ON ISOMORPHISMS OF CANONICAL $\mathcal{E}$-LATTICES

MARIUS TĂRNĂUCEANU

Faculty of Mathematics, ”Al.I. Cuza” University, Iași, Romania
E-mail: tarnauc@uaic.ro

Abstract. The aim of the present paper is to study isomorphisms of canonical $\mathcal{E}$-lattices. Some interesting results are obtained in the particular case of isomorphisms between two subgroup $\mathcal{E}$-lattices.

Key Words and Phrases: canonical $\mathcal{E}$-lattices, subgroup $\mathcal{E}$-lattices, isomorphisms.


1. Preliminaries

The starting point for our discussion is given by the paper [10], where there is introduced the category of $\mathcal{E}$–lattices and there are made some basic constructions in this category. Given a nonempty set $L$ and a map $\varepsilon : L \to L$, we denote by $\text{Ker} \varepsilon$ the kernel of $\varepsilon$ (i.e. $\text{Ker} \varepsilon = \{ (a, b) \in L \times L \mid \varepsilon(a) = \varepsilon(b) \}$), by $\text{Im} \varepsilon$ the image of $\varepsilon$ (i.e. $\text{Im} \varepsilon = \{ \varepsilon(a) \mid a \in L \}$) and by $\text{Fix} \varepsilon$ the set consisting of all fixed points of $\varepsilon$ (i.e. $\text{Fix} \varepsilon = \{ a \in L \mid \varepsilon(a) = a \}$). We say that $L$ is an $\mathcal{E}$–lattice (relative to $\varepsilon$) if there exist two binary operations $\land_{\varepsilon}$, $\lor_{\varepsilon}$ on $L$ which satisfy the following properties:

a) $a \land_{\varepsilon} (b \land_{\varepsilon} c) = (a \land_{\varepsilon} b) \land_{\varepsilon} c$, $a \lor_{\varepsilon} (b \lor_{\varepsilon} c) = (a \lor_{\varepsilon} b) \lor_{\varepsilon} c$, for all $a, b, c \in L$;

b) $a \land_{\varepsilon} b = b \land_{\varepsilon} a$, $a \lor_{\varepsilon} b = b \lor_{\varepsilon} a$, for all $a, b \in L$;

c) $a \land_{\varepsilon} a = a \lor_{\varepsilon} a = \varepsilon(a)$, for any $a \in L$;

d) $a \land_{\varepsilon} (a \lor_{\varepsilon} b) = a \lor_{\varepsilon} (a \land_{\varepsilon} b) = \varepsilon(a)$, for all $a, b \in L$.

Clearly, in an $\mathcal{E}$–lattice $L$ (relative to $\varepsilon$) the map $\varepsilon$ is idempotent and $\text{Im} \varepsilon = \text{Fix} \varepsilon$. Moreover, the set $\text{Fix} \varepsilon$ is closed under the binary operations $\land_{\varepsilon}$, $\lor_{\varepsilon}$ and, denoting by $\land_{\varepsilon}^0$, $\lor_{\varepsilon}^0$ the restrictions of $\land_{\varepsilon}$, $\lor_{\varepsilon}$ to $\text{Fix} \varepsilon$, we have that $(\text{Fix} \varepsilon, \land_{\varepsilon}^0, \lor_{\varepsilon}^0)$ is a lattice. The connection between the $\mathcal{E}$–lattice concept and the lattice concept is very powerful. So, if $(L, \land_{\varepsilon}, \lor_{\varepsilon})$ is an $\mathcal{E}$–lattice and $\sim$
is an equivalence relation on $L$ such that $\sim \subseteq \ker \varepsilon$, then the factor set $L/\sim$ is a lattice isomorphic to the lattice $\text{Fix}\varepsilon$. Conversely, if $L$ is a nonvoid set and $\sim$ is an equivalence relation on $L$ having the property that the factor set $L/\sim$ is a lattice, then the set $L$ can be endowed with an $\mathcal{E}$–lattice structure (relative to a map $\varepsilon : L \to L$) such that $\sim \subseteq \ker \varepsilon$ and $L/\sim \cong \text{Fix}\varepsilon$.

