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SOME FIXED POINT THEOREMS IN BANACH SPACES FOR THREE WEAKLY COMPATIBLE MAPPINGS

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Abstract. A fixed point theorem for three weakly compatible mappings on a Banach space is proved. An application is also given for the solvability of certain non-linear functional equations. The results generalize the corresponding theorems in [4].
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1. Introduction

Let F and G be two mappings of a metric space (X, d) into itself. Pathak [3] defined F and G to be weakly compatible mappings with respect to G if and only if whenever

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = t \in X,$$

$$\lim_{n\to\infty} d(FGx_n, GFx_n) \ \leq d(Ft, Gt)$$

for all sequences $\{x_n\}$ in X, and

$$d(Ft, Gt) \leq \lim_{n\to\infty} d(Gt, GFx_n)$$

for at least one sequence $\{x_n\}$ in X.

In [4], the authors obtained some results on common fixed points for two weakly compatible mappings on a Banach space. Their results generalized and improved some results in [1] and [2].

In this paper, we extend the results of [4] for three weakly compatible mappings on a Banach space.

The following lemma in [3] is useful in the sequel.

Lemma 1. *Let F and G be mappings of a metric space (X,d) into itself which are weakly compatible with respect to G.*

(P1) If $Ft = Gt$, then $FGt = GFt$.

(P2) Suppose that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = t \ for\ some\ t\ in\ X.$$

Then

(a) If F is continuous at t, then

$$\lim_{n\to\infty} d(GFx_n, Ft) \le d(Ft, Gt).$$

(b) If F and G are continuous at t, then

$$Ft = Gt \ and\ FGt = GFt.$$

2. Main Results

We prove the following result.

Theorem 1. *Let X be a Banach space and let F, G and H be three self mappings on X satisfying the following conditions:*

$$(1-k)G(X) + kF(X) \subset G(X),\ \forall\ k \in (0,1), \tag{1}$$

$$(1-\acute{k})G(X) + \acute{k}H(X) \subset G(X),\ \forall\ \acute{k} \in (0,1), \tag{2}$$

$$\{F,G\} \text{ and } \{H,G\} \text{ are weakly compatible pairs with respect to } G, \tag{3}$$

$$\|Fx - Hy\|^p \le \Phi(max\{\|Gx - Gy\|^p,\ \|Gx - Fx\|^p,$$
$$\|Gy - Hy\|^p,\ \|Gx - Hy\|^p,\ \|Gy - Fx\|^p\}), \tag{4}$$

for all $x, y \in X$, where $p > 0$ and the function Φ satisfies the following conditions:

(c) $\Phi : [0,\infty) \to [0,\infty)$ is nondecreasing and right continuous.

(d) For every $\alpha > 0$, $\Phi(\alpha) < \alpha$.

If for some x_0 in X, the sequence $\{x_n\}$ defined by

$$Gx_{2n+1} = (1 - c_{2n})Gx_{2n} + c_{2n}Fx_{2n}, \tag{5}$$

$$Gx_{2n+2} = (1 - c_{2n+1})Gx_{2n+1} + c_{2n+1}Hx_{2n+1}, \tag{6}$$

$n = 0, 1, 2, ...,$ where $c_0 = 1$ and $0 < c_n \le 1$ and $\lim_{n\to\infty} c_n = h > 0$,

converges to a point z in X and if G is continuous at z, then z is a coincidence point of F, G and H.

Proof. Note first of all that the points x_n in the theorem exist because of conditions (1) and (2). Now let z be a point in X such that $\lim_{n\to\infty} x_n = z$.

Since G is continuous at z, we have $\lim_{n\to\infty} Gx_n = Gz$
and so from (5) we have

$$\begin{aligned}\lim_{n\to\infty} Fx_{2n} &= \lim_{n\to\infty} C_{2n}^{-1}[Gx_{2n+1} - (1 - C_{2n})Gx_{2n}] \\ &= h^{-1}[Gz - (1-h)Gz] = Gz.\end{aligned}$$

It follows similarly, from (6) that $\lim_{n\to\infty} Hx_{2n+1} = Gz$.

