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APPROACHING NONLINEAR VOLTERRA NEUTRAL DELAY INTEGRO-DIFFERENTIAL EQUATIONS WITH THE PEROV'S FIXED POINT THEOREM

ALEXANDRU MIHAI BICA* AND SORIN MUREŞAN**

*Department of Mathematics, University of Oradea
Armatei Romane No.5, 410087, Oradea, Romania
E-mail: abica@uoradea.ro, smbica@yahoo.com

**Department of Mathematics, University of Oradea
Armatei Romane No.5, 410087, Oradea, Romania
E-mail: smuresan@uoradea.ro

Abstract. Using the Perov's fixed point theorem and a Bielecki's type norm on a functional space, here we prove the existence and uniqueness of the solution of a class of nonlinear Volterra neutral delay integro-differential equations. Afterwards, we obtain some Lipschitz properties and the error estimation in the approximation of the solution and of his derivative, by the sequence of successive approximations.
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1. Introduction

In [7] is presented the following initial value problem :

$$\begin{cases} x'(t) = f(t, x(t)) + \int\limits_{t-\tau}^{t} g(t, s, x(s))ds, & t \in [0, b] \\ x(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{1}$$

where is solved using spline functions. This problem was studied also in [4], where was obtained the existence and uniqueness on $C[0, b]$, of the solution of (1), using the Banach's fixed point theorem.

In this paper we propose a generalization of this integral equation obtaining a nonlinear Volterra neutral delay integro-differential equation and the following initial value problem :

$$\begin{cases} x'(t) = f(t, x(t), x'(t-\tau)) + \int\limits_{t-\tau}^{t} g(t, s, x(s), x'(s))ds, & t \in [0, b] \\ x(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases} \tag{2}$$

Here we propose a new point of view in the study of the existence and uniqueness of the solution of this initial value problem, using a bidimensional variant of the Perov's fixed point theorem (this theorem appear in [10] (where appear for the first time), [11], [12] and [5]). The use of the Perov's fixed point theorem is founded on the remark that the space $C^1[0, b]$ with the norm of uniform convergence is not complete. This manner was also used by us in [2] and [8], to obtain the existence, uniqueness and approximation of the positive solution and the smooth dependence by parameter of this solution, of the following initial value problem

$$\begin{cases} x(t) = \int\limits_{t-\tau}^{t} f(s, x(s), x'(s))\, ds, & t \in [0, T] \\ x(t) = \varphi(t), & t \in [-\tau, 0] \end{cases}$$

which is a model for the spread of certain infectious disease with seasonally contact rate.

Results on the existence and uniqueness of the solution of nonlinear Volterra integro-differential equations and of corresponding initial value problems was obtained in [3], [6], [9], [13] and [14], using classic tools. Potential applications of the problem (1) and of his generalization (2) can be found in [7].

By a generalized metric, denoted by d, on a nonempty set X we understand a function $d : X \times X \to \mathbb{R}^n$ which fulfills the conditions :

$$d(x, y) \geq 0_{\mathbb{R}^n}, \quad \forall x, y \in X, \text{ and } \quad d(x, y) = 0_{\mathbb{R}^n} \Leftrightarrow x = y$$
$$d(x, y) = d(y, x), \qquad \forall x, y \in X$$
$$d(x, y) \leq d(x, z) + d(z, y), \qquad \forall x, y, z \in X,$$

where for $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ from $\mathbb{R}^n$ we have by definition, that

$$x \leq y \Longleftrightarrow x_i \leq y_i, \ \forall i = \overline{1, n}.$$

The pair (X, d) will be called generalized metric space. To establish the existence and uniqueness of the solution for (1), we will use the following generalization of the Banach's fixed point theorem :