We say that an $\mathcal{E}$–lattice $(L, \wedge, \vee)$ is a canonical $\mathcal{E}$–lattice if $a \wedge \varepsilon b, a \vee \varepsilon b \in \text{Fix}\varepsilon$, for all $a, b \in L$. Three fundamental types of canonical $\mathcal{E}$–lattices have been identified in [10]. One of the most important example of a canonical $\mathcal{E}$–lattice is constituted by the lattice $L(G)$ of all subgroups of a group $G$ (called the subgroup $\mathcal{E}$–lattice of $G$). Here the map $\varepsilon$ is defined by $\varepsilon(H) = H_G$ (where $H_G$ is the core of $H$ in $G$), for any $H \in L(G)$ and the binary operations $\wedge, \vee$ are defined by $H_1 \wedge \varepsilon H_2 = \varepsilon(H_1 \cap \varepsilon(H_2)$, $H_1 \vee \varepsilon H_2 = \varepsilon(H_1)\varepsilon(H_2)$, for all $H_1, H_2 \in L(G)$. Mention that in this situation the lattice $\text{Fix}\varepsilon$ is just the normal subgroup lattice $N(G)$ of $G$.

Let $(L_1, \wedge, \vee_1)$ and $(L_2, \wedge, \vee_2)$ be two lattices. A map $f : L_1 \to L_2$ is called an $\mathcal{E}$–lattice homomorphism if:

\begin{itemize}
  \item[a)] $f \circ \varepsilon_1 = \varepsilon_2 \circ f$;
  \item[b)] for all $a, b \in L_1$, we have:
    \begin{itemize}
      \item[i)] $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$;
      \item[i)] $f(a \vee_1 b) = f(a) \vee_2 f(b)$.
    \end{itemize}
\end{itemize}

Moreover, if the map $f$ is one-to-one and onto, then we say that it is an $\mathcal{E}$–lattice isomorphism. An $\mathcal{E}$–lattice isomorphism of an $\mathcal{E}$–lattice into itself is called an $\mathcal{E}$–lattice automorphism. For an $\mathcal{E}$–lattice $L$, we shall denote by $\text{Aut}_\mathcal{E}(L)$ the group consisting of all $\mathcal{E}$–lattice automorphisms of $L$.

2. Main results

2.1. Isomorphisms of canonical $\mathcal{E}$–lattices. In this section we present some general results concerned to isomorphisms between canonical $\mathcal{E}$–lattices.

Let $(L_1, \wedge, \vee_1)$ and $(L_2, \wedge, \vee_2)$ be two canonical $\mathcal{E}$–lattices. For every element $a \in L_i$, we denote by $[a]_i$ the equivalence class of $a$ modulo $\ker \varepsilon_i$ (i.e. $[a]_i = \{b \in L_i \mid \varepsilon_i(b) = \varepsilon_i(a)\}$, $i = 1, 2$. First of all, we give a characterization of $\mathcal{E}$–lattice isomorphisms from $L_1$ to $L_2$. 

Proposition 1. Let \((L_1, \wedge_{\varepsilon_1}, \vee_{\varepsilon_1})\) and \((L_2, \wedge_{\varepsilon_2}, \vee_{\varepsilon_2})\) be two canonical \(\varepsilon\)-lattices and \(f : L_1 \rightarrow L_2\) be a map. Then the following two conditions are equivalent:

a) \(f\) is an \(\varepsilon\)-lattice isomorphism.

b) i) The restriction \(f_0\) of \(f\) to the set \(\text{Fix} \varepsilon_1\) is a lattice isomorphism from \(\text{Fix} \varepsilon_1\) to \(\text{Fix} \varepsilon_2\).

ii) \(f \mid [a]_1 : [a]_1 \rightarrow [f(a)]_2\) is one-to-one and onto, for each \(a \in \text{Fix} \varepsilon_1\).

