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DATA DEPENDENCE OF THE FIXED POINTS IN A SET WITH TWO METRICS

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Abstract. A new variant of Maia fixed point theorem is presented. This variant is useful to study data dependence of the fixed points. Some open problems are presented and some applications to integral equations are given.
Key Words and Phrases: generalized contractions, generalized metric spaces, Maia fixed point theorems, data dependence, well posedness of fixed point problem, limit shadowing property, integral equations.
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1. Introduction

The problem of fixed point principles in a set with two metrics has been investigated by several authors. See for example M.G. Maia [10], I.A. Rus [16], M. Albu [2], B. Rzepecki [23], D.K. Bayen [4], R. Precup [13], I.A. Rus, A.S. Mureşan and V. Mureşan [20], I.A. Rus, A. Petruşel and G. Petruşel [21] and the references therein (S.P. Singh (1970), B.K. Ray (1975), B. Rzepecki (1980), V. Berinde (1997), A.S. Mureşan (1988), V. Mureşan (1988), R. Precup (1988),...).

The aim of this paper is to study the data dependence of the fixed points in a set with two metrics. For this a new variant of Maia fixed point theorem is presented for generalized contractions in the generalized metric spaces. Some applications to integral equations are given.

Throughout of the paper we follow the notations and terminologies in [18]. See also [17] and [21].

2. Generalized contractions in metric spaces

We begin our considerations with

Theorem 2.1. *Let X be a nonempty set, d and ρ two metric on X and $A : X \to X$ an operator. We suppose that:*

(i) (X, d) is a complete metric space;

(ii) there exists $k \in \mathbb{N}$ such that $A^k : (X, \rho) \to (X, d)$ be uniformly continuous;

(iii) $A : (X, d) \to (X, d)$ is closed;

(iv) $A : (X, \rho) \to (X, \rho)$ is a α-contraction.

Then:

(a) $F_A = \{x_A^*\}$;

(b) $A^n(x) \xrightarrow{d} x_A^*$, as $n \to \infty$, $\forall\ x \in X$;

(c) $A^n(x) \xrightarrow{\rho} x_A^*$, as $n \to \infty$, $\forall\ x \in X$, and

$$\rho(A^n(x), x_A^*) \leq \alpha^n \rho(x, x_A^*),\ \ \forall\ n \in \mathbb{N}^*,\ \ \forall\ x \in X;$$

(d) $\rho(x, x_A^*) \leq \dfrac{1}{1-\alpha}\rho(x, A(x))$, $\forall\ x \in X$.

Proof. The implication, (i)-(iv) $\Rightarrow$ (a)+(b) is the fixed point theorem of Maia (see Maia [10], Rus [16] and Precup [13]).

(c) In the relation

$$\rho(A^n(x), A^n(y)) \leq \alpha^n \rho(x, y)$$

we take $y = x_A^*$.

(d) We have

$$\begin{aligned}\rho(x, x_A^*) &\leq \rho(x, A(x)) + \rho(A(x), x_A^*)\\ &\leq \rho(x, A(x)) + \alpha\rho(x, x_A^*).\end{aligned}$$

So,

$$\rho(x, x_A^*) \leq \frac{1}{1-\alpha}\rho(x, A(x)).$$

Remark 2.1. In the terms of Picard operators (see [18] and [21]) the conclusions of Theorem 2.1 take the following form:

- (a)+(b). The operator $A : (X, d) \to (X, d)$ is Picard operator.
- (c)+(d). The operator $A : (X, \rho) \to (X, \rho)$ is $\dfrac{1}{1-\alpha}$−Picard operator.

Remark 2.2. Condition $d(A^k(x), A^k(y)) \leq C\rho(x, y)$ implies condition (ii) in Theorem 2.1.

