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POSITIVE SOLUTIONS FOR ϕ-LAPLACIAN DIRICHLET BVPS

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Abstract. In this paper, we present new existence results of nontrivial positive solutions for ϕ-Laplacian Dirichlet boundary value problems on bounded intervals of the real line. The nonlinear terms encompasses the sub-linear and super-linear cases. The Krasnosel'skii's fixed point theorem on cone expansion and compression is used. Applications to $p-$Laplacian problems and to the case of the sum of $p-$Laplacian and $q-$Laplacian ($p \neq q$) operators are given.
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1. Introduction

In this paper, we are concerned with the existence of positive solutions to the boundary value problem:

$$\begin{cases} -(\phi(u'))'(x) &= f(x,u), \quad 0<x<1 \\ u(0)=u(1) &= 0 \end{cases} \tag{1.1}$$

where $f\colon [0,1]\times\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function, $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}$ is an odd and increasing homeomorphism, extending the usual $p-$Laplacian nonlinear operator. Throughout this paper, we set $\psi\colon = \phi^{-1}$, Our aim is to prove some existence results of nontrivial positive solutions for Problem (1.1) under suitable conditions on the functions f and ϕ. The existence of solutions with arbitrary sign is well studied in the literature (see [1], [6], [12] and the references therein). In this work, our approach is based on the application of Krasnosel'skii's fixed point theorem of cone compression and expansion in Banach spaces in order to get existence of positive solutions. By a solution to Problem (1.1), we mean a function $u \in C^1([0,1], \mathbb{R})$ such that $\phi(u')$ is also of C^1-class. In the sequel, $\mathbb{R}^+$ refers to the set of nonnegative real numbers. The notation $:=$ means throughout to be defined equal to.

Under various assumptions, the boundary value problem (1.1) is widely investigated in the literature. For variable-separated nonlinearity $f(x,u) = a(x)g(u)$, the existence of positive solutions either in the sub-linear case $g_0 = +\infty$ and $g_\infty = 0$ or in the super-linear case $g_0 = 0$ and $g_\infty = \infty$ is proved in [13] in case of the one-dimensional $p-$Laplacian operator $\phi(u) = |u|^{p-2}u$ where $p > 1$ is a real number. Here,

$$g_0 := \lim_{s \downarrow 0} \frac{g(s)}{s^{p-1}} \quad \text{and} \quad g_\infty := \lim_{s \uparrow +\infty} \frac{g(s)}{s^{p-1}}.$$

The results generalize previous ones obtained in [5].

The eigenvalue problem

$$\begin{cases} (\phi(u'))' + \lambda a(x)g(u) &= 0, \quad 0 < x < 1 \\ u(0) = u(1) &= 0, \end{cases} \tag{1.2}$$

where $a\colon [0,1] \to [0,\infty)$ and $g\colon [0,\infty) \to [0,\infty)$ are continuous and positive, has attracted a particular attention in the last couple of years. The existence of positive solutions is often discussed in terms of the eigenvalue λ. For second-order boundary value problems corresponding to $p = 2$, that is $\phi(s) = s$, existence of positive solutions is proved in [9] for any eigenvalue λ satisfying the bounds

$$\frac{4}{g_\infty \left(\int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,s)a(s)\,ds \right)} < \lambda < \frac{1}{g_0 \left(\int_0^1 s(1-s)a(s)\,ds \right)}.$$

Here G is the Green function for the problem $-u'' = u(0) = u(1) = 0$ and $\tau \in [0,1]$ is defined by

$$\int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,s)a(s)\,ds = \max_{0 \le x \le 1} \int_{\frac{1}{4}}^{\frac{3}{4}} G(x,s)a(s)\,ds.$$

A slight generalization regarding the nonlinear term $\lambda f(t,u)$ is provided in [3] for the $\phi-$Laplacian problem with an odd and increasing homeomorphism

ϕ satisfying the so-called lower $\sigma-$condition (see also [6])

$$\forall\, \sigma > 0, \quad \limsup_{s\to+\infty} \frac{\phi(\sigma s)}{\phi(s)} < \infty$$

while the nonlinearity f is assumed to satisfy

$$\forall\, t \in [c,d] \subset (a,b), \quad \lim_{s\to\infty} \frac{f(t,s)}{\phi(s)} = +\infty.$$

All of these works use the Krasnosel'skii's fixed point theorem apart from De Coster's [2] where the upper and lower-solutions method is used to get existence of positive solutions. More recently, the topological degree of Leray and Schauder has been employed in [7] to study Problem (1.1) in the particular case $\phi(u) = \omega u$, $f(x,u) = a(x)h(x,u)$ and general Sturm-Liouville boundary conditions on the interval $[0,1]$. Existence of positive solution is obtained for any eigenvalue such that

$$\lambda < \frac{r}{A \displaystyle\max_{0\le x\le 1,\, 0\le u\le r} h(x,u)}$$

for some positive r and $A = \max\limits_{0\le x\le 1} \int_0^1 \mathcal{G}(x,s)a(s)\,ds$ where $\mathcal{G}$ is the Green function associated with the problem $-(\omega u')' = 0$ with general boundary conditions.

The purpose of this paper is to complement and extend some of these results to the $\phi-$Laplacian case. The organization is as follows. After giving some preliminaries in Section 2, we prove three existence results of positive solutions in Section 3, with the use of fixed point theory. Distinct growth assumptions are considered: nonlinearity with local growth, sum of monotonic nonlinearities, sub-linear and super-linear nonlinearities. We illustrate the applicability of the obtained existence theorems in Section 4 where three examples of application to $p-$Laplacian problems and to the case where ϕ is the sum of $p-$Laplacian and $q-$Laplacian operators ($p \neq q$) are given. We end the paper with some comments in Section 5.

2. Auxiliary lemmas

Denote by $E\colon = C([0,1], \mathbb{R}^+)$ the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}^+$ with the norm $\|u\| = \sup\{|u(x)|,\ 0 \le x \le 1\}$ and recall that $L^1([0,1])$ is the Lebesgue space of integrable functions on $[0,1]$. The norm in this space is denoted by

$$|u|_1 = \int_0^1 |u(t)|\,dt.$$

In order to transform Problem (1.1) into a fixed point problem, we need some background material and preliminary results which are collected in this section:

Lemma 2.1. *For any function $u \in L^1(0,1)$ positive almost everywhere, the problem of seeking $0 < x < 1$ such that*

$$\int_0^x \psi\left(\int_s^x u(\tau)\,d\tau\right) ds = \int_x^1 \psi\left(\int_x^s u(\tau)\,d\tau\right) ds \tag{2.1}$$

has uniquely one solution θ such that $0 < \theta < 1$.