Proof. a) \(\Longrightarrow\) b) Let \(a \in \text{Fix} \varepsilon_1\). Then \(\varepsilon_1(a) = a\) and so \(f_0(a) = f(a) = f(\varepsilon_1(a)) = (f \circ \varepsilon_1)(a) = (\varepsilon_2 \circ f)(a) = \varepsilon_2(f(a)) \in \text{Fix} \varepsilon_2\). Thus \(\text{Im} f_0 \subseteq \text{Fix} \varepsilon_2\).

For all \(a, b \in L_1\), we have \(f_0(a \wedge_{\varepsilon_1} b) = f(a \wedge_{\varepsilon_1} b) = f(a) \wedge_{\varepsilon_2} f(b) = f_0(a) \wedge_{\varepsilon_2} f_0(b) = f_0(a \wedge_{\varepsilon_2} f_0(b)\text{ and } f_0(a) \vee_{\varepsilon_1} b = f(a) \vee_{\varepsilon_2} f(b) = f_0(a) \vee_{\varepsilon_2} f_0(b) = f_0(a) \vee_{\varepsilon_2} f_0(b)\), which show that \(f_0\) is a lattice homomorphism. Since \(f\) is one-to-one and onto, it is clear that \(f_0\) has the same properties and hence i) holds.

Now, let \(b \in [a]_1\). Then \(\varepsilon_1(b) = \varepsilon_1(a) = a\), which implies that \(\varepsilon_2(f(b)) = (\varepsilon_2 \circ f)(b) = (f \circ \varepsilon_1)(b) = f(\varepsilon_1(b)) = f(a)\). Therefore \(\text{Im}(f \mid [a]_1) \subseteq [f(a)]_2\). As \(f\) is one-to-one, \(f \mid [a]_1\) is one-to-one, too. If \(c \in [f(a)]_2\), then \(\varepsilon_2(c) = f(a)\) and so \(\varepsilon_1(f^{-1}(c)) = (\varepsilon_1 \circ f^{-1})(c) = (f^{-1} \circ \varepsilon_2)(c) = f^{-1}(\varepsilon_2(c)) = f^{-1}(f(a)) = a\).

It results that \(f^{-1}(c) \in [a]_1\). Hence \(f \mid [a]_1\) is onto.

b) \(\Longrightarrow\) a) For any \(a \in L_1\), we have \(a \in [\varepsilon_1(a)]_1\). From the condition ii) of b), it obtains \(f(a) \in [f(\varepsilon_1(a))]_2\), which implies that \((\varepsilon_2 \circ f)(a) = \varepsilon_2(f(a)) = f(\varepsilon_1(a)) = (f \circ \varepsilon_1)(a)\). Thus \(f \circ \varepsilon_1 = \varepsilon_2 \circ f\). Since both \(L_1\) and \(L_2\) are canonical \(\varepsilon\)-lattices, we have \(f(a \wedge_{\varepsilon_1} b) = f(\varepsilon_1(a \wedge_{\varepsilon_1} b)) = f(\varepsilon_1(a) \wedge_{\varepsilon_1} \varepsilon_1(b)) = f_0(\varepsilon_1(a) \wedge_{\varepsilon_1} \varepsilon_1(b)) = f_0(\varepsilon_1(a)) \wedge_{\varepsilon_2} f_0(\varepsilon_1(b)) = (f \circ \varepsilon_1)(a) \wedge_{\varepsilon_2} (f \circ \varepsilon_1)(b) = (\varepsilon_2 \circ f)(a) \wedge_{\varepsilon_2} (\varepsilon_2 \circ f)(b) = \varepsilon_2(f(a)) \wedge_{\varepsilon_2} \varepsilon_2(f(b)) = \varepsilon_2(f(a) \wedge_{\varepsilon_2} f(b)) = f(a) \wedge_{\varepsilon_2} f(b)\) and, in the same manner, \(f(a) \vee_{\varepsilon_1} b = f(a) \vee_{\varepsilon_2} f(b)\), for all \(a, b \in L_1\). Therefore \(f\) is an \(\varepsilon\)-lattice homomorphism. Clearly, the map \(f\) is one-to-one and onto and hence our proof if finished.

As a consequence of the above proposition, it obtains the next characterization of \(\varepsilon\)-lattice automorphisms of a canonical \(\varepsilon\)-lattice.

