mN}}, \text{ for all } x \in \mathbb{R}.$$

As it is proven in [21],
(i) $f$ *verifies the following inequality*:

$$|\Delta_y^N f(x) - N!f(y)| \le \varepsilon(|x|^N + |y|^N), \ \forall x, y \in \mathbb{R}; \tag{4.15}$$

(ii) *There is no real number c for which there could exist a monomial function* $g : \mathbb{R} \to \mathbb{R}$ of degree $N$ *such that* $|f(x) - g(x)| \le c\varepsilon|x|^N$, *for all* $x \in \mathbb{R}$.

From (4.15) it is clear that, for $\beta = 1$, $\varphi(x, y) = \varepsilon(|x|^N + |y|^N)$, hence

$$\psi(x) = \frac{\varepsilon|x|^N}{N!}\left(1 + \frac{1}{2^N} \cdot \sum_{i=0}^{N} \binom{N}{N-i} \cdot (i^N + 1)\right) = \varepsilon \cdot \gamma(N) \cdot |x|^N,$$

and **none** of the conditions $(\mathbf{H_j})$ and $(\mathbf{H_j^*})$ is satisfied.

Indeed, it is easy to see that

$$\frac{\psi(2x)}{2^N} = \psi(x) \ \text{ and } \lim_{m\to\infty} \frac{\varphi\left(r_j{}^m x, r_j{}^m y\right)}{r_j{}^{mN}} = \varphi\left(x, y\right) \neq 0.$$

On the other hand, by (ii), **there exists no monomial mapping** $g$ satisfying a relation of the form $|f(x) - g(x)| \le k \cdot \psi(x), \ \forall x \in \mathbb{R}$, of type $(\mathbf{Est_j})$. □

**Assertion 4.3.** *There exist* $f : \mathbb{R} \to \mathbb{R}$ *and* $\varphi : \mathbb{R} \to [0, \infty)$ *such that: (a) the relation (2.4) holds, (b) the conditions* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$ ***are verified**, (c) the mapping* $\varphi$ ***does not satisfy** the condition* $(\mathbf{H_j^*})$ *and (d)* ***no monomial mapping*** *can satisfy the relation* $(\mathbf{Est_j})$.

**Proof.** Indeed, if we take $X = Y = \mathbb{R}, || \cdot ||_\beta = | \cdot |$,

$$f(x) = \begin{cases} 0 \ , & \text{if } x \in \mathbb{Q} \\ x^N, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

and $\varphi(x, y) := |\Delta_y^N f(x) - N!f(y)|$, then the inequality (2.4) is verified. Moreover, $\psi(x) = 0$. Both conditions $(\mathbf{H}_0)$ and $(\mathbf{H}_1)$ hold for every $L \in (0, 1)$, while the mapping $\varphi$ does not satisfy the condition $(\mathbf{H_j^*})$. Since $\frac{f\left(r_j{}^m x\right)}{r_j{}^{mN}} = f(x)$, it is clear that *the unique fixed point* of the operator $S$ in the set $F = \{g \in E, \ d\left(f, g\right) < +\infty\}$ is $g = f$, which clearly is not monomial. □

**Assertion 4.4.** *There exist* $f : \mathbb{R} \to \mathbb{R}$ *and* $\varphi : \mathbb{R} \to [0, \infty)$ *such that (a) the relation (2.4) holds, (b)* ***none*** *of the conditions* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$ *is satisfied, (c) the relation* $(\mathbf{H_j^*})$ *holds* *and (d)* ***no monomial mapping*** *$g$ can satisfy the relation* $(\mathbf{Est_j})$.

**Proof.** Let us take $X = Y = \mathbb{R}, ||\cdot||_\beta = |\cdot|$, $f(x) := \sin^N x$ and

$$\varphi(x,y) := |\Delta_y^N f(x) - N!f(y)|.$$

Then the inequality (2.4) is trivially verified. Obviously

$$\psi(x) = \frac{1}{N!}\left(\left|\sum_{j=0}^{N}(-1)^{N-j}\begin{pmatrix} N \\ N-j \end{pmatrix}\sin^N(jx) - N!\sin^N x\right| + \right.$$