Theorem 1. *(Perov, see* [11], [5], [1]*)Let* (X, d) *a complete generalized metric space such that* $d(x, y) \in \mathbb{R}^n$. *Suppose that* $A : X \to X$ *is a map for which exists a matrix* $Q \in \mathcal{M}_n(\mathbb{R})$ *such that:*

$$d(A(x), A(y)) \leq Q \cdot d(x, y), \forall x, y \in X.$$

If all the eigenvalues of Q *lies in the open unit disc of* $\mathbb{R}^2$ *(that is the operator* A *became a* Q*-contraction), then* A *has an unique fixed point* x^* *and the sequence of successive approximations,* $x_m = A^m(x_0)$, *converges to* x^* *for any* $x_0 \in X$. *Moreover, for any* $m \in \mathbb{N}^*$ *the following estimation holds*

$$d(x_m, x^*) \leq Q^m (I_n - Q)^{-1} d(x_0, x_1). \tag{3}$$

A variant on normed spaces (with generalized complete norm of Tchebychev's type) of this theorem was used in [1], to obtain the existence and uniqueness of the boundary value problem :

$$\begin{cases} y''(x) = f(x, y(x), y'(x)), & x \in [a, b] \\ y(a) = 0, \quad y(b) = 0. \end{cases}$$

2. Main results

Consider the product functional space $X = C[-\tau, b] \times C[-\tau, b]$, where

$$C[-\tau, b] = \{f : [-\tau, b] \longrightarrow \mathbb{R} : f \text{ continuous }\}$$

On this space we define the generalized metric

$$d_B : X \times X \longrightarrow \mathbb{R}^2,$$

by

$$d_B((x_1, y_1), (x_2, y_2)) = (\|x_1 - x_2\|_B, \|y_1 - y_2\|_B), \tag{4}$$

where the Bielecki's type norm on $C[-\tau, b]$ is,

$$\|u\|_B = \max\{|u(t)| \cdot e^{-\theta(t+\tau)} : t \in [-\tau, b]\}, \quad \forall u \in C[-\tau, b], \tag{5}$$

where $\theta > 0$ is convenable chosen.

We observe that (X, d_B) is complete generalized metric space (see [5], [10]). Consider the following initial value problem

$$\begin{cases} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \varphi(0) + \int_0^t f(s, x(s), y(s-\tau))ds + \\ + \int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta, s, x(s), y(s))ds \right) d\eta \ , \\ f(t, x(t), y(t-\tau)) + \int_{t-\tau}^{t} g(t, s, x(s), y(s))ds \end{pmatrix}, \quad t \in [0, b] \\ (x(t), y(t)) = (\varphi(t), \varphi'(t)), \qquad t \in [-\tau, 0]. \end{cases} \tag{6}$$

We define the map $A : X \longrightarrow X$ by, $A = (A_1, A_2)$ with

$$(A_1(x(t), y(t)), A_2(x(t), y(t))) = (\varphi(t), \varphi'(t)), \qquad \forall t \in [-\tau, 0] \tag{7}$$

and

$$A_1(x(t), y(t)) = \varphi(0) + \int_0^t f(s, x(s), y(s-\tau))ds$$

$$+ \int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta, s, x(s), y(s))ds \right) d\eta \tag{8}$$

$$A_2(x(t), y(t)) = f(t, x(t), y(t-\tau)) + \int_{t-\tau}^{t} g(t, s, x(s), y(s))ds, \qquad \forall t \in [0, b]. \tag{9}$$

We will impose the following conditions :
(CC) (continuity conditions) :

$$f \in C([0, b] \times \mathbb{R} \times \mathbb{R}), \quad g \in C([0, b] \times [-\tau, b] \times \mathbb{R} \times \mathbb{R}), \quad \varphi \in C^1[-\tau, 0]$$

(BC) (boundedness condition) : $\exists M, K > 0$ such that

$$|f(t, u, v)| \leq M, \quad \forall (t, u, v) \in [0, b] \times \mathbb{R} \times \mathbb{R}$$

and

$$|g(t, s, u, v)| \leq K, \quad \forall (t, s, u, v) \in [0, b] \times [-\tau, b] \times \mathbb{R} \times \mathbb{R}.$$

(CPC) (compatibility condition) :

$$\varphi'(0) = f(0, \varphi(0), \varphi'(-\tau)) + \int_{-\tau}^{0} g(0, s, \varphi(s), \varphi'(s))ds \tag{10}$$

(LC) (Lipschitz conditions) : $\exists \alpha, \beta > 0$ and $\exists L_1, L_2 > 0$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \alpha |u_1 - u_2| + \beta |v_1 - v_2|, \\ , \forall t \in [0, b], \quad \forall u_1, u_2, v_1, v_2 \in \mathbb{R}$$

and

$$|g(t, s, u_1, v_1) - g(t, s, u_2, v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|, \\ , \forall (t, s) \in [0, b] \times [-\tau, b], \quad \forall u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

