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ON ISOMORPHISMS OF CANONICAL $\mathcal{E}$-LATTICES

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Abstract. The aim of the present paper is to study isomorphisms of canonical $\mathcal{E}$-lattices. Some interesting results are obtained in the particular case of isomorphisms between two subgroup $\mathcal{E}$-lattices.

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1. Preliminaries

The starting point for our discussion is given by the paper [10], where there is introduced the category of $\mathcal{E}$–lattices and there are made some basic constructions in this category. Given a nonempty set L and a map $\varepsilon : L \to L$, we denote by $\operatorname{Ker}\varepsilon$ the kernel of ε (i.e. $\operatorname{Ker}\varepsilon = \{(a,b) \in L \times L \mid \varepsilon(a) = \varepsilon(b)\}$), by $\operatorname{Im}\varepsilon$ the image of ε (i.e. $\operatorname{Im}\varepsilon = \{\varepsilon(a) \mid a \in L\}$) and by $\operatorname{Fix}\varepsilon$ the set consisting of all fixed points of ε (i.e. $\operatorname{Fix}\varepsilon = \{a \in L \mid \varepsilon(a) = a\}$). We say that L is an *$\mathcal{E}$–lattice* (relative to ε) if there exist two binary operations $\wedge_\varepsilon, \vee_\varepsilon$ on L which satisfy the following properties:

a) $a \wedge_\varepsilon (b \wedge_\varepsilon c) = (a \wedge_\varepsilon b) \wedge_\varepsilon c,\ a \vee_\varepsilon (b \vee_\varepsilon c) = (a \vee_\varepsilon b) \vee_\varepsilon c$, for all $a, b, c \in L$;
b) $a \wedge_\varepsilon b = b \wedge_\varepsilon a,\ a \vee_\varepsilon b = b \vee_\varepsilon a$, for all $a, b \in L$;
c) $a \wedge_\varepsilon a = a \vee_\varepsilon a = \varepsilon(a)$, for any $a \in L$;
d) $a \wedge_\varepsilon (a \vee_\varepsilon b) = a \vee_\varepsilon (a \wedge_\varepsilon b) = \varepsilon(a)$, for all $a, b \in L$.

Clearly, in an $\mathcal{E}$–lattice L (relative to ε) the map ε is idempotent and $\operatorname{Im}\varepsilon = \operatorname{Fix}\varepsilon$. Moreover, the set $\operatorname{Fix}\varepsilon$ is closed under the binary operations $\wedge_\varepsilon, \vee_\varepsilon$ and, denoting by $\wedge_\varepsilon^\circ, \vee_\varepsilon^\circ$ the restrictions of $\wedge_\varepsilon, \vee_\varepsilon$ to $\operatorname{Fix}\varepsilon$, we have that $(\operatorname{Fix}\varepsilon, \wedge_\varepsilon^\circ, \vee_\varepsilon^\circ)$ is a lattice. The connection between the $\mathcal{E}$–lattice concept and the lattice concept is very powerful. So, if $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is an $\mathcal{E}$–lattice and $\sim$

is an equivalence relation on L such that $\sim\subseteq \operatorname{Ker}\varepsilon$, then the factor set $L/\sim$ is a lattice isomorphic to the lattice $\operatorname{Fix}\varepsilon$. Conversely, if L is a nonvoid set and $\sim$ is an equivalence relation on L having the property that the factor set $L/\sim$ is a lattice, then the set L can be endowed with an $\mathcal{E}$–lattice structure (relative to a map $\varepsilon : L \longrightarrow L$) such that $\sim\subseteq \operatorname{Ker}\varepsilon$ and $L/\sim\cong \operatorname{Fix}\varepsilon$.