We will now show that $Fz = Gz = Hz$. From (4) we have

$$\begin{aligned}\|Fx_{2n} - Hz\|^p \le\ & \Phi(max\{\|Gx_{2n} - Gz\|^p, \|Gx_{2n} - Fx_{2n}\|^p, \|Gz - Hz\|^p, \\ & \|Gx_{2n} - Hz\|^p,\ \|Gz - Fx_{2n}\|^p\}).\end{aligned}$$

Letting n tend to infinity we get

$$\|Gz - Hz\|^p \le \Phi(max\{0,\ 0,\ \|Gz - Hz\|^p,\ \|Gz - Hz\|^p,\ 0\}),$$

we get a contradiction if $\|Gz - Hz\|^p > 0$, and so $\|Gz - Hz\|^p = 0$.

Thus $Gz = Hz$ and so z is a coincidence point of G and H.

Now suppose that $Fz \ne Gz$ so that for large enough $n, Fz \ne Gx_{2n}$.

Then using inequality (4) we have

$$\begin{aligned}\|Fz - Hx_{2n+1}\|^p \le \Phi(max\{&\|Gz - Gx_{2n+1}\|^p, \|Gz - Fz\|^p,\ \|Gx_{2n+1} - Hx_{2n+1}\|^p, \\ &\|Gz - Hx_{2n+1}\|^p, \|Gx_{2n+1} - Fz\|^p\}).\end{aligned}$$

Letting n tend to infinity, it follows that

$$\begin{aligned}\|Fz - Gz\|^p &\le \Phi(max\{0,\ \|Gz - Fz\|^p,\ 0,\ 0,\ \|Gz - Fz\|^p\}) \\ &= \Phi(\|Gz - Fz\|^p) < \|Gz - Fz\|^p,\end{aligned}$$

a contradiction. Thus $Gz = Fz = Hz$.

From (3), since F and G are weakly compatible with respect to G and $Fz = Gz$, we have $FGz = GFz$ by Lemma 1.

Similarly, $HGz = GHz$ since $Gz = Hz$ and H, G are weakly compatible with respect to G.

Hence, using (4), we have

$$\begin{aligned}\|F^2z - Hz\|^p &\le \Phi(max\{\|GFz - Gz\|^p, \|GFz - F^2z\|^p, \|Gz - Hz\|^p,\\ &\qquad \|GFz - Hz\|^p,\ \|Gz - F^2z\|^p\})\\ &= \Phi(max\{\|F^2z - Hz\|^p,\ 0,\ 0, \|F^2z - Fz\|^p,\ \|Hz - F^2z\|^p),\end{aligned}$$

which implies that, $FFz = Hz = Fz = FHz = HFz = Gz = GFz$. So $Fz = u$ is common fixed point of F, G and H.

Let v be a second common fixed point of F, G and H.

By (4), we have

$$\|u - v\|^p = \|Fu - Hv\|^p$$

$$\le \Phi(max\{\|Gu-Gv\|^p,\ \|Gu-Fu\|^p,\ \|Gv-Hv\|^p,\ \|Gu-Fv\|^p,\ \|Gv-Hu\|^p\})$$

$$= \Phi(max\{\|u - v\|^p,\ 0,\ 0,\ \|u - v\|^p,\ \|u - v\|^p\}),$$

which implies that $u = v$.

Completing the proof of the theorem.

We next investigate the solvability of a certain nonlinear functional equation in a Banach space.

Theorem 2. *Let $\{f_n\}$ and $\{g_n\}$ be sequences of elements in a Banach space X. Let $\{v_n\}$ be the unique solution of the system of equations $u - FGu = f_n$ and $u - HGu = g_n$, where F, G and H are self mappings on X satisfying the following conditions:*

(g) $\{F, G\}$ and $\{H, G\}$ are weakly compatible with respect to G,

(h) $F^2 = G^2 = H^2 = I$, where I denotes the identity mapping and

(i) $\|Fx - Hy\|^2 \le q\ max\ \{\|Gx - Gy\|^2,\ \|Gx - Fx\|^2,\ \|Gy - Hy\|^2,\ \|Gx - Fy\|^2,\ \|Gy - Hx\|^2\}$,

for all x, y in X, where $q \in (0, 1)$.