Corollary. Let \((L, \wedge, \vee, \varepsilon)\) be a canonical \(\varepsilon\)-lattice, \(\text{Aut}(\text{Fix} \varepsilon)\) be the group consisting of all automorphisms of the lattice \(\text{Fix} \varepsilon\) and \(f : L \rightarrow L\) be a map.
Then $f \in \text{Aut}_{\mathcal{E}}(L)$ if and only if $f \mid \text{Fix} \varepsilon \in \text{Aut}(\text{Fix} \varepsilon)$ and $f \mid [a] : [a] \rightarrow [f(a)]$ is one-to-one and onto, for each $a \in \text{Fix} \varepsilon$.

In the following we shall investigate the structure of the group $\text{Aut}_{\mathcal{E}}(L)$ associated to a canonical $\mathcal{E}$–lattice $(L, \wedge_{\varepsilon}, \vee_{\varepsilon})$. Suppose that $\text{Fix} \varepsilon = \{a_i \mid i \in I\}$. Let $S([a_i])$ be the symmetric group on the set $[a_i] = \{a \in L \mid \varepsilon(a) = a_i\}$ and $S'([a_i]) = \{u_i \in S([a_i]) \mid u_i(a_i) = a_i\}$ (of course, $S'([a_i])$ is a subgroup of $S([a_i])$, $i \in I$). Denote by $\prod_{i \in I} S'([a_i])$ the direct product of the groups $S'([a_i])$, $i \in I$. For every $(u_i)_{i \in I} \in \prod_{i \in I} S'([a_i])$, we construct an $\mathcal{E}$–lattice automorphism $f$ of $L$ by $f \mid \text{Fix} \varepsilon = 1_{\text{Fix} \varepsilon}$ and $f \mid [a_i] = u_i$, $i \in I$. In this way we defined a map $\varphi : \prod_{i \in I} S'([a_i]) \rightarrow \text{Aut}_{\mathcal{E}}(L)$. Moreover, note that $\varphi$ is a group monomorphism.

On the other hand, by the previous corollary, every element $f \in \text{Aut}_{\mathcal{E}}(L)$ induces a lattice automorphism $f \mid \text{Fix} \varepsilon \in \text{Aut}(\text{Fix} \varepsilon)$. Thus we defined another map $\psi : \text{Aut}_{\mathcal{E}}(L) \rightarrow \text{Aut}(\text{Fix} \varepsilon)$, which is a group epimorphism. Also, it is easy to see that $\text{Im} \varphi = \text{Ker} \psi$, therefore we have proved the next result.

**Proposition 2.** With the above notations, there exists an exact sequence:

$$1 \rightarrow \prod_{i \in I} S'([a_i]) \overset{\varphi}{\rightarrow} \text{Aut}_{\mathcal{E}}(L) \overset{\psi}{\rightarrow} \text{Aut}(\text{Fix} \varepsilon) \rightarrow 1.$$ 

Remark that we identified an important normal subgroup of $\text{Aut}_{\mathcal{E}}(L)$:

$$\text{Aut}_{\mathcal{E}}^0(L) = \{ f \in \text{Aut}_{\mathcal{E}}(L) \mid f|_{\text{Fix} \varepsilon} = 1_{\text{Fix} \varepsilon} \},$$

which is isomorphic to the direct product $\prod_{i \in I} S'([a_i])$. A case when the group $\text{Aut}_{\mathcal{E}}(L)$ itself is isomorphic to $\prod_{i \in I} S'([a_i])$ is described by the following corollary.

**Corollary 1.** Under the same notations as in Proposition 2, if the group $\text{Aut}(\text{Fix} \varepsilon)$ is trivial, then we have:

$$\text{Aut}_{\mathcal{E}}(L) \cong \prod_{i \in I} S'([a_i]).$$
There exist many situations in which the group of all automorphisms of a lattice is trivial. One of them is obtained when the lattice is finite and fully ordered.