Theorem 2. *In the conditions (CC), (CPC) and (LC), the initial value problem (6) have in $C[-\tau, b] \times C[-\tau, b]$ an unique solution (x^*, y^*) such that $x^* \in C^1[-\tau, b]$ and $(x^*)' = y^*$.*

Proof. For $t \in [-\tau, 0]$ we have

$$|A_1(x_1, y_1)(t) - A_1(x_2, y_2)(t)| = 0 \tag{11}$$

and

$$|A_2(x_1, y_1)(t) - A_2(x_2, y_2)(t)| = 0. \tag{12}$$

For $t \in [0, b]$ we obtain,

$$|A_1(x_1, y_1)(t) - A_1(x_2, y_2)(t)| \leq$$

$$\leq \int_0^t |f(s, x_1(s), y_1(s-\tau) - f(s, x_2(s), y_2(s-\tau))| ds +$$

$$+ \int_0^t \left(\int_{\eta-\tau}^{\eta} |g(\eta, s, x_1(s), y_1(s)) - g(\eta, s, x_2(s), y_2(s))| ds \right) d\eta \leq$$

$$\leq \int_0^t [\alpha |x_1(s) - x_2(s)| \cdot e^{-\theta(s+\tau)} \cdot e^{\theta(s+\tau)} + \beta |y_1(s-\tau) - y_2(s-\tau)| \cdot e^{-\theta s} \cdot e^{\theta s} \cdot$$

$$\cdot e^{\theta\tau} \cdot e^{-\theta\tau}]ds + \int_0^t (\int_{\eta-\tau}^{\eta} [L_1 |x_1(s) - x_2(s)| \cdot e^{-\theta(s+\tau)} \cdot e^{\theta(s+\tau)} +$$

$$+ L_2 |y_1(s) - y_2(s)| \cdot e^{-\theta(s+\tau)} \cdot e^{\theta(s+\tau)}]ds)d\eta \leq$$

$$\leq \int_0^t [\alpha \|x_1 - x_2\|_B \cdot e^{\theta(s+\tau)} + \beta \|y_1 - y_2\|_B \cdot e^{\theta(s+\tau)} \cdot e^{-\theta\tau}]ds+$$

$$+\int_0^t \left(\int_{\eta-\tau}^{\eta} [L_1 \|x_1 - x_2\|_B \cdot e^{\theta(s+\tau)} + L_2 \|y_1 - y_2\|_B \cdot e^{\theta(s+\tau)}]ds \right) \leq$$

$$\leq \left(\frac{\alpha}{\theta} \|x_1 - x_2\|_B + \frac{\beta}{\theta} e^{-\theta\tau} \cdot \|y_1 - y_2\|_B \right) \int_0^t \theta e^{\theta(s+\tau)} ds+$$

$$+\left(\frac{L_1}{\theta} \|x_1 - x_2\|_B + \frac{L_2}{\theta} \|y_1 - y_2\|_B \right) \int_0^t \left(\int_{\eta-\tau}^{\eta} \theta e^{\theta(s+\tau)}]ds \right) d\eta \leq$$

$$\leq \left(\frac{\alpha}{\theta} \|x_1 - x_2\|_B + \frac{\beta}{\theta} e^{-\theta\tau} \cdot \|y_1 - y_2\|_B \right) \cdot e^{\theta(t+\tau)}+$$

$$+\left(\frac{L_1}{\theta^2} \|x_1 - x_2\|_B + \frac{L_2}{\theta^2} \|y_1 - y_2\|_B \right) \cdot \left(e^{\theta(t+\tau)} - e^{\theta\tau} - \left(e^{\theta t} - 1 \right) \right) \leq$$

$$\leq \left[\left(\frac{\alpha}{\theta} + \frac{L_1}{\theta^2} \right) \|x_1 - x_2\|_B + \left(\frac{\beta}{\theta} \cdot e^{-\theta\tau} + \frac{L_2}{\theta^2} \right) \|y_1 - y_2\|_B \right] \cdot e^{\theta(t+\tau)},\ \forall t \in [0, b],$$

which lead to

$$\|A_1(x_1, y_1) - A_1(x_2, y_2)\|_B \leq$$

$$\leq \left(\frac{\alpha}{\theta} + \frac{L_1}{\theta^2} \right) \|x_1 - x_2\|_B + \left(\frac{\beta}{\theta} \cdot e^{-\theta\tau} + \frac{L_2}{\theta^2} \right) \|y_1 - y_2\|_B . \tag{13}$$