From Theorem 2.1 we have

Theorem 2.2. *Let X, d, ρ and $A : X \to X$ be as in Theorem 2.1. Let $B : X \to X$ and $\eta > 0$ be such that*

$$\rho(A(x), B(x)) \leq \eta, \ \forall \ x \in X.$$

Then:

$$x_B^* \in F_B \ \Rightarrow \ \rho(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

Proof. We take in the relation (d) of Theorem 2.1, $x = x_B^*$. We have

$$\begin{aligned}\rho(x_B^*, x_A^*) &\leq \frac{1}{1-\alpha}\rho(x_B^*, A(x_B^*)) \\ &= \frac{1}{1-\alpha}\rho(B(x_B^*), A(x_B^*)) \leq \frac{\eta}{1-\alpha}.\end{aligned}$$

Theorem 2.3. *Let X, d, ρ and $A : X \to X$ be as in Theorem 2.1. Then we have:*

(e) The fixed point problem for $A : (X, \rho) \to (X, \rho)$ is well posed;

(f) The operator $A : (X, \rho) \to (X, \rho)$ has the limit shadowing property.

Proof. (e). By definition (see, for example, [19]) the fixed point problem for $A : (X, \rho) \to (X, \rho)$ is well posed iff:

- $F_A = \{x_A^*\}$

and

- $x_n \in X$, $\rho(x_n, A(x_n)) \to 0$ as $n \to \infty \ \Rightarrow \ \rho(x_n, x_A^*) \to 0$ as $n \to \infty$.

So, let $x_n \in X$, $n \in \mathbb{N}$ be such that $\rho(x_n, A(x_n)) \to 0$ as $n \to \infty$. We have

$$\begin{aligned}\rho(x_n, x_A^*) &\leq \rho(x_n, A(x_n)) + \rho(A(x_n), x_A^*) \\ &\leq \rho(x_n, A(x_n)) + \alpha\rho(x_n, x_A^*).\end{aligned}$$

Since, $\rho(x_n, x_A^*) \leq \dfrac{1}{1-\alpha}\rho(x_n, A(x_n)) \to 0$ as $n \to \infty$.

(f). By definition (see, for example [19]) the operator $A : (X, \rho) \to (X, \rho)$ has the limit shadowing property iff $x_n \in X$, $n \in \mathbb{N}$, $\rho(x_{n+1}, A(x_n)) \to 0$ as $n \to \infty$ imply that there exists $x \in X$ such that $\rho(x_n, A^n(x)) \to 0$ as $n \to \infty$.

In our case, $F_A = \{x_A^*\}$ and $\rho(A^n(x), x_A^*) \to 0$ as $n \to \infty$, $\forall\ x \in X$. We have

$$\begin{aligned}\rho(x_n, x_A^*) &\le \rho(x_n, A(x_{n-1})) + \rho(A(x_{n-1}, x^*)) \\ &\le \rho(x_n, A(x_{n-1})) + \alpha\rho(x_{n-1}, x^*) \le \dots \\ &\le \rho(x_n, A(x_{n-1})) + \alpha\rho(x_{n-1}, A(x_{n-2})) + \dots \\ &+ \alpha^{n-1}\rho(x_1, A(x_0)) + \alpha^n\rho(x_0, x_A^*).\end{aligned}$$

From the Cauchy's lemma (see [18], p.208) we have that $\rho(x_n, x_A^*) \to 0$ as $n \to \infty$. So,

$$\rho(x_n, A^n(x_0)) \le \rho(x_n, x_A^*) + \rho(x_A^*, A^n(x_0)) \to 0$$

as $n \to \infty$.

There are many types of generalized contractions. See, for example B.E. Rhoades [15], W.A. Kirk [9], I.A. Rus [17], I.A. Rus, A. Petruşel and G. Petruşel [21] and references therein. The above considerations give rise to

Problem 2.1. Extend the above results for the case when $A : (X, \rho) \to (X, \rho)$ is a generalized contraction.