Proof. Consider the continuous functions

$$\alpha(x) = \int_0^x \psi\left(\int_s^x u(\tau)\,d\tau\right) ds \ \text{ and } \beta(x) = \int_x^1 \psi\left(\int_x^s u(\tau)\,d\tau\right) ds.$$

Then $\alpha(0) = \beta(1) = 0$ and the function α (respectively the function β) is increasing (respectively decreasing); whence comes the result.

Lemma 2.2. *Consider the following boundary value problem*

$$\begin{cases} -(\phi(v'))' &= u(x), \quad 0 < x < 1 \\ v(0) = v(1) &= 0 \end{cases} \tag{2.2}$$

where $u \in L^1(0,1)$ is positive almost everywhere. Then, Problem (2.2) has a unique solution given by

$$v(x) = \begin{cases} \int_0^x \psi\left(\int_s^\theta u(\tau)\,d\tau\right) ds & \text{if} \quad 0 \le x \le \theta \\ \int_x^1 \psi\left(\int_\theta^s u(\tau)\,d\tau\right) ds & \text{if} \quad \theta \le x \le 1 \end{cases} \tag{2.3}$$

where θ is as given in Lemma 2.1.

Proof. Let $\theta \in]0,1[$ be such that $v'(\theta) = 0$. Integrating the equation in (2.2) between x and θ, we get, since $\phi(0) = 0$:

$$\phi(v'(x)) = \int_x^\theta u(t)\,dt,$$

whence

$$v'(x) = \psi\left(\int_x^\theta u(t)\,dt\right).$$

Integrating (2.2) from θ to x, we find, since ψ is odd

$$v'(x) = \psi\left(-\int_\theta^x u(t)\,dt\right) = -\psi\left(\int_\theta^x u(t)\,dt\right).$$

Integrating again $v'(x)$ over $(0,x)$ and $(x,1)$ respectively yield the expression of the function v. Conversely, it is clear that v defined by (2.3) is solution of Problem (2.2). Moreover, u is positive implies that $\phi(v')$ is nonincreasing and so $\phi(v') \geq 0$ for $x \leq \theta$ and $\phi(v') \leq 0$ for $x \geq \theta$. From the properties of the homeomorphism ϕ, we deduce that v changes monotonicity at the point θ, and that θ satisfies (2.1).

Lemma 2.3. *Let θ be as defined in Lemma 2.1. Then, the mapping*

$$A\colon\ L^1((0,1),\mathbb{R}^+) \longrightarrow C([0,1],\mathbb{R}^+)$$

defined by:

$$Av(x) = \begin{cases} \int_0^x \psi\left(\int_s^\theta v(\tau)d\tau\right) ds & \text{if} \quad 0 \leq x \leq \theta \\ \int_x^1 \psi\left(\int_\theta^s v(\tau)d\tau\right) ds & \text{if} \quad \theta \leq x \leq 1 \end{cases} \tag{2.4}$$

is completely continuous.

Proof. **(a)** A is continuous. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence converging to some limit v in $L^1(0,1)$. Then, for any $s \in (0,1)$, it holds that

$$0 \leq \lim_{n\to\infty} \int_s^\theta |v_n(\tau) - v(\tau)|\,d\tau \leq \lim_{n\to\infty}\int_0^1 |v_n(\tau) - v(\tau)|\,d\tau = 0.$$

Therefore for any $x \in (0,1)$, the integral $\int_0^x \psi\left(\int_s^\theta v_n(\tau)\,d\tau\right) ds$ converges to the integral $\int_0^x \psi\left(\int_s^\theta v(\tau)\,d\tau\right) ds$ because ϕ is an homeomorphism. The same holds for the second term in (2.4), proving the continuity of A.

(b) A is totally bounded. Let B be a bounded subset of $L^1(0,1)$ and $M > 0$ a constant such that $|v|_1 \leq M$ for any $v \in B$. We have $\|Av\| \leq \psi(M)$, which implies the boundedness of $A(B)$. In addition, the set $\{Av,\ v \in B\}$ is equicontinuous. Indeed, if $x_1, x_2 \in (0,1)$, then we may distinguish between four cases according to the relative position of x_1, x_2 with respect to θ. For brevity, we only assume $0 \leq x_1, x_2 \leq \theta$, in which case, we have

$$\begin{aligned} |(Av)(x_1) - (Av)(x_2)| &= |\int_{x_1}^{x_2} \psi\left(\int_s^\theta v(\tau)d\tau\right) ds \\ &\leq \psi(M)|x_1 - x_2|, \end{aligned}$$

and our claim follows. The Arzéla-Ascoli theorem then implies that A is completely continuous.

Lemma 2.4. *(see also* [3]*) Let* $u \in L^1([0,1])$, $u \geq 0$ *a.e. and let* v *satisfy*

$$\begin{cases} -\left(\phi(v')\right)'(x) = u(x), & 0 < x < 1 \\ v(0) = v(1) = 0. \end{cases} \tag{2.5}$$

Then

$$v(x) \geq p(x)\|v\|_0, \quad \forall\, x \in [0,1]$$

where

$$p(x) = \min(x, 1-x),\ x \in [0,1].$$

Proof. Since ϕ is nondecreasing and $\phi(v')$ is nonincreasing, the function v' is also nonincreasing. Further, there exists some $0 < x_0 < 1$ such that $v'(x_0) = 0$. Therefore, v is positive, concave and admits a unique maximum at x_0. Its graph is then above the lines joining $v(x_0)$ to the endpoints. It follows that:

$$\begin{array}{lll} v(x) & \geq & x\frac{v(x_0)}{x_0} \geq x\ v(x_0) = x\|v\|_0, & \forall\, x \in [0, x_0], \\ v(x) & \geq & (1-x)\frac{v(x_0)}{1-x_0} \geq (1-x)v(x_0) = (1-x)\|v\|_0, & \forall\, x \in [x_0, 1]. \end{array}$$

The lemma is proved.