On the other hand, for $t \in [0, b]$, we have,

$$|A_2(x_1, y_1)(t) - A_2(x_2, y_2)(t)| \leq$$

$$\leq |f(t, x_1(t), y_1(t - \tau)) - f(t, x_2(t), y_2(t - \tau))| +$$

$$+\int_{t-\tau}^{t} |g(t, s, x_1(s), y_1(s)) - g(t, s, x_2(s), y_2(s))|\, ds \leq$$

$$\leq \alpha |x_1(t) - x_2(t)| \cdot e^{-\theta(t+\tau)} \cdot e^{\theta(t+\tau)} + \beta |y_1(t - \tau) - y_2(t - \tau)| \cdot e^{-\theta t} \cdot e^{\theta t} \cdot e^{\theta\tau} \cdot e^{-\theta\tau} +$$

$$+\int_{t-\tau}^{t} [L_1 |x_1(s) - x_2(s)| \cdot e^{-\theta(s+\tau)} \cdot e^{\theta(s+\tau)} + L_2 |y_1(s) - y_2(s)| \cdot e^{-\theta(s+\tau)} \cdot e^{\theta(s+\tau)}]ds$$

$$\leq \alpha \|x_1 - x_2\|_B \cdot e^{\theta(t+\tau)} + \beta \|y_1 - y_2\|_B \cdot e^{\theta(t+\tau)} \cdot e^{-\theta\tau}+$$

$$+\left(\frac{L_1}{\theta}\left\|x_1-x_2\right\|_B+\frac{L_2}{\theta}\left\|y_1-y_2\right\|_B\right)\int\limits_{t-\tau}^{t}\theta e^{\theta(s+\tau)}ds\le$$

$$\le\left(\alpha\left\|x_1-x_2\right\|_B+\beta e^{-\theta\tau}\left\|y_1-y_2\right\|_B\right)e^{\theta(t+\tau)}$$

$$+\left(\frac{L_1}{\theta}\left\|x_1-x_2\right\|_B+\frac{L_2}{\theta}\left\|y_1-y_2\right\|_B\right)[e^{\theta(t+\tau)}-e^{\theta t}]$$

$$<\left[\left(\alpha+\frac{L_1}{\theta}\right)\left\|x_1-x_2\right\|_B+\left(\beta\cdot e^{-\theta\tau}+\frac{L_2}{\theta}\right)\left\|y_1-y_2\right\|_B\right]\cdot e^{\theta(t+\tau)}.$$

Then,

$$\left\|A_2(x_1,y_1)-A_2(x_2,y_2)\right\|_B\le$$

$$\le\left(\alpha+\frac{L_1}{\theta}\right)\left\|x_1-x_2\right\|_B+\left(\beta\cdot e^{-\theta\tau}+\frac{L_2}{\theta}\right)\left\|y_1-y_2\right\|_B. \qquad (14)$$

From (11), (12), (13) and (14) we obtain for any $(x_1,y_1),(x_2,y_2)\in X$, the inequality,

$$d_B(A(x_1,y_1),A(x_2,y_2))\le\begin{pmatrix}\frac{\alpha}{\theta}+\frac{L_1}{\theta^2} & \frac{\beta}{\theta}\cdot e^{-\theta\tau}+\frac{L_2}{\theta^2}\\ \alpha+\frac{L_1}{\theta} & \beta\cdot e^{-\theta\tau}+\frac{L_2}{\theta}\end{pmatrix}\cdot d_B((x_1,y_1),(x_2,y_2)). \qquad (15)$$

