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NOTE ON FIXED POINT THEOREM OF CHEN

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Abstract. W. A. Kirk [9] first introduced the notion of asymptotic contractions and proved the fixed point theorem for this class of mappings. In this paper we present one fixed point theorem of Kirk's type which generalizes recent results of Y.-Z. Chen [5] and I. Aranđelović [2].

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1. Introduction

W. A. Kirk [9] introduced the notion of asymptotic contractions and proved fixed point theorem for this class of mappings. In note [1] we present a new short and simple proof of Kirk's theorem. Further results on this class of mappings was obtained by: J. Jachymski, I. Jóźwik [8], Y.-Z. Chen [5], P. Gerhardy [6], [7], T. Suzuki [11], H. K. Xu [14], M. Arav, F. E. C. Santos, S. Reich, A. Zaslavski [3] and K. Włodarczyk, D. Klim, R. Plebaniak [12], K. Włodarczyk, R. Plebaniak, C. Obczyński [13], E. M. Briseid [4], A. Razani, E. Nabizadeh, M. Beyg Mohamadi, S. Homaei Pour [10] and I. Aranđelović [2]. The papers [12] and [13] presents some ideas for application of the theory of asymptotic contractions in the analysis of set-valued dynamic systems.

In this paper we present one fixed point theorem of Kirk's type which generalizes recent results of Y.-Z. Chen [5] and I. Aranđelović [2].

Let X be a nonempty set and $f : X \to X$ arbitrary mapping. $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that a sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) is the orbit of f at point x_0.

In [3] M. Arav, F. E. C. Santos, S. Reich and A. Zaslavski proved the following result:

Proposition 1. *Let (X, d) be a metric space, $f : X \to X$ continuous function and (φ_i) sequence of functions such that $\varphi_i : [0, \infty) \to [0, \infty)$ and for each $x, y \in X$*

$$d(f^i(x), f^i(y)) \leq \varphi_i(d(x, y)).$$

Assume also that there exists upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ such that for any $r > 0$ $\varphi(r) < r$, $\varphi(0) = 0$ and $\varphi_i \to \varphi$ uniformly on any bounded interval $[0, b]$. If there exists $y \in X$ such that $y = f(y)$ then all sequences of Picard iterates defined by f converge to y, uniformly on each bounded subset of X.

2. Results

Now we present our results.

Theorem 1. *Let (X, d) be a complete metric space, $f : X \to X$ continuous function and (φ_i) sequence of functions such that $\varphi_i : [0, \infty) \to [0, \infty)$ and for each $x, y \in X$*

$$d(f^i(x), f^i(y)) \leq \varphi_i(d(x, y)).$$

Assume also that there exists upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ such that for any $r > 0$ $\varphi(r) < r$, $\varphi(0) = 0$ and $\varphi_i \to \varphi$ uniformly on any bounded interval $[0, b]$. If

$$\lim_{t \to \infty} \frac{\varphi(t)}{t} < 1,$$

then f has an unique fixed point $y \in X$ and all sequences of Picard iterates defined by f converge to y, uniformly on each bounded subset of X.

Proof. For any $x, y \in X$, $x \neq y$, we have:

$$\overline{\lim} d(f^n(x), f^n(y)) \leq \overline{\lim} \varphi_n(d(x, y)) = \varphi(d(x, y)) < d(x, y).$$

Suppose that there exist $x, y \in X$ and $\varepsilon > 0$ such that $\overline{\lim} d(f^n(x), f^n(y)) = \varepsilon$. Then there exists sequence of integers (m_j), such that

$$\lim d(x_{m_j}, y_{m_j}) = \overline{\lim} d(x_m, y_m).$$

If $\overline{\lim} d(f^k(x), f^k(y)) \geq \varepsilon$, for each $k \in (m_j)$, then from upper semicontinuity of φ follows $\overline{\lim} d(x_{m_j}, y_{m_j}) \leq \varphi(\varepsilon) < \varepsilon$, which is a contradiction. So there exists $k \in (m_j)$ such that

$$\varphi(d(f^k(x), f^k(y))) < \varepsilon.$$

This implies that

$$\overline{\lim} d(f^n(x), f^n(y)) = \overline{\lim_n} d(f^n(f^k(x)), f^n(f^k(y))) \leq \overline{\lim_n} \varphi_n(d(f^k(x), f^k(y))) = \\ = \varphi(d(f^k(x), f^k(y))) < \varepsilon,$$

which is a contradiction. So we obtain that

$$\lim d(f^n(x), f^n(y)) = 0, \tag{1}$$

for any $x, y \in X$, which implies that all sequences of Picard iterates defined by f, are equiconvergent.

Now let $a \in X$ be arbitrary, (a_n) be a sequence of Picard iterates of f at point a, $Y = \overline{(a_n)}$ and $F_n = \{x \in Y : d(x, f^k(x)) \leq 1/n \quad k = 1, \ldots, n\}$. From (1) follows that F_n is nonempty and since f is continuous F_n is closed, for any n. Also, we have $F_{n+1} \subseteq F_n$. Let (x_n) and (y_n) be arbitrary sequences, such that $x_n, y_n \in F_n$. Let (n_j) be a sequence of integers, such that $\lim d(x_{n_j}, y_{n_j}) = \overline{\lim} d(x_n, y_n)$.