**Corollary 2.** Let \((L, \wedge, \vee)\) be a canonical \(E\)-lattice having a finite fully ordered lattice of fixed points \(\text{Fix} \, \varepsilon = \{a_1, a_2, \ldots, a_n\}\). Then the following group isomorphism holds:

\[
\text{Aut}_E(L) \cong \prod_{i=1}^{n} S'(\{a_i\}).
\]

Moreover, if \(L\) itself is finite, we can estimate its number of \(E\)-lattice automorphisms.

**Corollary 3.** Let \((L, \wedge, \vee)\) be a finite canonical \(E\)-lattice having a fully ordered lattice of fixed points \(\text{Fix} \, \varepsilon = \{a_1, a_2, \ldots, a_n\}\). If \(m_i = |\{a_i\}|, i = 1, n\), then the following equality holds:

\[
|\text{Aut}_E(L)| = \prod_{i=1}^{n} (m_i - 1)!. 
\]

### 2.2. Isomorphisms of subgroup \(E\)-lattices

In this section we investigate isomorphisms between subgroup \(E\)-lattices.

Let \(G_1, G_2\) be two groups and \(L(G_1), L(G_2)\) be their subgroups lattices. Remind that \(G_1\) and \(G_2\) are called \(L\)-isomorphic if \(L(G_1) \cong L(G_2)\) (a lattice isomorphism from \(L(G_1)\) to \(L(G_2)\) will be called an \(L\)-isomorphism). Now, let us consider the \(E\)-lattice structure on \(L(G_i), i = 1, 2\). As we have seen above, in this situation the lattices of fixed points associated to \(L(G_1), L(G_2)\) are the normal subgroup lattices \(N(G_1), N(G_2)\) of \(G_1\) and \(G_2\), respectively. We say that \(G_1\) and \(G_2\) are \(EL\)-isomorphic if the \(E\)-lattices \(L(G_1)\) and \(L(G_2)\) are isomorphic (an \(E\)-lattice isomorphism from \(L(G_1)\) to \(L(G_2)\) will be called an \(EL\)-isomorphism).

Our first goal is to establish some connections between these different types of isomorphisms from \(G_1\) to \(G_2\). Clearly, if \(G_1 \cong G_2\), then \(G_1, G_2\) are both \(L\)-isomorphic and \(EL\)-isomorphic. Also, if \(G_1\) and \(G_2\) are \(EL\)-isomorphic,
then, by Proposition 1, 2.1, $N(G_1) \cong N(G_2)$, but they are not necessarily $L$-isomorphic. Conversely, $N(G_1) \cong N(G_2)$ does not imply that $G_1$ and $G_2$ are $\mathcal{EL}$-isomorphic (for example, take $G_1$ the quaternion group and $G_2$ the dihedral group of order 8). Moreover, even the lattice isomorphism $L(G_1) \cong L(G_2)$ does not assure that $G_1$ and $G_2$ are $\mathcal{EL}$-isomorphic (for example, take $G_1$ a finite elementary abelian $p$-group and $G_2$ the nonabelian $P$-group which is $L$-isomorphic to $G_1$ (see [7], page 11)).

The following result indicates us some classes of groups which are preserved by $\mathcal{EL}$-isomorphisms.

**Proposition 1.** Let $G_1$ and $G_2$ be two $\mathcal{EL}$-isomorphic groups. If $G_1$ is a simple group or a Dedekind (in particular abelian) group, then $G_2$ is also simple or Dedekind, respectively.

**Proof.** Let $f : L(G_1) \rightarrow L(G_2)$ be an $\mathcal{EL}$-isomorphism.

If $G_1$ is simple, then $|N(G_1)| = 2$. Since $N(G_1) \cong N(G_2)$, it results that $|N(G_2)| = 2$ and hence $G_2$ is simple, too.

If $G_1$ is a Dedekind group, then any subgroup $H_1 \in L(G_1)$ is normal in $G_1$ and its congruence class $[H_1]_1$ consists only of $H_1$. Because $f$ maps normal subgroups into normal subgroups and induces an one-to-one and onto map between the congruence classes $[H_1]_1$ and $[f(H_1)]_2$ of the $\mathcal{E}$–lattices $L(G_1)$ and $L(G_2)$, respectively, it obtains $[H_2]_2 = \{H_2\}$, for all $H_2 \in N(G_2)$. This implies that every subgroup of $G_2$ is normal and hence $G_2$ is also Dedekind.