We say that an *$\mathcal{E}$–lattice* $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is a *canonical $\mathcal{E}$–lattice* if $a \wedge_\varepsilon b, a \vee_\varepsilon b \in \operatorname{Fix}\varepsilon$, for all $a, b \in L$. Three fundamental types of canonical $\mathcal{E}$–lattices have been identified in [10]. One of the most important example of a canonical $\mathcal{E}$–lattice is constituted by the lattice $L(G)$ of all subgroups of a group G (called the *subgroup $\mathcal{E}$–lattice* of G). Here the map ε is defined by $\varepsilon(H) = H_G$ (where H_G is the core of H in G), for any $H \in L(G)$ and the binary operations $\wedge_\varepsilon, \vee_\varepsilon$ are defined by $H_1 \wedge_\varepsilon H_2 = \varepsilon(H_1) \cap \varepsilon(H_2)$, $H_1 \vee_\varepsilon H_2 = \varepsilon(H_1)\varepsilon(H_2)$, for all $H_1, H_2 \in L(G)$. Mention that in this situation the lattice $\operatorname{Fix}\varepsilon$ is just the normal subgroup lattice $N(G)$ of G.

Let $(L_1, \wedge_{\varepsilon_1}, \vee_{\varepsilon_1})$ and $(L_2, \wedge_{\varepsilon_2}, \vee_{\varepsilon_2})$ be two $\mathcal{E}$–lattices. A map $f : L_1 \to L_2$ is called an *$\mathcal{E}$–lattice homomorphism* if:

a) $f \circ \varepsilon_1 = \varepsilon_2 \circ f$;
b) for all $a, b \in L_1$, we have:
  i) $f(a \wedge_{\varepsilon_1} b) = f(a) \wedge_{\varepsilon_2} f(b)$;
  i) $f(a \vee_{\varepsilon_1} b) = f(a) \vee_{\varepsilon_2} f(b)$.

Moreover, if the map f is one-to-one and onto, then we say that it is an *$\mathcal{E}$–lattice isomorphism*. An $\mathcal{E}$–lattice isomorphism of an $\mathcal{E}$–lattice into itself is called an *$\mathcal{E}$–lattice automorphism*. For an $\mathcal{E}$–lattice L, we shall denote by $\operatorname{Aut}_{\mathcal{E}}(L)$ the group consisting of all $\mathcal{E}$–lattice automorphisms of L.

2. Main results

2.1. Isomorphisms of canonical $\mathcal{E}$–lattices. In this section we present some general results concerned to isomorphisms between canonical $\mathcal{E}$–lattices.

Let $(L_1, \wedge_{\varepsilon_1}, \vee_{\varepsilon_1})$ and $(L_2, \wedge_{\varepsilon_2}, \vee_{\varepsilon_2})$ be two canonical $\mathcal{E}$–lattices. For every element $a \in L_i$, we denote by $[a]_i$ the equivalence class of a modulo $\operatorname{Ker}\varepsilon_i$ (i.e. $[a]_i = \{b \in L_i \mid \varepsilon_i(b) = \varepsilon_i(a)\}$), $i = 1, 2$. First of all, we give a characterization of $\mathcal{E}$–lattice isomorphisms from L_1 to L_2.

Proposition 1. *Let* $(L_1, \wedge_{\varepsilon_1}, \vee_{\varepsilon_1})$ *and* $(L_2, \wedge_{\varepsilon_2}, \vee_{\varepsilon_2})$ *be two canonical* $\mathcal{E}$*–lattices and* $f : L_1 \to L_2$ *be a map. Then the following two conditions are equivalent:*

a) f *is an* $\mathcal{E}$*–lattice isomorphism.*

b) i) *The restriction* f_0 *of* f *to the set* $\mathrm{Fix}\,\varepsilon_1$ *is a lattice isomorphism from* $\mathrm{Fix}\,\varepsilon_1$ *to* $\mathrm{Fix}\,\varepsilon_2$*.*

ii) $f \mid [a]_1 : [a]_1 \longrightarrow [f(a)]_2$ *is one-to-one and onto, for each* $a \in \mathrm{Fix}\,\varepsilon_1$*.*