For any $\varepsilon > 0$ there exists positive integer k such that

$$\varphi(t) + \varepsilon \geq \varphi_k(t)$$

for all $t \in [0, +\infty)$ and $m \geq k$, because $\varphi_n \to \varphi$ uniformly on the rang of d. Now we have:

$$\begin{aligned}\lim d(x_{n_j}, y_{n_j}) &\leq \overline{\lim}(d(x_{n_j}, f^{n_j}(x_{n_j})) + d(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})) + \\ &+ d(y_{n_j}, f^{n_j}(y_{n_j}))) = \overline{\lim} d(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})) \leq \\ &\leq \overline{\lim} \varphi_{n_j}(d(x_{n_j}, y_{n_j})) \leq \varepsilon + \overline{\lim} \varphi(d(x_{n_j}, y_{n_j})) \leq \\ &\leq \varepsilon + \varphi(\lim d(x_{n_j}, y_{n_j})),\end{aligned}$$

for $n_j \geq k$ and so $\lim d(x_{n_j}, y_{n_j}) = \varphi(\lim d(x_{n_j}, y_{n_j}))$, which implies that $\lim d(x_{n_j}, y_{n_j}) \in \{0, +\infty\}$. Now we have following two cases:

Let $\lim d(x_{n_j}, y_{n_j}) = 0$. Thus $\overline{\lim} d(x_n, y_n) = 0$ and so $\lim d(x_n, y_n) = 0$. This implies that $\lim \operatorname{diam} F_n = 0$. By completeness of Y follows that there exists $z \in X$ such that

$$\bigcap_{i=1}^{\infty} F_n = \{z\}.$$

Since $d(z, f(z)) \leq 1/n$ for any n, we have $f(z) = z$. From (1) follows that all sequences of Picard iterates defined by f converge to z. From Proposition 1 follows that this convergence is uniform on bounded subsets of X.

Let $\lim d(x_{n_j}, y_{n_j}) = +\infty$. Then from

$$d(x_{n_j}, y_{n_j}) \leq d(x_{n_j}, f^{n_j}(x_{n_j})) + d(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})) + d(y_{n_j}, f^{n_j}(y_{n_j}))$$

follows

$$1 \leq \frac{d(x_{n_j}, f^{n_j}(x_{n_j})) + d(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})) + d(y_{n_j}, f^{n_j}(y_{n_j}))}{d(x_{n_j}, y_{n_j})} \leq$$

$$\leq \frac{d(x_{n_j}, f^{n_j}(x_{n_j})) + \varphi_{n_j}(d(x_{n_j}, y_{n_j})) + d(y_{n_j}, f^{n_j}(y_{n_j}))}{d(x_{n_j}, y_{n_j})} \leq$$

$$\leq \frac{d(x_{n_j}, f^{n_j}(x_{n_j})) + \varphi(d(x_{n_j}, y_{n_j})) + \varepsilon + d(y_{n_j}, f^{n_j}(y_{n_j}))}{d(x_{n_j}, y_{n_j})},$$

for $n_j \geq k$. So

$$1 \leq \underline{\lim} \frac{d(x_{n_j}, f^{n_j}(x_{n_j})) + \varphi(d(x_{n_j}, y_{n_j})) + d(y_{n_j}, f^{n_j}(y_{n_j}))}{d(x_{n_j}, y_{n_j})} \leq$$

$$\leq \overline{\lim} \frac{d(x_{n_j}, f^{n_j}(x_{n_j})) + d(y_{n_j}, f^{n_j}(y_{n_j}))}{d(x_{n_j}, y_{n_j})} + \underline{\lim} \frac{\varphi(d(x_{n_j}, y_{n_j}))}{d(x_{n_j}, y_{n_j})}$$

$$= \underline{\lim} \frac{\varphi(d(x_{n_j}, y_{n_j}))}{d(x_{n_j}, y_{n_j})}.$$

Then from

$$\underline{\lim}_{t \to \infty} \frac{\varphi(t)}{t} < 1$$

follows

$$1 \leq \underline{\lim} \frac{\varphi(d(x_{n_j}, y_{n_j}))}{d(x_{n_j}, y_{n_j})} < 1$$

which is a contradiction.

The statement of Y.-Z. Chen [5] - Corollary 2.4 has additional assumptions that one of (φ_i) is upper semicontinuous. In [2] first author proved that this condition can be omitted.

The statements of Y.-Z. Chen [5] - Corollary 2.4 and I. Aranđelović [2] - Theorem 1.C has condition

$$\overline{\lim_{t\to\infty}}\frac{\varphi(t)}{t} < 1$$

which is stronger then our condition

$$\underline{\lim_{t\to\infty}}\frac{\varphi(t)}{t} < 1.$$

In the statements of W. A. Kirk [9] - Theorem 2.1 and Y.-Z. Chen [5] - Theorem 2.2, the assumption "f is continuous" was inadvertently left out, but it was used in the proofs of theorems. J. Jachymski, I. Jóźwik [8], give the following example for necessity of this condition.

Example 1. *Let $X = [0, 1]$ an $f : X \to X$ defined by*

$$f(x) = \begin{cases} 1, & x = 0, \\ x/2, & x \neq 0. \end{cases}$$

So $f(X) \subseteq (0, 1]$ which implies $f^n(X) \subseteq (0, 1/2^{n-1}]$. A sequence of functions $\varphi_n = 1/2^{n-1}$ satisfies the conditions of Theorem 2.1 because $\varphi_n \to 0$ uniformly, but f is fixed point free.

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