The eigenvalues of the matrix

$$Q=\begin{pmatrix}\frac{\alpha}{\theta}+\frac{L_1}{\theta^2} & \frac{\beta}{\theta}\cdot e^{-\theta\tau}+\frac{L_2}{\theta^2}\\ \alpha+\frac{L_1}{\theta} & \beta\cdot e^{-\theta\tau}+\frac{L_2}{\theta}\end{pmatrix}$$

are $\lambda_1=0$ and

$$\lambda_2=\frac{\alpha}{\theta}+\frac{L_1}{\theta^2}+\beta\cdot e^{-\theta\tau}+\frac{L_2}{\theta}>0.$$

We have,

$$0<\lambda_2<1\Longleftrightarrow h(\theta)=\theta^2-(\alpha+L_2)\theta-L_1>\theta^2\beta\cdot e^{-\theta\tau}.$$

The equation $\theta^2-(\alpha+L_2)\theta-L_1=0$ have the roots $\theta_1<0$ and $\theta_2>0$, and the the peak $V(\frac{\alpha+L_2}{2},-\frac{\Delta}{4})$, where

$$\Delta=(\alpha+L_2)^2+4L_1.$$

If we represent geometric the graphs of the functions $h(\theta)$ and $u(\theta)=\theta^2\beta\cdot e^{-\theta\tau}$, then we see that there exists an unique point $\theta^*>\theta_2$ such that $h(\theta^*)=u(\theta^*)$ and $h(\theta)>\theta^2\beta\cdot e^{-\theta\tau},\quad\forall\theta>\theta^*$ (on the other hand, this fact follows from the properties :

$$h(\theta)<0,\forall\theta\in[0,\theta_2),\quad u(\theta)>0,\forall\theta>0,$$

$$\lim_{\theta\to\infty} h(\theta)=\infty, \quad \lim_{\theta\to\infty} \theta^2\beta\cdot e^{-\theta\tau}=0$$

and because the function $u(\theta)=\theta^2\beta\cdot e^{-\theta\tau}$ have in $\theta=0$ global minimum and in $\theta=\frac{2}{\tau}$ a local maximum). If we choose a value $\theta>\theta^*$, then the operator $A=(A_1,A_2)$ given by (7), (8), (9) is Q-contraction, and has an unique fixed point $(x^*,y^*)\in X$, according to Theorem 1. The pair (x^*,y^*) will be the unique solution of the initial value problem (6), hence for any $t\in[0,b]$ and any $\eta\in[0,b]$,

$$x^*(t)=\varphi(0)+\int_0^t f(s,x^*(s),y^*(s-\tau))ds+\int_0^t\left(\int_{\eta-\tau}^{\eta} g(\eta,s,x^*(s),y^*(s))ds\right)d\eta \tag{16}$$

and

$$y^*(t)=f(t,x^*(t),y^*(t-\tau))+\int_{t-\tau}^{t} g(t,s,x^*(s),y^*(s))ds, \qquad \forall t\in[0,b]. \tag{17}$$

Using the continuity conditions (CC) and the compatibility condition (CPC), since $x^*,y^*\in C[-\tau,b]$ and $x^*(t)=\varphi(t), \quad \forall t\in[-\tau,0]$, we infer that $x^*\in C^1[-\tau,b]$. If we derive by t the equality (16), we obtain,

$$(x^*)'(t)=f(t,x^*(t),y^*(t-\tau))+\int_{t-\tau}^{t} g(t,s,x^*(s),y^*(s))ds, \qquad \forall t\in[0,b]$$

and together the equality (17) follows that $(x^*)'=y^*$.
Now, at the final of the proof, let see how can obtain the point θ^*.
We have

$$h(\theta)=\theta^2\beta\cdot e^{-\theta\tau}\Longleftrightarrow \theta=H(\theta)=\alpha+L_2+\theta\beta\cdot e^{-\theta\tau}+\frac{L_1}{\theta}$$

that is θ^* a fixed point of H. Moreover,

$$H'(\theta)<0\Longleftrightarrow -\frac{L_1}{\theta^2}+\beta\cdot e^{-\theta\tau}(1-\theta\tau)<0.$$