Next, consider the operator $T\colon\ C([0,1], \mathbb{R}^+) \longrightarrow C([0,1], \mathbb{R}^+)$ defined by

$$Tu(x) = \begin{cases} \int_0^x \psi\left(\int_s^\theta f(\tau, u(\tau))\, d\tau\right) ds & \text{if} \quad 0 \leq x \leq \theta \\ \int_x^1 \psi\left(\int_\theta^s f(\tau, u(\tau))\, d\tau\right) ds & \text{if} \quad \theta \leq x \leq 1, \end{cases} \tag{2.6}$$

where θ is as defined in Lemma 2.1 with u replaced by $f(., u(.))$. We have

Lemma 2.5. *The operator* T *is completely continuous.*

Proof. The Nemytskii operator $N\colon\ C([0,1], \mathbb{R}^+) \longrightarrow L^1(0,1)$ defined by $Nv(x) = f(x, v(x))$ is continuous by Lebesgue dominated convergence theorem. The operator $T = AN\colon\ C([0,1], \mathbb{R}^+) \longrightarrow C([0,1], \mathbb{R}^+)$ is the composition of the completely continuous mapping A introduced in Lemma 2.3 with N; whence it is completely continuous.

Lemma 2.6. *Let T and θ be as defined in (2.6) and $0 < \sigma < \frac{1}{2}$ a real number. Then the operator T verifies*

$$\|Tu\| \geq \begin{cases} \sigma\psi\left(\int_\theta^{1-\sigma} f(\tau,u(\tau))\,d\tau\right), & \text{if } \sigma \geq \theta \\ \sigma\psi\left(\int_\sigma^{\theta} f(\tau,u(\tau))\,d\tau\right), & \text{if } \sigma \geq 1-\theta \\ \frac{\sigma}{2}\psi\left(\int_\sigma^{\theta} f(\tau,u(\tau))\,d\tau\right) + \frac{\sigma}{2}\psi\left(\int_\theta^{1-\sigma} f(\tau,u(\tau))\,d\tau\right), & \\ \quad \text{if } \sigma \leq \theta \leq 1-\sigma. & \end{cases}$$

Proof. **(a)** If $\sigma > \theta$, then

$$\begin{aligned} \|Tu\| \geq Tu(1-\sigma) &= \int_{1-\sigma}^{1} \psi\left(\int_\theta^{s} f(\tau,u(\tau))\,d\tau\right) ds \\ &\geq \int_{1-\sigma}^{1} \psi\left(\int_\theta^{1-\sigma} f(\tau,u(\tau))\,d\tau\right) ds \\ &= \sigma\psi\left(\int_\theta^{1-\sigma} f(\tau,u(\tau))\,d\tau\right). \end{aligned}$$

(b) If $\sigma > 1-\theta$, then

$$\begin{aligned} \|Tu\| \geq Tu(\sigma) &= \int_0^{\sigma} \psi\left(\int_s^{\theta} f(\tau,u(\tau))\,d\tau\right) ds \\ &\geq \int_0^{\sigma} \psi\left(\int_\sigma^{\theta} f(\tau,u(\tau))\,d\tau\right) ds \\ &= \sigma\psi\left(\int_\sigma^{\theta} f(\tau,u(\tau))\,d\tau\right). \end{aligned}$$

(c) If $\theta \in [\sigma, 1-\sigma]$, then write $2\|Tu\| \geq Tu(\sigma) + Tu(1-\sigma)$ and the proof follows identically, ending the claim of the lemma.

3. Existence results

3.1. **Introduction.** In this section, we seek for positive fixed points for the mapping T introduced in (2.6) and prove three existence theorems. For this, the continuous nonlinear function f is assumed nonnegative. First, recall

Definition 3.1. *A nonempty subset K of a Banach space E is called a cone if K is convex, closed, and satisfies*
(a) *$\alpha u \in K$ for all $u \in K$ and any real positive number α,*
(b) *$u, -u \in K$ imply $u = 0$.*

The following celebrated theorem, known as Krasnosel'skii's Fixed Point Theorem in cones will be the main tool used throughout. Many works are based on this result to prove existence of positive nontrivial solutions to boundary value problems (see [5, 9, 13]).

Theorem A. ([8, 10, 11]) Let E be a Banach space, $K \subset E$ a cone and Ω_1, Ω_2 two bounded open subsets satisfying $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. Let $T\colon K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that:
• either $\|Tv\| \leq \|v\|$ for $v \in K \cap \partial\Omega_1$ and $\|Tv\| \geq \|v\|$ for $v \in K \cap \partial\Omega_2$,
• or $\|Tv\| \geq \|v\|$ for $v \in K \cap \partial\Omega_1$ and $\|Tv\| \leq \|v\|$ for $v \in K \cap \partial\Omega_2$.
Then T has at least a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Finally, introduce the positive cone

$$K = \left\{u \in C([0,1], \mathbb{R}^+); \quad u(x) \geq p(x)\|u\|, \ x \in [0,1]\right\} \tag{3.1}$$

with $p(x)\colon = \min(x, 1-x)$. We remark that, by Lemma 2.4, the mapping T maps K into itself. Indeed, Tu verifies

$$\begin{cases} -(\phi((Tu)'))'(x) &= f(x, u(x)) \geq 0, \quad 0 < x < 1 \\ Tu(0) = Tu(1) &= 0. \end{cases}$$

It is clear that fixed points of T are solutions for the boundary value problem (1.1) and conversely.

In the sequel, $0 < \sigma < \frac{1}{2}$ will be any real number. All of the existence results in the paper will involve this parameter.

3.2. **Local growth restrictions.** The main result in this subsection is:

Theorem 3.1. *Suppose that* $f\colon [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is continuous and satisfies:*
(a) $\exists\, M_1 > 0$ *such that* $f(x,u) \leq M_1$ *for* $x \in [0,1]$ *and* $0 \leq u \leq \psi(M_1) = \eta$.
(b) $\exists\, M_2 > 0$ *such that* $f(x,u) \geq M_2$ *for* $x \in [\sigma, 1-\sigma]$ *and* $\sigma\mu \leq u \leq \mu$ *with* $\mu \neq \eta$, $\mu = \frac{1}{2}\sigma D_1^*$, $D_1^*\colon = \min\limits_{\sigma \leq x \leq 1-\sigma} D_1(x)$ *and the function* D_1 *is defined by*

$$D_1(x)\colon = \psi\left[M_2(x-\sigma)\right] + \psi\left[M_2(1-\sigma-x)\right].$$

Then Problem (1.1) has a positive solution satisfying

$$\min(\mu, \eta) \leq \|u\| \leq \max(\mu, \eta).$$

Remark 3.1. *In case of the $p-$Laplacian mapping $\phi(s) = |s|^{p-2}s$ $(p > 1)$, a straightforward computation yields*

$$D_1^* = \min\left(D_1(\sigma), D_1(1/2)\right) = \begin{cases} M_2^{q-1}(1-2\sigma)^{q-1}, & \text{if } 1 < q \le 2 \\ 2^{2-q}M_2^{q-1}(1-2\sigma)^{q-1}, & \text{if } q \ge 2, \end{cases}$$

in short

$$D_1^* = M_2^{q-1}(1-2\sigma)^{q-1}2^{\min(2-q,0)}.$$

Here $q = \frac{p}{p-1}$ is the conjugate of the real number p. Also notice that $D_1(\sigma) = D_1(1-\sigma)$.