Note that, for two groups of the above types, we are able to indicate some necessary and sufficient conditions in order to be $\mathcal{EL}$-isomorphic. In this way, two finite simple groups are $\mathcal{EL}$-isomorphic iff they have the same number of subgroups and two Dedekind (in particular abelian) groups are $\mathcal{EL}$-isomorphic iff they are $L$-isomorphic.

Next, we shall present a property satisfied by $\mathcal{EL}$-isomorphisms between two finite groups in the case when one of them is nilpotent.
Proposition 2. Let $G_1, G_2$ be two finite groups, $\Phi(G_1), \Phi(G_2)$ be their Frattini subgroups and $f : L(G_1) \rightarrow L(G_2)$ be an $\mathcal{E}L$-isomorphism. If $G_1$ is nilpotent, then $f(\Phi(G_1)) \supseteq \Phi(G_2)$.

Proof. Let $M_1$ be a maximal subgroup of $G_1$. Since $G_1$ is nilpotent, it follows that $M_1$ is normal in $G_1$ and so $M_2 = f(M_1)$ is a normal subgroup of $G_2$. Let $H_2 \in L(G_2)$ such that $M_2 \subseteq H_2 \subseteq G_2$ and assume that $H_2 \neq G_2$. Then $M_2$ is contained in the core $C_2$ of $H_2$ in $G_2$. As $f$ induces a lattice isomorphism between $N(G_1)$ and $N(G_2)$, it results that $C_1 = f^{-1}(C_2)$ is a normal subgroup of $G_1$ and $M_1 \subseteq C_1 \subseteq G_1$. By the maximality of $M_1$, it obtains $M_1 = C_1$ and therefore $M_2 = C_2$. This shows that $H_2 \in [M_2]_2$. But $f$ induces also an one-to-one and onto map between $[M_1]_1 = \{M_1\}$ and $[M_2]_2$, thus $H_2 = M_2$. Because $f$ maps the maximal subgroups of $G_1$ into maximal subgroups of $G_2$, we have $f(\Phi(G_1)) \supseteq \Phi(G_2)$.

Corollary. Let $G$ be a finite nilpotent group, $\Phi(G)$ be its Frattini subgroup and $f$ be an $\mathcal{E}L$-automorphism of $G$. Then $\Phi(G)$ is a fixed point of $f$.

Proof. By Proposition 2, we have $f(\Phi(G)) \supseteq \Phi(G)$. On the other hand, applying Proposition 2 to $f^{-1}$, it results that $f^{-1}(\Phi(G)) \supseteq \Phi(G)$ and therefore $\Phi(G) \supseteq f(\Phi(G))$. Hence the equality $f(\Phi(G)) = \Phi(G)$ holds.

Remark. Assume that $G_1, G_2$ are two finite groups and let $f : L(G_1) \rightarrow L(G_2)$ be an $\mathcal{E}L$-isomorphism. By a well-known result of H. Heineken (see [4]), under the additional conditions that $G_1$ is a noncyclic $p$-group and the derived subgroup of $G_2$ is nilpotent, it obtains that $G_2$ is also a $p$-group of the same order as $G_1$. In this case $f$ maps any principal series of $G_1$ into a principal series of $G_2$ and induces an one-to-one and onto map between the sets of maximal subgroups of $G_1$ and $G_2$. By Proposition 2, we have $f(\Phi(G_1)) = \Phi(G_2)$ and thus $|\Phi(G_1)| = |\Phi(G_2)|$. It follows that the vector spaces (over $F_p$) $G_1/\Phi(G_1)$ and $G_2/\Phi(G_2)$ have the same dimension. Hence $G_1$ and $G_2$ can be generated by exactly the same number of generators.

Let $G_1, G_2$ be two groups, $f : L(G_1) \rightarrow L(G_2)$ be an $\mathcal{E}L$-isomorphism and $H_1$ be a (normal) subgroup of $G_1$. Since a normal subgroup of $H_1$ is not necessarily mapped by $f$ into a normal subgroup of $f(H_1)$, $f$ induces not
an $\mathcal{EL}$-isomorphism between $H_1$ and $f(H_1)$. The situation is different with respect to the factor groups of our two groups, as shows the following lemma.