Proof. a) $\Longrightarrow$ b) Let $a \in \mathrm{Fix}\,\varepsilon_1$. Then $\varepsilon_1(a) = a$ and so $f_0(a) = f(a) = f(\varepsilon_1(a)) = (f \circ \varepsilon_1)(a) = (\varepsilon_2 \circ f)(a) = \varepsilon_2(f(a)) \in \mathrm{Fix}\,\varepsilon_2$. Thus $\mathrm{Im}\, f_0 \subseteq \mathrm{Fix}\,\varepsilon_2$. For all $a, b \in L_1$, we have $f_0(a \wedge^\circ_{\varepsilon_1} b) = f(a \wedge_{\varepsilon_1} b) = f(a) \wedge_{\varepsilon_2} f(b) = f_0(a) \wedge_{\varepsilon_2} f_0(b) = f_0(a) \wedge^\circ_{\varepsilon_2} f_0(b)$ and $f_0(a \vee^\circ_{\varepsilon_1} b) = f(a \vee_{\varepsilon_1} b) = f(a) \vee_{\varepsilon_2} f(b) = f_0(a) \vee_{\varepsilon_2} f_0(b) = f_0(a) \vee^\circ_{\varepsilon_2} f_0(b)$, which show that f_0 is a lattice homomorphism. Since f is one-to-one and onto, it is clear that f_0 has the same properties and hence i) holds.

Now, let $b \in [a]_1$. Then $\varepsilon_1(b) = \varepsilon_1(a) = a$, which implies that $\varepsilon_2(f(b)) = (\varepsilon_2 \circ f)(b) = (f \circ \varepsilon_1)(b) = f(\varepsilon_1(b)) = f(a)$. Therefore $\mathrm{Im}(f \mid [a]_1) \subseteq [f(a)]_2$. As f is one-to-one, $f \mid [a]_1$ is one-to-one, too. If $c \in [f(a)]_2$, then $\varepsilon_2(c) = f(a)$ and so $\varepsilon_1(f^{-1}(c)) = (\varepsilon_1 \circ f^{-1})(c) = (f^{-1} \circ \varepsilon_2)(c) = f^{-1}(\varepsilon_2(c)) = f^{-1}(f(a)) = a$. It results that $f^{-1}(c) \in [a]_1$. Hence $f \mid [a]_1$ is onto.

b) $\Longrightarrow$ a) For any $a \in L_1$, we have $a \in [\varepsilon_1(a)]_1$. From the condition ii) of b), it obtains $f(a) \in [f(\varepsilon_1(a))]_2$, which implies that $(\varepsilon_2 \circ f)(a) = \varepsilon_2(f(a)) = f(\varepsilon_1(a)) = (f \circ \varepsilon_1)(a)$. Thus $f \circ \varepsilon_1 = \varepsilon_2 \circ f$. Since both L_1 and L_2 are canonical $\mathcal{E}$–lattices, we have $f(a \wedge_{\varepsilon_1} b) = f(\varepsilon_1(a \wedge_{\varepsilon_1} b)) = f(\varepsilon_1(a) \wedge_{\varepsilon_1} \varepsilon_1(b)) = f_0(\varepsilon_1(a) \wedge^\circ_{\varepsilon_1} \varepsilon_1(b)) = f_0(\varepsilon_1(a)) \wedge^\circ_{\varepsilon_2} f_0(\varepsilon_1(b)) = (f \circ \varepsilon_1)(a) \wedge_{\varepsilon_2} (f \circ \varepsilon_1)(b) = (\varepsilon_2 \circ f)(a) \wedge_{\varepsilon_2} (\varepsilon_2 \circ f)(b) = \varepsilon_2(f(a)) \wedge_{\varepsilon_2} \varepsilon_2(f(b)) = \varepsilon_2(f(a) \wedge_{\varepsilon_2} f(b)) = f(a) \wedge_{\varepsilon_2} f(b)$ and, in the same manner, $f(a \vee_{\varepsilon_1} b) = f(a) \vee_{\varepsilon_2} f(b)$, for all $a, b \in L_1$. Therefore f is an $\mathcal{E}$–lattice homomorphism. Clearly, the map f is one-to-one and onto and hence our proof if finished.

As a consequence of the above proposition, it obtains the next characterization of $\mathcal{E}$–lattice automorphisms of a canonical $\mathcal{E}$–lattice.