Proof. Without restriction in the proof, assume $\eta < \mu$; otherwise invert the roles played by the parameters μ and η. Consider the open sets:

$$\Omega_1 : = \left\{u \in C([0,1], \mathbb{R}^+);\ \|u\| < \eta\right\}$$

and

$$\Omega_2 : = \left\{u \in C([0,1], \mathbb{R}^+);\ \|u\| < \mu\right\}.$$

(a) Let $u \in K \cap \partial\Omega_1$, that is $u \in K$ and $\|u\| = \eta$. For any $x \in [0,1]$, we have

$$\begin{aligned} |Tu(x)| = Tu(x) &\le \psi\left(\int_0^1 f(s,u(s))\,ds\right) \\ &\le \psi(M_1) \\ &= \eta = \|u\|. \end{aligned}$$

Passing to the supremum, we get $\|Tu\| \le \|u\|$ for any $u \in K \cap \partial\Omega_1$.

(b) Let $u \in K \cap \partial\Omega_2$, that is $u \in K$ and $\|u\| = \mu$. We have that $u(x) \ge p(x)\|u\| \ge \sigma\|u\|$ for $x \in [\sigma, 1-\sigma]$ and then u verifies

$$\sigma\mu = \sigma\|u\| \le u(x) \le \|u\| = \mu,\ \forall\, x \in [\sigma, 1-\sigma]. \tag{3.2}$$

In addition, the following discussion holds true:

- If $\theta < \sigma$ or $\theta > 1-\sigma$, then, from Assumption **(b)** and Lemma 2.6

$$\begin{aligned} \|Tu\| &\ge \sigma\psi\left(\int_\sigma^{1-\sigma} f(\tau,u(\tau))\,d\tau\right) \\ &\ge \sigma\psi[(1-2\sigma)M_2] = \sigma D_1(\sigma) \\ &\ge \sigma D_1^* \ge \mu = \|u\|. \end{aligned}$$

• If $\theta \in [\sigma, 1-\sigma]$, then we get similarly the estimates

$$\begin{aligned} 2\|Tu\| &\geq \sigma\psi\left(\int_\sigma^\theta f(\tau,u(\tau))d\tau\right) + \sigma\psi\left(\int_\theta^{1-\sigma} f(\tau,u(\tau))d\tau\right) \\ &\geq \sigma\psi\left(M_2(\theta-\sigma)\right) + \sigma\psi\left(M_2(1-\sigma-\theta)\right) = \sigma D_1(\theta) \\ &\geq \sigma D_1^* \geq 2\mu = 2\|u\|. \end{aligned}$$

It follows that $\|Tu\| \geq \|u\|$ for any $u \in K \cap \partial\Omega_2$.
By Theorem A, Problem (1.1) has a positive solution such that $\eta \leq \|u\| \leq \mu$.

Remark 3.2. *We can see from the proof of Theorem 3.1 that the constant* μ *may be any constant satisfying* $0 < \mu \leq \frac{\sigma D_1^*}{2}$.

3.3. **The sum of two nonlinearities.**

Theorem 3.2. *Suppose that*

(a) *There exist* F_1, $F_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$: F_1 *is nonincreasing, strictly positive,* $\frac{F_2}{F_1}$ *is nondecreasing and*

$$\exists\, r_0 > 0 : \quad \left(1 + \frac{F_2(r_0)}{F_1(r_0)}\right)\int_0^1 F_1(r_0 p(s))\, ds \leq \phi(r_0) \tag{3.3}$$

such that $0 \leq f(x,u) \leq F_1(u) + F_2(u)$, *for any* $x \in [0,1]$ *and* $0 \leq u \leq r_0$.

(b) *There exist* G_1, $G_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$: G_1 *is nonincreasing, strictly positive,* $\frac{G_2}{G_1}$ *is nondecreasing and satisfy*

$$\exists\, R_0 > 0,\ R_0 \neq r_0 \text{ such that } \sigma D_2^* \geq 2R_0 \tag{3.4}$$

such that $f(x,u) \geq G_1(u) + G_2(u)$, *for any* $x \in [0,1]$ *and* $0 \leq u \leq R_0$. *Here*

$$\begin{aligned} D_2(x) &= \psi\left(G_1(R_0)\left(1 + \tfrac{G_2(\sigma R_0)}{G_1(\sigma R_0)}\right)(x-\sigma)\right) \\ &+ \psi\left(G_1(R_0)\left(1 + \tfrac{G_2(\sigma R_0)}{G_1(\sigma R_0)}\right)(1-\sigma-x)\right) \end{aligned}$$

and $D_2^* = \min\limits_{\sigma \leq x \leq 1-\sigma} D_2(x)$.

Then Problem (1.1) has a positive solution satisfying

$$\min(r_0, R_0) \leq \|u\| \leq \max(r_0, R_0).$$

Remark 3.3. *Recall that* $p(x) = \min(x, 1-x)$. *Since* $0 \leq p(s) \leq 1$, $\forall\, s \in [0,1]$, *it is easy to check that (3.3) implies that*

$$\exists\, r_0 > 0,\ 0 \leq f(x, r_0) \leq \phi(r_0),\ \forall\, x \in [0,1]$$

which is nothing but a weak sub-linear growth condition with respect to ϕ.

Proof. Without loss of generality, suppose that $0 < r_0 < R_0$ and consider the open sets

$$\Omega_1 = \{u \in C([0,1]),\ \|u\| < r_0\} \quad \text{and} \quad \Omega_2 = \{u \in C([0,1]),\ \|u\| < R_0\}\,.$$

(a) Let $u \in K \cap \partial\Omega_1$, that is $u \in K$ and $\|u\| = r_0$. For any $x \in [0,1]$, we have, by Assumption (3.3)

$$0 \le Tu(x) \le \psi\left(\int_0^1 [F_1(u(x)) + F_2(u(x))]\, dx\right).$$

With Lemma 2.4 and $r_0 \ge u(x) \ge p(x)r_0$, we infer the upper bound

$$\begin{aligned} 0 \le Tu(x) \quad &\le \quad \psi\left(\left(1 + \frac{F_2(r_0)}{F_1(r_0)}\right)\int_0^1 F_1(r_0 p(x))\, dx\right) \\ &\le \quad \psi(\phi(r_0)) = r_0 = \|u\|. \end{aligned}$$

Thus, $\|Tu\| \le \|u\|$ for any $u \in K \cap \partial\Omega_1$.