If $\theta\ge\frac{1}{\tau}$ then $H'(\theta)<0$ and $H'(\frac{1}{\tau})=-\tau^2L_1<0$. So, $H'(\theta)<0 \ \ \forall\theta\ge\frac{1}{\tau}$.
If $\frac{1}{\tau}<\theta_2$ then we can take $\overline{\theta}=H(\theta_2)>\theta^*$ and for any $\theta>\overline{\theta}$ we have $0<\lambda_2<1$.
If $\frac{1}{\tau}>\theta_2$ then we have two possibilities :
1) If $h(\frac{1}{\tau})<\frac{1}{\tau^2}\cdot\beta e^{-1}$ then we take $\overline{\theta}=H(\frac{1}{\tau})>\theta^*$ and for any $\theta>\overline{\theta}$ we have

$0 < \lambda_2 < 1$.
2) If $h(\frac{1}{\tau}) > \frac{1}{\tau^2} \cdot \beta e^{-1}$ then it is clear that $\frac{1}{\tau} > \theta^*$ and for any $\theta > \frac{1}{\tau}$ we will have $0 < \lambda_2 < 1$.
Consequently, in the conditions of the enunciation we can choose $\overline{\theta}$ (which can be $H(\theta_2)$, or $H(\frac{1}{\tau})$, or $\frac{1}{\tau}$) such that $0 < \lambda_2 < 1, \quad \forall \theta > \overline{\theta}$. □

Corollary 3. *In the conditions (CC), (CPC) and (LC), the initial value problem (2) have in $C[-\tau, b]$ an unique solution x^*, such that the pair $(x^*, (x^*)')$ is approximated by the sequence of successive approximations $((x_m, y_m))_{m\in\mathbb{N}}$, where*

$$(x_0(t)), y_0(t)) = \begin{cases} (\varphi(t), \varphi'(t)), & t \in [-\tau, 0] \\ (\varphi(0), \varphi'(0)), & t \in [0, b] \end{cases} \tag{18}$$

and

$$(x_m, y_m) = A(x_{m-1}, y_{m-1}), \quad \forall m \in \mathbb{N}^*, \tag{19}$$

with the following error estimation :

$$d_B\left(\begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} x^* \\ (x^*)' \end{pmatrix}\right) \leq \frac{\lambda_2^{m-1}}{1-\lambda_2} \cdot \begin{pmatrix} \frac{\alpha}{\theta} + \frac{L_1}{\theta^2} & \frac{\beta}{\theta} \cdot e^{-\theta\tau} + \frac{L_2}{\theta^2} \\ \alpha + \frac{L_1}{\theta} & \beta \cdot e^{-\theta\tau} + \frac{L_2}{\theta} \end{pmatrix} \cdot$$

$$\cdot d_B\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right), \qquad \forall m \in \mathbb{N}^*. \tag{20}$$

Proof. From Theorem 2 and Theorem 1 (inequality (3)), we obtain for any $m \in \mathbb{N}^*$, the estimation :

$$d_B\left(\begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} x^* \\ (x^*)' \end{pmatrix}\right) \leq Q^m (I_2 - Q)^{-1} \cdot d_B\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right). \tag{21}$$

After elementary calculus we find

$$\det(I_2 - Q) = 1 - \beta \cdot e^{-\theta\tau} - \frac{\alpha}{\theta} - \frac{L_2}{\theta} - \frac{L_1}{\theta^2} = 1 - \lambda_2$$

$$Q^m = \left(\beta \cdot e^{-\theta\tau} + \frac{\alpha}{\theta} + \frac{L_2}{\theta} + \frac{L_1}{\theta^2}\right)^{m-1} \cdot Q = \lambda_2^{m-1} \cdot Q, \qquad \forall m \in \mathbb{N}^*$$

and

$$(I_2 - Q)^{-1} = \frac{1}{1-\lambda_2} \cdot \begin{pmatrix} 1 - \beta \cdot e^{-\theta\tau} - \frac{L_2}{\theta} & \frac{\beta}{\theta} \cdot e^{-\theta\tau} + \frac{L_2}{\theta^2} \\ \alpha + \frac{L_1}{\theta} & 1 - \frac{\alpha}{\theta} - \frac{L_1}{\theta^2} \end{pmatrix}.$$

Hence,

$$Q^m (I_2 - Q)^{-1} = \frac{\lambda_2^{m-1}}{1-\lambda_2} \cdot Q, \qquad \forall m \in \mathbb{N}^*,$$

from the inequality (21) we infer that the estimation (20) holds. Using the Theorem 1 we obtain the following uniform convergence on (X, d_B) :

$$(x_m, y_m) \rightrightarrows (x^*, (x^*)'), \qquad \text{for} \quad m \to \infty.$$