Corollary. *Let* $(L, \wedge_\varepsilon, \vee_\varepsilon)$ *be a canonical* $\mathcal{E}$*–lattice,* $\mathrm{Aut}(\mathrm{Fix}\,\varepsilon)$ *be the group consisting of all automorphisms of the lattice* $\mathrm{Fix}\,\varepsilon$ *and* $f : L \longrightarrow L$ *be a map.*

Then $f \in \mathrm{Aut}_{\mathcal{E}}(L)$ *if and only if* $f \mid \mathrm{Fix}\,\varepsilon \in \mathrm{Aut}(\mathrm{Fix}\,\varepsilon)$ *and* $f \mid [a] : [a] \longrightarrow [f(a)]$ *is one-to-one and onto, for each* $a \in \mathrm{Fix}\,\varepsilon$.

In the following we shall investigate the structure of the group $\mathrm{Aut}_{\mathcal{E}}(L)$ associated to a canonical $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$. Suppose that $\mathrm{Fix}\,\varepsilon=\{a_i \mid i\in I\}$. Let $S([a_i])$ be the symmetric group on the set $[a_i] = \{a \in L \mid \varepsilon(a) = a_i\}$ and $S'([a_i]) = \{u_i \in S([a_i]) \mid u_i(a_i) = a_i\}$ (of course, $S'([a_i])$ is a subgroup of $S([a_i]))$, $i \in I$. Denote by $\bigtimes_{i\in I} S'([a_i])$ the direct product of the groups $S'([a_i])$, $i \in I$. For every $(u_i)_{i\in I} \in \bigtimes_{i\in I} S'([a_i])$, we construct an $\mathcal{E}$–lattice automorphism f of L by $f \mid \mathrm{Fix}\,\varepsilon = 1_{\mathrm{Fix}\,\varepsilon}$ and $f \mid [a_i] = u_i$, $i \in I$. In this way we defined a map $\varphi : \bigtimes_{i\in I} S'([a_i]) \longrightarrow \mathrm{Aut}_{\mathcal{E}}(L)$. Moreover, note that φ is a group monomorphism.

On the other hand, by the previous corollary, every element $f \in \mathrm{Aut}_{\mathcal{E}}(L)$ induces a lattice automorphism $f \mid \mathrm{Fix}\,\varepsilon \in \mathrm{Aut}(\mathrm{Fix}\,\varepsilon)$. Thus we defined another map $\psi : \mathrm{Aut}_{\mathcal{E}}(L) \longrightarrow \mathrm{Aut}(\mathrm{Fix}\,\varepsilon)$, which is a group epimorphism. Also, it is easy to see that $\mathrm{Im}\,\varphi = \mathrm{Ker}\,\psi$, therefore we have proved the next result.

Proposition 2. *With the above notations, there exists an exact sequence:*

$$1 \longrightarrow \bigtimes_{i\in I} S'([a_i]) \xrightarrow{\varphi} \mathrm{Aut}_{\mathcal{E}}(L) \xrightarrow{\psi} \mathrm{Aut}(\mathrm{Fix}\,\varepsilon) \longrightarrow 1.$$

Remark that we identified an important normal subgroup of $\mathrm{Aut}_{\mathcal{E}}(L)$:

$$\mathrm{Aut}^0_{\mathcal{E}}(L) = \{f \in \mathrm{Aut}_{\mathcal{E}}(L) \mid f|\mathrm{Fix}\,\varepsilon = 1_{\mathrm{Fix}\,\varepsilon}\},$$

which is isomorphic to the direct product $\bigtimes_{i\in I} S'([a_i])$. A case when the group $\mathrm{Aut}_{\mathcal{E}}(L)$ itself is isomorphic to $\bigtimes_{i\in I} S'([a_i])$ is described by the following corollary.

Corollary 1. *Under the same notations as in Proposition* 2*, if the group* $\mathrm{Aut}(\mathrm{Fix}\,\varepsilon)$ *is trivial, then we have:*

$$\mathrm{Aut}_{\mathcal{E}}(L) \cong \bigtimes_{i\in I} S'([a_i]).$$

There exist many situations in which the group of all automorphisms of a lattice is trivial. One of them is obtained when the lattice is finite and fully ordered.