(b) Let $u \in K \cap \partial\Omega_2$, that is $u \in K$ and $\|u\| = R_0$. We have the discussion:

• If $\theta < \sigma$ or $\theta > 1-\sigma$, then, by Lemma 2.6 and $u(x) \ge \sigma\|u\|$ for $x \in [\sigma, 1-\sigma]$, we get

$$\begin{aligned} \|Tu\| \quad &\ge \quad \sigma\psi\left(\int_\sigma^{1-\sigma} f(\tau, u(\tau))d\tau\right) \\ &\ge \quad \sigma\psi\left(\int_\sigma^{1-\sigma} G_1(u)\left(1 + \frac{G_2(u)}{G_1(u)}\right)\, d\tau\right) \\ &\ge \quad \sigma\psi\left(\int_\sigma^{1-\sigma} G_1(\|u\|)\left(1 + \frac{G_2(\sigma\|u\|)}{G_1(\sigma\|u\|)}\right)\, d\tau\right) \\ &= \quad \sigma\psi\left((1-2\sigma)G_1(R_0)\left(1 + \frac{G_2(\sigma R_0)}{G_1(\sigma R_0)}\right)\right) \\ &= \quad \sigma D_2(\sigma) \ge \sigma D_2^* \ge 2R_0 > \|u\|. \end{aligned}$$

• If $\theta \in [\sigma, 1-\sigma]$, then Lemma 2.6 again yields the following estimates:

$$\begin{aligned} 2\|Tu\| \quad &\ge \quad \sigma\psi\left(\int_\sigma^\theta f(\tau, u(\tau))\, d\tau\right) + \sigma\psi\left(\int_\theta^{1-\sigma} f(\tau, u(\tau))\, d\tau\right) \\ &\ge \quad \sigma\psi\left((\theta-\sigma)G_1(R_0)\left(1 + \frac{G_2(\sigma R_0)}{G_1(\sigma R_0)}\right)\right) \\ &+ \quad \sigma\psi\left((1-\sigma-\theta)G_1(R_0)\left(1 + \frac{G_2(\sigma R_0)}{G_1(\sigma R_0)}\right)\right) = \sigma D_2(\theta) \\ &\ge \quad \sigma D_2^* \ge 2R_0 = 2\|u\|. \end{aligned}$$

Consequently, $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$. Therefore, all assumptions of Theorem A are met and so Problem (1.1) admits a positive solution such that $r_0 \leq \|u\| \leq R_0$.

Remark 3.4. *In Assumptions* **(a)** *and* **(b)** *respectively, one may choose the functions F_1 and F_2 (respectively the functions G_1 and G_2) be such that F_1 and $\frac{F_2}{F_1}$ are nondecreasing (respectively G_1 and $\frac{G_2}{G_1}$ are nonincreasing).*

3.4. **The sublinear and superlinear-like cases.** In this subsection, we suppose further that the operator ϕ satisfies the following condition

$$\exists\,\alpha, \beta \in (0, +\infty),\ \forall\, t \in [0, 1],\ \forall\, x \in \mathbb{R}^+, \quad t^\beta \phi(x) \leq \phi(tx) \leq t^\alpha \phi(x). \tag{3.5}$$

Remark 3.5. **(a)** *Condition (3.5) implies that*

$$\forall\, t \in [0, 1],\ \forall\, x \in \mathbb{R}^+, \quad \psi(t^\beta x) \leq t\psi(x) \leq \psi(t^\alpha x), \tag{3.6}$$

that is

$$\forall\, t \in [0, 1],\ \forall\, x \in \mathbb{R}^+,\ t^{\frac{1}{\alpha}}\psi(x) \leq \psi(tx) \leq t^{\frac{1}{\beta}}\psi(x). \tag{3.7}$$

(b) *Let $x,\, y \in \mathbb{R}^+$ be such that $x + y > 0$. From (3.7), we infer that*

$$\begin{array}{c} \left(\frac{x}{x+y}\right)^{\frac{1}{\alpha}} \psi(x+y) \leq \psi(x) \leq \left(\frac{x}{x+y}\right)^{\frac{1}{\beta}} \psi(x+y) \\ \left(\frac{y}{x+y}\right)^{\frac{1}{\alpha}} \psi(x+y) \leq \psi(y) \leq \left(\frac{y}{x+y}\right)^{\frac{1}{\beta}} \psi(x+y), \end{array}$$

which yields

$$\begin{aligned} \left(\left(\frac{x}{x+y}\right)^{\frac{1}{\alpha}} + \left(\frac{y}{x+y}\right)^{\frac{1}{\alpha}}\right)\psi(x+y) &\leq \psi(x) + \psi(y) \\ &\leq \left(\left(\frac{x}{x+y}\right)^{\frac{1}{\beta}} + \left(\frac{y}{x+y}\right)^{\frac{1}{\beta}}\right)\psi(x+y). \end{aligned}$$

Depending on the position of the positive parameter γ with respect to unity, the function $t \hookrightarrow t^\gamma + (1-t)^\gamma$, defined on the interval $(0, 1)$, has either the maximum $2\left(\frac{1}{2}\right)^\gamma$ and a minimum equal to unity or the converse. Setting

$$\alpha_*: \ = \min\left(1, 2\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right) \quad \textit{and} \quad \beta^*: \ = \max\left(1, 2\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right),$$

we finally arrive at the useful estimate

$$\alpha_*\psi(x+y) \leq \psi(x) + \psi(y) \leq \beta^*\psi(x+y). \tag{3.8}$$

Remark 3.6. *Here are some functions which satisfy Condition (3.5):*

(a) *The* $p-$*Laplacian operator* $\phi(x) = |x|^{p-2}x$ $(p > 1)$ *is homogenous* $\phi(tx) = \phi(t)\phi(x) = t^{p-1}\phi(x),\ \forall\,(t,x) \in [0,1] \times \mathbb{R}^+$.