If $\lim_{n\to\infty} \|f_n\| = \lim_{n\to\infty} \|g_n\| = 0$,

then the sequence $\{v_n\}$ converges to the solution of the equations

$$u = Fu = Gu = Hu.$$

Proof. We will show that $\{v_n\}$ is a Cauchy sequence.

We have

$$\|v_n - v_m\|^2 \le [\|v_n - FGv_n\| + \|FGv_n - HGv_m\| + \|HGv_m - v_m\|]^2$$

$$\le \{\|v_n - FGv_n\| + \|HGv_m - v_m\|\}^2 + 2\{\|v_n - FGv_n\| + \|HGv_m - v_m\|\}$$

$$[\|FGv_n - v_n\| + \|v_n - v_m\| + \|v_m - HGv_m\|]+$$
$$q \max \{\|G^2v_n - G^2v_m\|, \|G^2v_n - FGv_n\|, \|G^2v_m - HGv_m\|^2,$$
$$\|G^2v_n - FGv_m\|^2, \|G^2v_m - HGv_n\|^2\}$$
$$\leq \{\|v_n - FGv_n\| + \|HGv_m - v_m\|\}^2 + 2\{\|v_n - FGv_n\| + \|HGv_m - v_m\|\}$$
$$[\|FGv_n - v_n\| + \|v_n - v_m\| + \|v_m - HGv_m\|]+$$
$$q \max\{\|v_n - v_m\|^2, \|v_n - FGv_n\|^2, \|v_m - HGv_m\|^2,$$
$$[\|v_n - v_m\| + \|v_m - FGv_m\|]^2, [\|v_m - v_n\| + \|v_n - HGv_n\|]^2\}.$$

Letting n tend to infinity with $m > n$, we have

$$\lim_{m,n\to\infty} \|v_n - v_m\|^2 \leq q \lim_{m,n\to\infty} \|v_n - v_m\|^2,$$

which implies that

$$\lim_{m,n\to\infty} \|v_n - v_m\|^2 = 0.$$

Thus $\{v_n\}$ is a Cauchy sequence and converges to a point v in X.

Further,

$$\|v - HGv\| \leq \|v - v_n\| + \|v_n - FGv_n\| + \|FGv_n - HGv\|$$
$$\leq \|v - v_n\| + \|v_n - FGv_n\| + \{q \max \{\|v_n - v\|^2, \|v_n - FGv_n\|^2, \|v - HGv\|^2,$$
$$[\|v_n - v\| + \|v - FGv\|]^2, [\|v_n - v\| + \|v_n - HGv_n\|]^2\}\}^{\frac{1}{2}}.$$

Letting n tend to infinity we get $v = HGv$, which from (h) implies that $Hv = Gv$. Similarly, $Gv = Fv$.

From (g), we now have $FGv = GFv = v = HGv = GHv$.

Using (i) and (h), we have

$$\begin{aligned}
\|v - Hv\|^2 &= \|F^2v - Hv\|^2 \\
&\leq q \max \{\|GFv - Gv\|^2, \|GFv - F^2v\|^2, \\
&\quad \|Gv - Hv\|^2, \|GFv - Fv\|^2, \|Gv - HFv\|^2\} \\
&\leq q \max \{\|v - Hv\|^2, 0, 0, \|v - Hv\|^2, \|v - Hv\|^2\},
\end{aligned}$$

which implies that $v = Hv$. It follows that $Gv = GFv = FGv = v = GHv = HGv$, completing the proof of the theorem.

Remark 1. *Theorems 2.1 and 3.1 of* [4] *follows from the above theorems respectively by putting* $F = H$.

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