Corollary 2. *Let* $(L, \wedge_\varepsilon, \vee_\varepsilon)$ *be a canonical* $\mathcal{E}$*–lattice having a finite fully ordered lattice of fixed points* $\operatorname{Fix}\varepsilon = \{a_1, a_2, ..., a_n\}$. *Then the following group isomorphism holds:*

$$\operatorname{Aut}_{\mathcal{E}}(L) \cong \bigtimes_{i=1}^{n} S'([a_i]).$$

Moreover, if L itself is finite, we can estimate its number of $\mathcal{E}$–lattice automorphisms.

Corollary 3. *Let* $(L, \wedge_\varepsilon, \vee_\varepsilon)$ *be a finite canonical* $\mathcal{E}$*–lattice having a fully ordered lattice of fixed points* $\operatorname{Fix}\varepsilon = \{a_1, a_2, ..., a_n\}$. *If* $m_i = |[a_i]|$, $i = \overline{1, n}$, *then the following equality holds:*

$$|\operatorname{Aut}_{\mathcal{E}}(L)| = \prod_{i=1}^{n} (m_i - 1)!.$$

2.2. **Isomorphisms of subgroup $\mathcal{E}$–lattices.** In this section we investigate isomorphisms between subgroup $\mathcal{E}$–lattices.

Let G_1, G_2 be two groups and $L(G_1), L(G_2)$ be their subgroups lattices. Remind that G_1 and G_2 are called *L-isomorphic* if $L(G_1) \cong L(G_2)$ (a lattice isomorphism from $L(G_1)$ to $L(G_2)$ will be called an *L-isomorphism*). Now, let us consider the $\mathcal{E}$–lattice structure on $L(G_i)$, $i = 1, 2$. As we have seen above, in this situation the lattices of fixed points associated to $L(G_1)$, $L(G_2)$ are the normal subgroup lattices $N(G_1), N(G_2)$ of G_1 and G_2, respectively. We say that G_1 and G_2 are *$\mathcal{E}L$-isomorphic* if the $\mathcal{E}$–lattices $L(G_1)$ and $L(G_2)$ are isomorphic (an $\mathcal{E}$–lattice isomorphism from $L(G_1)$ to $L(G_2)$ will be called an *$\mathcal{E}L$-isomorphism*).

Our first goal is to establish some connections between these different types of isomorphisms from G_1 to G_2. Clearly, if $G_1 \cong G_2$, then G_1, G_2 are both L-isomorphic and $\mathcal{E}L$-isomorphic. Also, if G_1 and G_2 are $\mathcal{E}L$-isomorphic, then, by Proposition 1, 2.1, $N(G_1) \cong N(G_2)$, but they are not necessarily L-isomorphic. Conversely, $N(G_1) \cong N(G_2)$ does not imply that G_1 and G_2 are $\mathcal{E}L$-isomorphic (for example, take G_1 the quaternion group and G_2 the dihedral group of order 8). Moreover, even the lattice isomorphism $L(G_1) \cong L(G_2)$ does not assure that G_1 and G_2 are $\mathcal{E}L$-isomorphic (for example, take G_1 a finite elementary abelian p-group and G_2 the nonabelian P-group which is L-isomorphic to G_1 (see [7], page 11)).

The following result indicates us some classes of groups which are preserved by $\mathcal{E}L$-isomorphisms.

Proposition 1. *Let G_1 and G_2 be two $\mathcal{E}L$-isomorphic groups. If G_1 is a simple group or a Dedekind (in particular abelian) group, then G_2 is also simple or Dedekind, respectively.*

Proof. Let $f : L(G_1) \longrightarrow L(G_2)$ be an $\mathcal{E}L$-isomorphism.

If G_1 is simple, then $|N(G_1)| = 2$. Since $N(G_1) \cong N(G_2)$, it results that $|N(G_2)| = 2$ and hence G_2 is simple, too.

If G_1 is a Dedekind group, then any subgroup $H_1 \in L(G_1)$ is normal in G_1 and its congruence class $[H_1]_1$ consists only of H_1. Because f maps normal subgroups into normal subgroups and induces an one-to-one and onto map between the congruence classes $[H_1]_1$ and $[f(H_1)]_2$ of the $\mathcal{E}$–lattices $L(G_1)$ and $L(G_2)$, respectively, it obtains $[H_2]_2 = \{H_2\}$, for all $H_2 \in N(G_2)$. This implies that every subgroup of G_2 is normal and hence G_2 is also Dedekind.