(b) *The sum of a* p *and a* q *Laplacian* $\phi(x) = |x|^{p-2}x + |x|^{q-2}x$ $(p \neq q$ *and* $p, q > 1)$. *Indeed, for every* $x \in \mathbb{R}^+$ *and* $t \in [0,1]$, *we have:*

$$t^{\max(p,q)-1}\phi(x) \leq \phi(tx) \leq t^{\min(p,q)-1}\phi(x).$$

(c) $\phi(x) = |x|^{p-1}x\ln(1+|x|)$ $(p > 1)$ *and* $\phi(x) = |x|^p \arctan x$ $(p > 0)$ *satisfy*

$$t^{p+1}\phi(x) \leq \phi(tx) \leq t^p\phi(x),\ \forall\,(t,x) \in [0,1] \times \mathbb{R}^+.$$

(d) $\phi(x) = \frac{x|x|^{p-1}}{1+x^2}$ $(p > 2)$ *satisfies*

$$t^p\phi(x) \leq \phi(tx) \leq t^{p-2}\phi(x),\ \forall\,(t,x) \in [0,1] \times \mathbb{R}^+.$$

The main existence result in this sub-section is

Theorem 3.3. *Let*

$$\begin{array}{ll} \liminf\limits_{u\to 0^+}\left(\min\limits_{x\in[0,1]}\frac{f(x,u)}{\phi(u)}\right) = q_0, & \limsup\limits_{u\to 0^+}\left(\max\limits_{x\in[0,1]}\frac{f(x,u)}{\phi(u)}\right) = q^0, \\ \liminf\limits_{u\to +\infty}\left(\min\limits_{x\in[0,1]}\frac{f(x,u)}{\phi(u)}\right) = q_\infty, & \limsup\limits_{u\to +\infty}\left(\max\limits_{x\in[0,1]}\frac{f(x,u)}{\phi(u)}\right) = q^\infty. \end{array} \tag{3.9}$$

Then, Problem (1.1) has at least one positive nontrivial solution provided one of the following conditions holds true:

$$\text{either} \quad q_0 > \frac{2^\beta}{\sigma^{2\beta}(1-2\sigma)\alpha_*^\beta} \quad \text{and} \quad q^\infty < 1 \tag{3.10}$$

$$\text{or} \quad q^0 < 1 \quad \text{and} \quad q_\infty > \frac{2^\beta}{\sigma^{2\beta}(1-2\sigma)\alpha_*^\beta}. \tag{3.11}$$

Proof. (a) The sublinear-like case.

• **Claim 1.** Let $\varepsilon > 0$ be such that $q_0 - \varepsilon \geq \frac{2^\beta}{\sigma^{2\beta}(1-2\sigma)\alpha_*^\beta}$. By definition of q_0, there exists an $r_\varepsilon > 0$ such that $f(t,u) \geq (q_0 - \varepsilon)\phi(u)$ for every $(t,u) \in [0,1] \times [0, r_\varepsilon]$. Consider the open ball $\Omega_1 : = B(0, r_\varepsilon)$ and let $u \in K \cap \partial\Omega_1$, that is $u \in K$ and $\|u\| = r_\varepsilon$. Then, in one hand, we have that $u(x) \geq p(x)\|u\| \geq \sigma\|u\|$ for any $x \in [\sigma, 1-\sigma]$ and in the other hand, the following discussion holds true:

• If $\theta < \sigma$ or $\theta > 1-\sigma$, then, by Lemma 2.6, we get, since ϕ is increasing

$$\begin{aligned} \|Tu\| &\geq \sigma\psi\left(\int_\sigma^{1-\sigma} f(\tau,u(\tau))\,d\tau\right) \\ &\geq \sigma\psi\left(\int_\sigma^{1-\sigma} (q_0-\varepsilon)\phi(u(\tau))\,d\tau\right) \\ &\geq \sigma\psi\left((q_0-\varepsilon)(1-2\sigma)\phi(\sigma\|u\|)\right). \end{aligned}$$

Owing to (3.5) and (3.6) together with $2^\beta > \alpha_*^\beta$, we deduce

$$\|Tu\| \geq \psi\left(\sigma^{2\beta}(1-2\sigma)(q_0-\varepsilon)\phi(\|u\|)\right) \geq \|u\|.$$

• If $\theta \in [\sigma, 1-\sigma]$, then again by Lemmas 2.4, 2.6, we have successively the estimates:

$$\begin{aligned} \|Tu\| &\geq \frac{\sigma}{2}\psi\left(\int_\sigma^{\theta} f(\tau,u(\tau))\,d\tau\right) + \frac{\sigma}{2}\psi\left(\int_\theta^{1-\sigma} f(\tau,u(\tau))\,d\tau\right) \\ &\geq \frac{\sigma}{2}\psi\left(\int_\sigma^{\theta} (q_0-\varepsilon)\phi(u(\tau))\,d\tau\right) + \frac{\sigma}{2}\psi\left(\int_\theta^{1-\sigma} (q_0-\varepsilon)\phi(u(\tau))\,d\tau\right) \\ &\geq \frac{\sigma}{2}\psi\left((\theta-\sigma)(q_0-\varepsilon)\phi(\sigma\|u\|)\right) + \frac{\sigma}{2}\psi\left((1-\sigma-\theta)(q_0-\varepsilon)\phi(\sigma\|u\|)\right), \end{aligned}$$

which, with (3.8), imply

$$\|Tu\| \geq \frac{\sigma}{2}\alpha_*\psi\left((1-2\sigma)(q_0-\varepsilon)\phi(\sigma\|u\|)\right).$$

Turning back to (3.5) and (3.6) together with the choice of ε, we arrive at

$$\|Tu\| \geq \psi\left(\frac{\sigma^{2\beta}}{2^\beta}(1-2\sigma)\alpha_*^\beta(q_0-\varepsilon)\phi(\|u\|)\right) \geq \|u\|.$$

Therefore, in both cases $\|Tu\| \geq \|u\|$, $\forall\, u \in K \cap \partial\Omega_2$.

• **Claim 2.** Let $\varepsilon > 0$ be such that $q^\infty + \varepsilon < 1$. By definition of q^∞, there exists a $C > 0$ such that $f(t,u) \leq (q^\infty+\varepsilon)\phi(u)+C$ for every $(t,u) \in [0,1]\times\mathbb{R}^+$. Let the open ball $\Omega_2 := B(0,R)$ be such that $(q^\infty+\varepsilon)\phi(R)+C < \phi(R)$ and let $u \in K \cap \partial\Omega_2$, that is $u \in K$ and $\|u\| = R$. If $v = Tu$, then v verifies

$$\begin{cases} -(\phi(v'))'(x) &= f(x,u), \quad 0 < x < 1 \\ v(0) = v(1) &= 0. \end{cases}$$

Let θ be such that $v'(\theta) = 0$. Given some $s \in [0,1]$, we have

$$\phi(v'(s)) = \int_s^\theta f(\tau,u(\tau))\,d\tau$$

and

$$\begin{array}{rcl} |\phi(v'(s))| = \phi(|v'(s)|) & \leq & \int_0^1 f(\tau, u(\tau))\, d\tau \\ & \leq & \int_0^1 \left((q^\infty + \varepsilon)\phi(u(\tau)) + C\right)\, d\tau \\ & \leq & (q^\infty + \varepsilon)\phi(\|u\|) + C \leq \phi(R) \end{array}$$

whence, $|v'(s)| \leq R,\ \forall\, s \in [0,1]$ and then

$$v(t) = \int_0^t v'(s)\, ds \leq \sup_{s\in[0,1]} |v'(s)| \leq R = \|u\|,$$

that is

$$\|Tu\| \leq \|u\|.$$

(b) The superlinear-like case.