**Lemma.** If $G_1, G_2$ are two groups, $f : L(G_1) \rightarrow L(G_2)$ is an $\mathcal{EL}$-isomorphism and $H_1$ is a normal subgroup of $G_1$, then the map

$$\bar{f} : L(G_1/H_1) \rightarrow L(G_2/f(H_1))$$

defined by $\bar{f}(K_1/H_1) = f(K_1)/f(H_1)$, for all $K_1/H_1 \in L(G_1/H_1)$, is also an $\mathcal{EL}$-isomorphism.

In the hypothesis of the above lemma consider $H_1$ be the derived subgroup $D(G_1)$ of $G_1$. Then the groups $G_1/D(G_1)$ and $G_2/f(D(G_1))$ are $\mathcal{EL}$-isomorphic. But $G_1/D(G_1)$ is abelian and an $\mathcal{EL}$-isomorphism maps abelian groups into Dedekind groups (see Proposition 1, 2.1), therefore $G_2/f(D(G_1))$ is a Dedekind group $L$-isomorphic to $G_1/D(G_1)$. It is well-known (for example, see [7], Theorem 6, page 39) that a primary hamiltonian group cannot be $L$-isomorphic to an abelian group. Thus, under a supplementary condition of type

(*)  every hamiltonian quotient of $G_2$ is primary,

it follows that $G_2/f(D(G_1))$ is also abelian and therefore $D(G_2) \subseteq f(D(G_1))$ (mention that the author has not be able to decide if without a condition of type (*) it obtains the commutativity of $G_2/f(D(G_1))$). Hence we have proved the next proposition.

**Proposition 3.** Let $G_1, G_2$ be two groups, $D(G_1), D(G_2)$ be their derived subgroups and $f : L(G_1) \rightarrow L(G_2)$ be an $\mathcal{EL}$-isomorphism. If $G_2$ satisfies the condition $(*)$, then $f(D(G_1)) \supseteq D(G_2)$.

By Proposition 3, we can easily see that the following result holds.

**Corollary.** Let $G$ be a group which satisfies the condition $(*)$, $D(G)$ be its derived subgroup and $f$ be an $\mathcal{EL}$-automorphism of $G$. Then $D(G)$ is a fixed point of $f$.

As we have already seen in 2.1, an important normal subgroup of the group $\text{Aut}_{\mathcal{E}}(L(G))$ associated to a group $G$ is $\text{Aut}_{\mathcal{E}}^0(L(G)) = \{ f \in \text{Aut}_{\mathcal{E}}(L(G))$ |
f|N(G) = 1_{N(G)}\}. We finish this section by indicating another two remarkable subgroups of Aut_{E}(L(G)):

- the subgroup Aut_{1}{E}(L(G)) consisting of all L-automorphisms of G induced by group automorphisms;
- the subgroup Aut_{2}{E}(L(G)) consisting of all L-automorphisms of G of the type \(f_a : L(G) \rightarrow L(G), f_a(H) = H^a\), for all \(H \in L(G)\) (\(a \in G\)).

Note that Aut_{1}{E}(L(G)) is a normal subgroup of Aut_{2}{E}(L(G)) and, also, it is contained in Aut_{0}{E}(L(G)).

Finally, we find these subgroups in two situations.

Examples.

1) For the symmetric group \(S_3\) of degree 3, we have:

\[\text{Aut}_{0}{E}(L(S_3)) = \text{Aut}_{1}{E}(L(S_3)) = \text{Aut}_{2}{E}(L(S_3)) \cong S_3.\]

2) For the dihedral group \(D_4\) of order 8, we have:

\[\text{Aut}_{0}{E}(L(D_4)) \cong S_4, \text{ Aut}_{1}{E}(L(D_4)) \cong D_4, \text{ Aut}_{2}{E}(L(D_4)) \cong \mathbb{Z}_2.\]

References


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