□

Remark 4. *(i) In the conditions (CC)-(LC), the initial value problem (2) have an unique bounded solution in* $C[-\tau, b]$. *Indeed,*

$$|x^*(t)| \le |\varphi(0)| + \int_0^t \left| f(s, x^*(s), (x^*)'(s-\tau)) \right| ds$$

$$+ \int_{t-\tau}^t \left| g(t, s, x^*(s), (x^*)'(s)) \right| ds$$

$$\le |\varphi(0)| + Mb + K\tau, \quad \forall t \in [0, b]$$

and $x^*(t) = \varphi(t), \quad \forall t \in [-\tau, 0]$.
(ii) Also, we obtain a positive solution if we consider the above conditions and the conditions $\varphi(t) > 0, \quad \forall t \in [-\tau, 0]$,

$$f(t, u, v) > 0, \quad \forall (t, u, v) \in [0, b] \times \mathbb{R} \times \mathbb{R}$$

and

$$g(t, s, u, v) > 0, \quad \forall (t, s, u, v) \in [0, b] \times [-\tau, b] \times \mathbb{R} \times \mathbb{R}.$$

In this case we obtain $0 < x^*(t) \le \varphi(0) + Mb + K\tau, \quad \forall t \in [0, b]$.

3. Uniform Lipschitz properties

Definition 5. *Let* $I \subset \mathbb{R}$, *interval and* $F(I, \mathbb{R})$ *the set of all functions* $f : I \to \mathbb{R}$. *A subset* $Y \subset F(I, \mathbb{R})$ *is uniform Lipschitz on* I *if there exists* $\dot{L} > 0$ *such that* $\forall f \in Y$ *we have :*

$$|f(u) - f(v)| \le L |u - v|, \qquad \forall u, v \in I.$$

Theorem 6. *In the conditions (CC)-(LC), if* φ' *is Lipschitzian on* $[-\tau, 0]$, $0 < \beta < 1$ *and if there exists* $\gamma > 0$ *such that*

$$|f(t_1, u, v) - f(t_2, u, v)| \le \gamma |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, b], \quad \forall u, v \in \mathbb{R},$$

then the sequence of successive approximations $(y_m)_{m\in\mathbb{N}}$, *given by (19) and (18), is uniform Lipschitz on* $[0, b]$ *and the derivative of the solution of the initial value problem (2) is Lipschitzian on* $[-\tau, b]$.

Proof. Consider the functions $F_m : [0, b] \to \mathbb{R}$, given by

$$F_m(t) = f(t, x_m(t), y_m(t - \tau)), \qquad t \in [0, b].$$

Let $\gamma' > 0$ the Lipschitz constant of φ' and $\delta > 0$ such that $|\varphi'(t)| \le \delta, \quad \forall t \in [-\tau, 0]$.

Since,

$$x_m(t) = \varphi(0) + \int_0^t f(s, x_{m-1}(s), y_{m-1}(s - \tau))ds +$$
$$+ \int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta, s, x_{m-1}(s), y_{m-1}(s))ds \right) d\eta$$

and

$$y_m(t) = f(t, x_{m-1}(t), y_{m-1}(t - \tau)) + \int_{t-\tau}^{t} g(t, s, x_{m-1}(s), y_{m-1}(s))ds, \ \forall t \in [0, b]$$

we have, for any $m \in \mathbb{N}^*$,

$$|x_m(t_1) - x_m(t_2)| \le \int_{t_1}^{t_2} |f(s, x_{m-1}(s), y_{m-1}(s - \tau))| \, ds +$$
$$+ \int_{t_1}^{t_2} \left(\int_{\eta-\tau}^{\eta} |g(\eta, s, x_{m-1}(s), y_{m-1}(s))| \, ds \right) d\eta$$
$$\le (M + \tau K) \cdot |t_1 - t_2|, \ \forall t_1, t_2 \in [0, b].$$