$$\left. + \sum_{i=0}^{N}\begin{pmatrix} N \\ N-i \end{pmatrix}\cdot\left|\sum_{j=0}^{N}(-1)^{N-j}\begin{pmatrix} N \\ N-j \end{pmatrix}\sin^N\left(\frac{(i+j)x}{2}\right) - N!\sin^N\frac{x}{2}\right|\right),$$

for all $x \in \mathbb{R}$. For $x \to 0$ we obtain $\frac{\psi(2x)}{2^N\cdot\psi(x)} \to 4$, and for $x \to \pi$ we have that $\frac{\psi(2x)}{2^N\cdot\psi(x)} \to 0$. Therefore neither $(\mathbf{H_0})$ nor $(\mathbf{H_1})$ is satisfied. Clearly the mapping $\varphi$ satisfies the condition $\left(\mathbf{H_j^*}\right)$. Let us suppose, for a contradiction, that a monomial mapping $g$ satisfies

$(\mathbf{Est_j})$ $\qquad |f(x) - g(x)| \leq k\psi(x),$ for all $x \in \mathbb{R}, k$ a real constant.

Then $g$ is a bounded mapping on $\mathbb{R}$. Therefore (see [33]), $g(x) = \eta\cdot x^N$, for $x \in \mathbb{R}$, where $\eta$ is a constant, hence $g(x) = 0$ for $x \in \mathbb{R}$. Now, the inequality $(\mathbf{Est_j})$ implies

$$\left|\sin^N x\right| \leq \frac{k}{N!}\left(\left|\sum_{j=0}^{N}(-1)^{N-j}\begin{pmatrix} N \\ N-j \end{pmatrix}\sin^N(jx) - N!\sin^N x\right| + \right.$$

$$\left. + \sum_{i=0}^{N}\begin{pmatrix} N \\ N-i \end{pmatrix}\cdot\left|\sum_{j=0}^{N}(-1)^{N-j}\begin{pmatrix} N \\ N-j \end{pmatrix}\sin^N\left(\frac{(i+j)x}{2}\right) - N!\sin^N\frac{x}{2}\right|\right),$$

for all $x \in \mathbb{R}$, which is impossible. Therefore, there exists no monomial mapping $g$ which satisfies the estimation relation $(\mathbf{Est_j})$. □

**Remark 4.1.** In (Forti, [16]) a stability result for equations of the form $SF = F$ is presented, with $S$ a functional operator of the Schröder type: $SF(x) =$

$(H \circ F \circ G)(x)$. The hypotheses (as well as the proofs) insinuate a Banach-Caccioppoli type condition:

$$\sum_{n=0}^{\infty} dist(S^n f, S^{n+1} f) < \infty.$$

We only remark that such a condition and the continuity of $S$ together ensure the existence of a fixed point function for $S$, which is the limit of $S^n f$. It is worth noting that in (Baker, [4]) the Banach contraction principle is applied to obtain the Hyers-Ulam stability for nonlinear functional equations of a similar form: $\varphi(x) = F(x, \varphi(f(x)))$ (see also [1] for other details and examples).

**Remark 4.2.** In [10] we used the direct method to prove, among others, the following Ulam-Hyers-Bourgin stability property for monomial equations:
*Let there be given a complete $\beta-$normed space $Y$, an Abelian group $G$ and* ***a controlling mapping*** $\varphi : G \times G \to [0, \infty)$ *such that*

$$\Phi_i(x) := \sum_{k=0}^{\infty} \frac{\varphi(2^k ix, 2^k x)}{(2^{N\beta})^k} < \infty, \forall x \in G, \text{ for } i = 0, 1, ..., N,$$

*and*

$$\lim_{m\to\infty} \frac{\varphi(2^m x, 2^m y)}{(2^{N\beta})^m} = 0 \ , \ \forall x, y \in G.$$

*Then, for every mapping $f : G \to Y$ which verifies*

$$\|\Delta_y^N f(x) - N! f(y))\|_\beta \leq \varphi(x, y) \ , \ \forall x, y \in G,$$

*there exists* ***a unique monomial function*** $g : G \to Y$ *of degree $N$ such that, for all $x \in G$,*

$$\|f(x) - g(x)\|_\beta \leq \frac{1}{2^{N\beta} \cdot (N!)^\beta} \left( \Phi_0(2x) + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \Phi_i(x) \right).$$

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