It can be treated in a similar way. In this case, consider some $\varepsilon > 0$ such that $q^0 + \varepsilon \leq 1$ so that there exists an $R_\varepsilon > 0$ such that $f(t,u) \leq (q^0 + \varepsilon)\phi(u)$ for every $(t,u) \in [0,1] \times [0, R_\varepsilon]$. For $\Omega_1 : = B(0, R_\varepsilon)$ and $u \in K \cap \partial\Omega_1$, we get

$$\forall\, x \in [0,1],\ Tu(x) \leq \psi\left(\int_0^1 f(\tau, u(\tau))\, d\tau\right) \leq \psi\left(\phi(\|u\|)\right) = \|u\|.$$

Let $\varepsilon > 0$ be such that $q_\infty - \varepsilon \geq \frac{2^\beta}{\sigma^{2\beta}(1-2\sigma)\alpha_*^\beta}\cdot$ Then there exists an $R_\varepsilon > 0$ such that $f(t,u) \leq (q_\infty - \varepsilon)\phi(u)$ for every $(t,u) \in [0,1] \times [R_\varepsilon, +\infty)$. Let $\Omega_2 : = B(0,R)$ with $R = \frac{R_\varepsilon}{\sigma}\cdot$ Then, as in Part **(a)**, we have to distinguish between two cases and derive the estimate

$$\|Tu\| \geq \|u\|,\ \forall\, u \in K \cap \partial\Omega_2.$$

By Theorem A, Problem (1.1) admits a positive solution such that $R \leq \|u\| \leq R_\varepsilon$.

As a consequence, we recover a classical result:

Corollary 3.4. *Let $q \in C([0,1], \mathbb{R}^+)$ with $\min_{x\in[0,1]} q(x) > 0$ and F satisfies*

$$\textit{either (sub-linear case)}\quad \liminf_{s\to 0^+} \frac{F(s)}{\phi(s)} = +\infty \quad \textit{and} \quad \limsup_{s\to+\infty} \frac{F(s)}{\phi(s)} = 0,$$

$$\textit{or (super-linear case)}\quad \limsup_{s\to 0^+} \frac{F(s)}{\phi(s)} = 0 \quad \textit{and} \quad \liminf_{s\to+\infty} \frac{F(s)}{\phi(s)} = +\infty.$$

Then, the boundary value problem

$$\left\{ \begin{array}{rcl} -(\phi(u'))' & = & q(x)F(u), \quad 0 < x < 1, \\ u(0) = u(1) & = & 0 \end{array} \right. \tag{3.12}$$

has at least one nontrivial positive solution.

Remark 3.7. *The real function* $\sigma \hookrightarrow \zeta(\sigma) = \frac{2^\beta}{\sigma^{2\beta}(1-2\sigma)\alpha_*^\beta}$ *defined for* $\sigma \in (0, \frac{1}{2})$ *achieves its minimum at the point* $\sigma_0 = \frac{\beta}{2\beta+1}$ *and assumes the value* $\zeta(\sigma_0) = \left(\frac{2}{\alpha_*^\beta}\right)^\beta \frac{(2\beta+1)^{2\beta+1}}{\beta^{2\beta}}\cdot$ *Theorem 3.3 may then be changed as follows; we omit the proof.*

Theorem 3.5. *With the notations in (3.9), assume that*

$$\text{either} \quad q_0 > \zeta(\sigma_0) \;\; \text{and} \;\; q^\infty < 1$$

$$\text{or} \quad q^0 < 1 \;\; \text{and} \;\; q_\infty > \zeta(\sigma_0).$$

Then, Problem (1.1) has at least one positive nontrivial solution.

4. Applications: Problems involving the $p-$Laplacian

In the first two examples, we assume $\phi(s) = \phi_p(s) = |s|^{p-2}s$ for some $p > 1$. The number $q = \frac{p}{p-1}$ will denote the conjugate of p. The third example is concerned with the sum of two $p-$Laplacian operators.

4.1. **Example 1.** Consider the function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(s) = c_1 s^\alpha + \frac{1}{c_1} s^\beta$ with some $0 < \alpha < p-1 = \frac{1}{q-1} < \beta$ and c_1 a positive real number. For any $0 \le s \le \eta = \psi(M_1) = M_1^{q-1}$, Assumption (a) in Theorem 3.1, namely the condition $f(s) \le M_1$, is satisfied whenever

$$\exists\, M_1 > 0, \quad c_1 M_1^a + \frac{1}{c_1} M_1^b \le 1, \tag{4.1}$$

where we have set $a\colon = \alpha(q-1) - 1$ and $b\colon = \beta(q-1) - 1$. Hereafter, we assume $a < 0$, $b > 0$ and $a + b > 0$, that is

$$\alpha(q-1) < 1,\ \beta(q-1) > 1 \;\text{ and }\; (\alpha+\beta)(q-1) > 2. \tag{4.2}$$

Equivalently,

$$0 < \alpha < p-1 < \beta \;\text{ and }\; 0 < p-1 < \frac{\alpha+\beta}{2}\cdot$$

The graph of the function χ defined on $\mathbb{R}^+$ by $\chi(x) = c_1 x^a + \frac{1}{c_1} x^b$ looks like a convex parabola with a minimum achieved at some point $x_0 = \left(\frac{-ac_1^2}{b}\right)^{\frac{1}{b-a}}$.