Note that, for two groups of the above types, we are able to indicate some necessary and sufficient conditions in order to be $\mathcal{E}L$-isomorphic. In this way, two finite simple groups are $\mathcal{E}L$-isomorphic iff they have the same number of subgroups and two Dedekind (in particular abelian) groups are $\mathcal{E}L$-isomorphic iff they are L-isomorphic.

Next, we shall present a property satisfied by $\mathcal{E}L$-isomorphisms between two finite groups in the case when one of them is nilpotent.

Proposition 2. *Let G_1, G_2 be two finite groups, $\Phi(G_1), \Phi(G_2)$ be their Frattini subgroups and $f : L(G_1) \longrightarrow L(G_2)$ be an $\mathcal{E}L$-isomorphism. If G_1 is nilpotent, then $f(\Phi(G_1)) \supseteq \Phi(G_2)$.*

Proof. Let M_1 be a maximal subgroup of G_1. Since G_1 is nilpotent, it follows that M_1 is normal in G_1 and so $M_2 = f(M_1)$ is a normal subgroup of G_2. Let $H_2 \in L(G_2)$ such that $M_2 \subseteq H_2 \subseteq G_2$ and assume that $H_2 \neq G_2$. Then M_2 is contained in the core C_2 of H_2 in G_2. As f induces a lattice isomorphism between $N(G_1)$ and $N(G_2)$, it results that $C_1 = f^{-1}(C_2)$ is a normal subgroup of G_1 and $M_1 \subseteq C_1 \subseteq G_1$. By the maximality of M_1, it obtains $M_1 = C_1$ and therefore $M_2 = C_2$. This shows that $H_2 \in [M_2]_2$. But f induces also an one-to-one and onto map between $[M_1]_1 = \{M_1\}$ and $[M_2]_2$, thus $H_2 = M_2$. Because f maps the maximal subgroups of G_1 into maximal subgroups of G_2, we have $f(\Phi(G_1)) \supseteq \Phi(G_2)$.

Corollary. *Let G be a finite nilpotent group, $\Phi(G)$ be its Frattini subgroup and f be an $\mathcal{E}L$-automorphism of G. Then $\Phi(G)$ is a fixed point of f.*

Proof. By Proposition 2, we have $f(\Phi(G)) \supseteq \Phi(G)$. On the other hand, applying Proposition 2 to f^{-1}, it results that $f^{-1}(\Phi(G))) \supseteq \Phi(G)$ and therefore $\Phi(G) \supseteq f(\Phi(G))$. Hence the equality $f(\Phi(G)) = \Phi(G)$ holds.

Remark. Assume that G_1, G_2 are two finite groups and let $f : L(G_1) \longrightarrow L(G_2)$ be an $\mathcal{E}L$-isomorphism. By a well-known result of H. Heineken (see [4]), under the additional conditions that G_1 is a noncyclic p-group and the derived subgroup of G_2 is nilpotent, it obtains that G_2 is also a p-group of the same order as G_1. In this case f maps any principal series of G_1 into a principal series of G_2 and induces an one-to-one and onto map between the sets of maximal subgroups of G_1 and G_2. By Proposition 2, we have $f(\Phi(G_1)) = \Phi(G_2)$ and thus $|\Phi(G_1)| = |\Phi(G_2)|$. It follows that the vector spaces (over F_p) $G_1/\Phi(G_1)$ and $G_2/\Phi(G_2)$ have the same dimension. Hence G_1 and G_2 can be generated by exactly the same number of generators.

Let G_1, G_2 be two groups, $f : L(G_1) \longrightarrow L(G_2)$ be an $\mathcal{E}L$-isomorphism and H_1 be a (normal) subgroup of G_1. Since a normal subgroup of H_1 is not necessarily mapped by f into a normal subgroup of $f(H_1)$, f induces not an $\mathcal{E}L$-isomorphism between H_1 and $f(H_1)$. The situation is different with respect to the factor groups of our two groups, as shows the following lemma.