For example

Theorem 2.1'. *Let X be a nonempty set, d, ρ two metrics on X and $A : X \to X$ an operator. We suppose that:*

(i) (X, d) is a complete metric space;

(ii) There exists $k \in \mathbb{N}$ such that $A^k : (X, \rho) \to (X, d)$ be uniformly continuous;

(iii) $A : (X, d) \to (X, d)$ is closed;

(iv') There exists a strict comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ (see [17], p.69) such that

$$\rho(A(x), A(y)) \le \varphi(d(x, y)),\ \forall\ x, y \in X.$$

Then:

(a) $F_A = \{x_A^*\}$;

(b) $A^n(x) \xrightarrow{d} x_A^*$ as $n \to \infty$, $\forall\ x \in X$;

(c') $A^n(x) \xrightarrow{\rho} x_A^*$ as $n \to \infty$, $\forall\ x \in X$ and

$$\rho(A^n(x), x_A^*) \le \varphi^n(\rho(x, x_A^*)),\ \forall\ n \in \mathbb{N}^*,\ \forall\ x \in X;$$

(d') $\rho(x, x_A^*) \le \sup\{t \in \mathbb{R}_+ \mid t - \varphi(t) \le \rho(x, A(x))\}$.

Theorem 2.2'. *Let X, d, ρ and $X \to X$ be as in Theorem 2.1'. Let $B : X \to X$ and $\eta > 0$ be such that*

$$\rho(A(x), B(x)) \leq \eta, \ \forall\ x \in X.$$

Then:

$$x_B^* \in F_B \ \Rightarrow \ \rho(x_A^*, x_B^*) \leq \sup\{t \in \mathbb{R}_+ \mid t - \varphi(t) \leq \eta\}.$$

Theorem 2.3'. *Let X, d, ρ and $A : X \to X$ be as in Theorem 2.1'. Then we have:*

(e') The fixed point problem for $A : (X, \rho) \to (X, \rho)$ is well posed.

Remark 2.2. For to have a (f') as in Theorem 2.3 we need some supplemental conditions on the comparison function φ.

3. Generalized contractions in generalized metric spaces

We begin with the following suggestive example.

Theorem 3.1. *Let X be a nonempty set, d and ρ two generalized metrics on X ($d(x, y) \in \mathbb{R}_+^m$, $\rho(x, y) \in \mathbb{R}_+^m$) and $A : X \to X$ an operator. We suppose that:*

(i) (X, d) is a complete metric space;

(ii) There exists $k \in \mathbb{N}$ such that $A^k : (X, \rho) \to (X, d)$ be uniformly continuous;

(iii) $A : (X, d) \to (X, d)$ is closed;

(iv) $A : (X, \rho) \to (X, \rho)$ is a S-contraction, i.e., the matrix S is convergent toward zero (see [17], p.96) and

$$\rho(A(x), A(y)) \leq S\rho(x, y), \ \forall\ x, y \in X.$$

Then:

(a) $F_A = \{x_A^*\}$;

(b) $A^n(x) \xrightarrow{d} x_A^*$ as $n \to \infty$, $\forall\ x \in X$;

(c) $A^n(x) \xrightarrow{\rho} x_A^*$ as $n \to \infty$, $\forall\ x \in X$ and

$$\rho(A^n(x), x_A^*) \leq S^n\rho(x, x_A^*), \ \forall\ n \in \mathbb{N}, \ \forall\ x \in X;$$

(d) $\rho(x, x_A^*) \leq (E - S)^{-1}\rho(x, A(x))$, $\forall\ x \in X$.

Proof. The proof is similar with that of Theorem 2.1.

Theorem 3.1. For the case when $k = 1$ and $A : (X, d) \to (X, d)$ is continuous see A.I. Perov [12], M. Albu [2] and R. Precup [13]. See also, D. O'Regan and R. Precup [11].

Theorem 3.2. *Let X, d, ρ and $A : X \to X$ be as in Theorem 3.1. Let $B : X \to X$ and $\eta \in \mathbb{R}^m_+$ be such that*

$$\rho(A(x), B(x)) \leq \eta, \ \forall \ x \in X.$$

Then:

$$x^*_B \in F_B \ \Rightarrow \ \rho(x^*_A, x^*_B) \leq (E - S)^{-1}\eta.$$

Theorem 3.3. *Let X, d, ρ and $A : X \to X$ be as in Theorem 3.1. Then we have:*

(e) The fixed point problem for $A : (X, \rho) \to (X, \rho)$ is well posed;

(f) The operator $A : (X, \rho) \to (X, \rho)$ has the limit shadowing property.