On the other hand,

$$|F_0(t_1) - F_0(t_2)| \le \gamma |t_1 - t_2| + \alpha |x_0(t_1) - x_0(t_2)| + \beta |y_0(t_1) - y_0(t_2)| \le$$
$$\le (\gamma + \alpha\delta + \beta\gamma') \cdot |t_1 - t_2| = L_0 \cdot |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, b]$$

and

$$|F_1(t_1) - F_1(t_2)| \leq \gamma |t_1 - t_2| + \alpha |x_1(t_1) - x_1(t_2)| + \beta |y_1(t_1) - y_1(t_2)| .$$

with

$$|y_1(t_1) - y_1(t_2)| \leq |f(t_1, x_0(t_1), y_0(t_1 - \tau)) - f(t_2, x_0(t_2), y_0(t_2 - \tau))| +$$
$$+K \cdot |t_1 - t_2| \leq (K + L_0) \cdot |t_1 - t_2| , \quad \forall t_1, t_2 \in [0, b].$$

Then,

$$|F_1(t_1) - F_1(t_2)| \leq [\gamma + \alpha(M + \tau K) + \beta(K + L_0)] \cdot |t_1 - t_2| , \quad \forall t_1, t_2 \in [0, b].$$

By induction we infer that,

$$|F_m(t_1) - F_m(t_2)| \leq [\gamma + \alpha(M + \tau K) + \beta(K + L_{m-1})] \cdot |t_1 - t_2| =$$
$$= L_m \cdot |t_1 - t_2| , \quad \forall t_1, t_2 \in [0, b].$$

We see that,

$$L_1 = \gamma + \alpha(M + \tau K) + \beta(K + L_0)$$
$$L_2 = \gamma + \alpha(M + \tau K) + \beta(K + L_1) = \gamma + \alpha(M + \tau K) + \beta K+$$
$$+\beta[\gamma + \alpha(M + \tau K) + \beta K] + \beta^2 L_0 =$$
$$= [\gamma + \alpha(M + \tau K) + \beta K](1 + \beta) + \beta^2 L_0$$
$$L_3 = [\gamma + \alpha(M + \tau K) + \beta K](1 + \beta + \beta^2) + \beta^3 L_0$$

and

$$L_m = [\gamma + \alpha(M + \tau K) + \beta K](1 + \beta + ... + \beta^{m-1}) + \beta^m L_0 = \beta^m L_0+$$
$$+[\gamma + \alpha(M + \tau K) + \beta K] \cdot \frac{1 - \beta^m}{1 - \beta} < \frac{\gamma + \alpha(M + \tau K) + \beta K}{1 - \beta} + L_0 = \overline{L}.$$

Then,

$$L_m \leq \overline{L}, \quad \forall m \in \mathbb{N}^*$$

and

$$\lim_{m \to \infty} L_m = \frac{\gamma + \alpha(M + \tau K) + \beta K}{1 - \beta}.$$

Consequently, for any $m \in \mathbb{N}^*$ and $t_1, t_2 \in [0, b]$ we have

$$|y_m(t_1) - y_m(t_2)| \leq L_{m-1} \cdot |t_1 - t_2| + K \cdot |t_1 - t_2| \leq (\overline{L} + K) \cdot |t_1 - t_2| \quad (22)$$

and then,

$$-(\overline{L} + K) \cdot |t_1 - t_2| \leq y_m(t_1) - y_m(t_2) \leq (\overline{L} + K) \cdot |t_1 - t_2| , \quad \forall m \in \mathbb{N}^*. \quad (23)$$

Since, according to Theorem 2, we have

$$\lim_{m\to\infty} y_m(t) = y^*(t) = (x^*)'(t), \qquad \forall t \in [0, b],$$

from inequality (23) we infer that

$$-(\overline{L} + K) \cdot |t_1 - t_2| \leq (x^*)'(t_1) - (x^*)'(t_2) \leq (\overline{L} + K) \cdot |t_1 - t_2|$$

and so,

$$\left|(x^*)'(t_1) - (x^*)'(t_2)\right| \leq (\max(\gamma', \overline{L} + K)) \cdot |t_1 - t_2|, \quad \forall t_1, t_2 \in [-\tau, b]. \quad (24)$$

With the inequalities (22) and (24) the proof is complete. □

Remark 7. *From the proof of the above theorem we see that, in the conditions of this theorem, the sequence of successive kernels* $(F_m)_{m\in\mathbb{N}}$, *is uniform Lipschitz on* $[0, b]$.

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