Lemma. *If G_1, G_2 are two groups, $f : L(G_1) \longrightarrow L(G_2)$ is an $\mathcal{E}L$-isomorphism and H_1 is a normal subgroup of G_1, then the map*

$$\bar{f} : L(G_1/H_1) \longrightarrow L(G_2/f(H_1))$$

defined by $\bar{f}(K_1/H_1) = f(K_1)/f(H_1)$, for all $K_1/H_1 \in L(G_1/H_1)$, is also an $\mathcal{E}L$-isomorphism.

In the hypothesis of the above lemma consider H_1 be the derived subgroup $D(G_1)$ of G_1. Then the groups $G_1/D(G_1)$ and $G_2/f(D(G_1))$ are $\mathcal{E}L$-isomorphic. But $G_1/D(G_1)$ is abelian and an $\mathcal{E}L$-isomorphism maps abelian groups into Dedekind groups (see Proposition 1, 2.1), therefore $G_2/f(D(G_1))$ is a Dedekind group L-isomorphic to $G_1/D(G_1)$. It is well-known (for example, see [7], Theorem 6, page 39) that a primary hamiltonian group cannot be L-isomorphic to an abelian group. Thus, under a supplementary condition of type

$(*)$ every hamiltonian quotient of G_2 is primary,

it follows that $G_2/f(D(G_1))$ is also abelian and therefore $D(G_2) \subseteq f(D(G_1))$ (mention that the author has not be able to decide if without a condition of type $(*)$ it obtains the commutativity of $G_2/f(D(G_1))$). Hence we have proved the next proposition.

Proposition 3. *Let G_1, G_2 be two groups, $D(G_1), D(G_2)$ be their derived subgroups and $f : L(G_1) \longrightarrow L(G_2)$ be an $\mathcal{E}L$-isomorphism. If G_2 satisfies the condition $(*)$, then $f(D(G_1)) \supseteq D(G_2)$.*

By Proposition 3, we can easily see that the following result holds.

Corollary. *Let G be a group which satisfies the condition $(*)$, $D(G)$ be its derived subgroup and f be an $\mathcal{E}L$-automorphism of G. Then $D(G)$ is a fixed point of f.*

As we have already seen in 2.1, an important normal subgroup of the group $\mathrm{Aut}_{\mathcal{E}}(L(G))$ associated to a group G is $\mathrm{Aut}^0_{\mathcal{E}}(L(G)) = \{f \in \mathrm{Aut}_{\mathcal{E}}(L(G)) \mid$

$f|N(G) = 1_{N(G)}\}$. We finish this section by indicating another two remarkable subgroups of $\mathrm{Aut}_{\mathcal{E}}(L(G))$:

- the subgroup $\mathrm{Aut}^1_{\mathcal{E}}(L(G))$ consisting of all L-automorphisms of G induced by group automorphisms;
- the subgroup $\mathrm{Aut}^2_{\mathcal{E}}(L(G))$ consisting of all L-automorphisms of G of the type $f_a : L(G) \longrightarrow L(G)$, $f_a(H) = H^a$, for all $H \in L(G)$ $(a \in G)$.

Note that $\mathrm{Aut}^2_{\mathcal{E}}(L(G))$ is a normal subgroup of $\mathrm{Aut}^1_{\mathcal{E}}(L(G))$ and, also, it is contained in $\mathrm{Aut}^0_{\mathcal{E}}(L(G))$.

Finally, we find these subgroups in two situations.

Examples.

1) For the symmetric group S_3 of degree 3, we have:
   $\mathrm{Aut}^0_{\mathcal{E}}(L(S_3)) = \mathrm{Aut}^1_{\mathcal{E}}(L(S_3)) = \mathrm{Aut}^2_{\mathcal{E}}(L(S_3)) \cong S_3$.
2) For the dihedral group D_4 of order 8, we have:
   $\mathrm{Aut}^0_{\mathcal{E}}(L(D_4)) \cong S_4,\ \mathrm{Aut}^1_{\mathcal{E}}(L(D_4)) \cong D_4,\ \mathrm{Aut}^2_{\mathcal{E}}(L(D_4)) \cong \mathbb{Z}_2$.

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