Condition (4.1) is then satisfied if and only if

$$\chi(x_0) = c_1^{\frac{b+a}{b-a}} \left(\left(\frac{-a}{b} \right)^{\frac{a}{b-a}} + \left(\frac{-a}{b} \right)^{\frac{b}{b-a}} \right) \leq 1$$

which is fulfilled for every

$$0 < c_1 \leq \frac{1}{\left(\left(\frac{-a}{b}\right)^{\frac{a}{b-a}} + \left(\frac{-a}{b}\right)^{\frac{b}{b-a}} \right)^{\frac{b-a}{b+a}}}. \tag{4.3}$$

As for Assumption (b) in Theorem 3.1, notice that $s \geq \sigma\mu$ implies that $c_1 s^\alpha + \frac{1}{c_1} s^\beta \geq c_1 (\sigma\mu)^\alpha + \frac{1}{c_1} (\sigma\mu)^\beta \geq M_2$ which in turn yields $f(s) \geq M_2$. Keeping in mind the value of μ (see Remark 3.1), this is equivalent to finding an $M_2 > 0$ such that

$$C_1 M_2^a + C_2 M_2^b \geq 1 \tag{4.4}$$

where

$$\begin{array}{rcl} C_1 & := & c_1 \sigma^{2\alpha} (1-2\sigma)^{\alpha(q-1)} 2^{\alpha \min(2-q,0)} \\ C_2 & := & \frac{1}{c_1} \sigma^{2\beta} (1-2\sigma)^{\beta(q-1)} 2^{\beta \min(2-q,0)}. \end{array}$$

Given the graph of the function $\varrho(x) = C_1 x^a + C_2 x^b$, Condition (4.4) is satisfied whenever M_2 is either small enough or large enough; in particular, this ensures that $\mu \neq \eta$, that is

$$\left(\frac{\sigma}{2}\right)^{\frac{1}{q-1}} (1-2\sigma) 2^{\frac{\min(2-q,0)}{q-1}} M_2 \neq M_1,$$

as required in Theorem 3.1. To summarize, we have proved that the autonomous problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)'(x) & = & c_1 u^\alpha + \frac{1}{c_1} u^\beta, \quad 0 < x < 1 \\ u(0) = u(1) & = & 0 \end{cases}$$

has a positive nontrivial solution under Condition (4.2) and for some constant c_1 obeying (4.3). Note that the nonlinear right-hand term encompasses sublinear and super-linear parts with respect to the $p-$Laplacian.

4.2. **Example 2.** Let the right-hand term be variable-separated $f(x,s) = g(x)h(s)$ with a nonnegative continuous function g, hence bounded over $[0,1]$, and a real positive continuous function h which satisfies the bounding $G_1(s) + G_2(s) \leq h(s) \leq F_1(s) + F_2(s)$ with $F_1(s) = \frac{1}{\sqrt{s}}$, $F_2(s) = s^\alpha$, $G_1(s) = \frac{1}{\sqrt{s+1}}$, $G_2(s) = s^\alpha$ for $s > 0$ and some positive real number α. Thus

$\frac{F_2}{F_1} = s^{\alpha+\frac{1}{2}}$ and $\frac{G_2}{G_1} = s^{\alpha}(s+1)^{\frac{1}{2}}$. Notice that $F_1 + F_2$ is neither bounded at positive infinity nor in the vicinity of the origin.

After computing $\int_0^1 F_1(r_0 p(s))\,ds = \frac{4}{\sqrt{2r_0}}$, Assumption (a) in Theorem 3.2 becomes

$$\exists\, r_0 > 0, \quad \frac{4\left(1 + r_0^{\alpha+\frac{1}{2}}\right)}{\sqrt{2r_0}} \le r_0^{p-1} \tag{4.5}$$

which is satisfied for every $0 < \alpha < p-1$ and r_0 large enough. As for Condition (b) in Theorem 3.2, it is fulfilled whenever

$$2^{2-q}\sigma\left(K(1-2\sigma)\right)^{q-1} \ge 2R_0 \tag{4.6}$$

where $q = \frac{p}{p-1}$ and $K = K(R_0) = G_1(R_0)\left(1 + \frac{G_2(\sigma R_0)}{G_1(\sigma R_0)}\right)$ that is

$$K = \frac{1}{\sqrt{R_0+1}}\left(1 + (\sigma R_0)^{\alpha}\sqrt{1+\sigma R_0}\right).$$

Since $\lim_{R_0 \to 0^+} K(R_0) = 1$, Condition (4.6) is satisfied for small R_0. Therefore, all assumptions in Theorem 3.2 are met and Problem (1.1) has a positive solution u with $R_0 \le \|u\| \le r_0$.

4.3. **Example 3.** Consider the boundary value problem:

$$\begin{cases} -\left(\phi(u')\right)'(x) &= a(x)f(u(x)), \quad 0 < x < 1 \\ u(0) = u(1) &= 0 \end{cases} \tag{4.7}$$

with $a \in C([0,1],\mathbb{R}^+)$ satisfies $\min_{x\in[0,1]} a(x) > 0$, $\phi(u) = \phi_p(u) + \phi_q(u)$ and $f(u) = k_1|\phi_s(u)| + k_2|\phi_t(u)|$ for some positive constants k_1, k_2 and $p, q, s, t > 1$. Then, Problem (4.7) has a positive nontrivial solution if

$$\begin{aligned} &\text{either } \ k_1 > 0,\ 1 < s < p < q \quad \text{and } s < t < q \\ &\text{or } \ k_2 > 0,\ 1 < p < q < t \quad \text{and } p < s < t. \end{aligned}$$

Indeed, the ratio

$$\frac{f(u)}{\phi(u)} = \frac{k_1 u^{s-1} + k_2 u^{t-1}}{u^{p-1} + u^{q-1}}, \quad u > 0$$

behaves as u^{t-q} when u goes to $+\infty$ and as u^{s-p} when u approaches 0^+. Thus Corollary 3.4 applies.

5. Concluding remarks

(a) We may notice that in Theorem 3.1 no condition is assumed on the function ϕ. Assumptions **(a)** and **(b)** rather provide local growth conditions on the nonlinearity ϕ, which are weaker than usual polynomial growth conditions. As Example 3.1 shows, this theorem provides existence of solution when the nonlinearity is the sum of the sub-linear and the super-linear case with respect to $(p-1)$. Clearly, the case $p=2$ is reminiscent of second-order boundary value problems.

(b) Thanks to Theorem 3.2, we can see that the nonlinearity f may be the sum of an increasing function and a decreasing one (see example 2 in Section 4). Similar results may be extended to obtain existence results for singular $\phi-$Laplacian boundary value problems.

(c) Theorem 3.3 not only gives existence of positive solutions for a new class of $\phi-$Laplacian Dirichlet boundary value problems but also allows for the nonlinear operator ϕ to be the sum of p and $q-$Laplacian mappings. We believe that the results obtained in this paper can make a contribution to the existence theory of positive solutions for $\phi-$Laplacian Dirichlet boundary value problems.

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