The above results give rise to

Problem 3.1. Extend the above results for the case of K-metric spaces and generalized contractions.

References: P.P. Zabreiko [25], V. Berinde [4], I.A. Rus, A.S. Mureşan and V. Mureşan [20], I.A. Rus, A. Petruşel and M.A. Şerban [22].

Problem 3.2. Extend the above results for the case of generalized metric spaces (ultrametric space, partial metric space, 2-metric space, probabilistic metric space,...) and generalized contractions.

References: W.A. Kirk [9], M. Frigon [7], R. Precup [13], D.K. Bayen, V. Radu [14], I.A. Rus [17], I.A. Rus, A. Petruşel and G. Petruşel [21], B. Rzepecki [23], E. Schörner [24].

Problem 3.3. Extend the above results for the case of gauge spaces and generalized contractions.

References: W.A. Kirk [9], I.A. Rus [17], Precup [13], R.P. Agarwal and D. O'Regan [1], V.G. Angelov [3], A. Chiş and R. Precup [6], M. Frigon [7], N. Gheorghiu [8], B. Rzepecki [23].

4. Applications to integral equations

Let $\Omega \subset \mathbb{R}^p$ be a bounded domain. We consider the following integral equations

$$x(t) = k(t) + \int_\Omega K(t,s,x(s))ds \tag{4.1}$$

and

$$x(t) = h(t) + \int_\Omega H(t,s,x(s))ds \tag{4.2}$$

We suppose that

(i) $k, h \in C(\overline{\Omega}),\ K, H \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R})$;

(ii) there exists $L \in C(\overline{\Omega} \times \overline{\Omega})$ such that

$$|K(t,s,u) - K(t,s,v)| \leq L(t,s)|u - v|$$

for all $t, s \in \overline{\Omega}$ and $u, v \in \mathbb{R}$;

(iii) $\displaystyle\int_{\Omega\times\Omega} |L(t,s)|^2 dtds < 1$;

(iv) there exist $\eta_1 > 0$, $\eta_2 > 0$ such that

$$|k(t) - h(t)| \leq \eta_1, \quad |K(t,s,u) - H(t,s,u)| \leq \eta_2$$

for all $t, s \in \overline{\Omega}$, $u \in \mathbb{R}$.

We have

Theorem 4.1. *In the above conditions:*

(a) equation (4.1) has in $C(\overline{\Omega})$ a unique solution, x^*.

(b) if $y^* \in C(\overline{\Omega})$ is a solution of the equation (4.2), then

$$\|x^* - y^*\|_{L^2(\Omega)} \leq \frac{(m(\Omega))^{\frac{1}{2}}}{1 - \|L\|_{L^2(\Omega\times\Omega)}}[\eta_1 + \eta_2 m(\Omega)].$$

Proof. We take $X = C(\overline{\Omega})$, $d(x,y) = \|x-y\|_{C(\overline{\Omega})}$ and $\rho(x,y) = \|x-y\|_{L^2(\Omega)}$. Let $A, B : C(\overline{\Omega}) \to C(\overline{\Omega})$ be given by

$$A(x)(t) := k(t) + \int_\Omega K(t,s,x(s))ds$$

$$B(x)(t) := h(t) + \int_\Omega H(t,s,x(s))ds$$

Then we are in the conditions of the Theorem 2.1 and 2.2. The proof follows from these theorems.

Remark 4.1. If the function H is bounded then the equation (4.2) has in $C(\overline{\Omega})$ at least a solution. Indeed, if $h \in C(\overline{\Omega})$, $H \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R})$ and H is bounded then the operator $B : (C(\overline{\Omega}), \|\cdot\|_C) \to (C(\overline{\Omega}), \|\cdot\|_C)$ is complete continuous and bounded. So, we are in the conditions of the Schauder fixed point theorem.

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