Fixed Point Theory, Volume 8, No. 1, 2007, 105-113
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

COMMON FIXED POINTS THROUGH IMPLICIT ITERATION PROCESS WITH ERRORS

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Abstract. We establish a general theorem to approximate common fixed points of quasi-contractive operators on a normed space through the implicit iteration process with errors in the sense of Xu [20]. Our result generalizes and improves upon, among others, the corresponding results of [1, 2, 3, 5, 14, 18].

Key Words and Phrases: common fixed point, implicit iteration process with errors, strong convergence.
2000 Mathematics Subject Classification: 47H10, 47H17, 54H25.

1. Introduction and Preliminaries

Let C be a nonempty convex subset of a normed space E, $T : C \to C$ be a mapping and $F(T)$ be the set of fixed points.

Let $\{b_n\}$ and $\{b'_n\}$ be two sequences in $[0, 1]$.

The Mann iteration process is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ (see [11]):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n T x_n, \ n \geq 0. \end{cases} \tag{1.1}$$

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n T y_n, \\ y_n = (1 - b'_n)\, x_n + b'_n T x_n, \quad n \geq 0 \end{cases} \tag{1.2}$$

is known as the Ishikawa iteration process [6].

Liu [10] introduced the concept of Ishikawa iteration process with errors by the sequence $\{x_n\}_{n=0}^{\infty}$ defined as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-b_n)\, x_n + b_n T y_n + u_n, \\ y_n = (1-b'_n)\, x_n + b'_n T x_n + v_n, \quad n \geq 0 \end{cases} \tag{1.3}$$

where $\{b_n\}$and $\{b'_n\}$ are sequences in $[0,1]$ and $\{u_n\}$ and $\{v_n\}$ satisfy $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$. This surely contains both (1.1) and (1.2). Also this contains the Mann process with error terms

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-b_n)x_n + b_n T x_n + u_n, \; n \geq 0. \end{cases} . \tag{1.4}$$

In 1998, Xu [20] introduced more satisfactory error terms in the sequence defined by:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \; n \geq 0 \end{cases} \tag{1.5}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. Clearly, this iteration process contains the processes (1.1), (1.2) as its special cases. Also it contains the Mann process with error terms:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \; n \geq 0. \end{cases} \tag{1.6}$$

For two self mappings S and T of C, the Ishikawa iteration processes have been generalized by Das and Debata [5] as follows

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-b_n)\, x_n + b_n S y_n \\ y_n = (1-b'_n)\; x_n + b'_n T x_n, \quad n \geq 0. \end{cases} \tag{1.7}$$

They used this iteration process to find the common fixed points of quasi-nonexpansive mappings in a uniformly convex Banach space. Takahashi and Tamura [18] studied it for the case of two nonexpansive mappings under different conditions in a strictly convex Banach space.

Recently, Agarwal et al [1] studied the iteration process for two quasi-contractive mappings using errors in the sense of Xu [20]:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \ n \geq 0, \end{cases} \tag{1.8}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

Inspired and motivated by the above said facts, we suggest the following implicit iteration process with errors and define the sequence $\{x_n\}$ as follows

$$\begin{cases} x_0 \in C, \\ x_n = a_n x_{n-1} + b_n S y_n + c_n u_n, \\ y_n = a'_n x_{n-1} + b'_n T x_n + c'_n v_n, \ n \geq 1, \end{cases} \tag{1.9}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

Remark. *1. For $a'_n = 0 = c'_n$ and $b'_n = 1$, (1.9) implies the following iteration process with errors*

$$\begin{cases} x_0 \in C, \\ x_n = a_n x_{n-1} + b_n ST x_n + c_n u_n, \ n \geq 0. \end{cases} \tag{1.10}$$

2. We can also deduce from (1.9) the implicit iteration processes in the sense of (1.1-1.8).

We recall the following definitions in a metric space (X, d). A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq a d(x, y) \text{ for all } x, y \in X, \tag{1.11}$$

where $a \in (0, 1)$.

The map T is called Kannan mapping [7] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \tag{1.12}$$

A similar definition is due to Chatterjea [4]: there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X. \tag{1.13}$$

Combining these three definitions, Zamfirescu [21] proved the following important result.

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exists the real numbers a, b and c satisfying $a \in (0, 1)$, $b, c \in (0, \frac{1}{2})$ such that for any pair $x, y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx, Ty) \le ad(x, y)$,
(z_2) $d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)]$,
(z_3) $d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \; n = 0, \; 2, ...$$

converges to p for any arbitrary but fixed $x_0 \in X$.

An operator T satisfying the contractive conditions $(z_1) - (z_3)$ in the above theorem is called Zamfirescu operator (condition Z).

In 2004, Berinde [2] introduced a new class of operators on an arbitrary Banach space E satisfying

$$\|Tx - Ty\| \le \delta \|x - y\| + 2\delta \|Tx - x\| \tag{1.14}$$

for any $x, y \in E$, $0 \le \delta < 1$.

He proved that this class is wider than the class of Zamfiresu operators and used the Ishikawa iteration process (1.2) to approximate fixed points of this class of operators in an arbitrary Banach space given in the form of following theorem:

Theorem 2. *Let C be a nonempty closed convex subset of an arbitrary Banach space E and $T : C \to C$ be an operator satisfying (1.14). Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process (1.2) and $x_0 \in C$, where $\{b_n\}$ and $\{b'_n\}$ are sequences of positive numbers in $[0, 1]$ with $\{b_n\}$ satisfying $\sum_{n=0}^{\infty} b_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

In this paper, a convergence theorem of Rhoades [15] regarding the approximation of fixed points of some quasi contractive operators in uniformly convex Banach spaces using the Mann iteration process, is extended to the approximation of common fixed points of some quasi contractive operators in normed spaces using the iteration process (1.9). Our result generalizes and improves upon, among others, the corresponding results of [1, 2, 3, 5, 14, 18].

The following lemma is proved in [19].

Lemma 1. *If there exists a positive integer* N *such that for all* $n \geq N, n \in \mathbb{N}$,

$$\rho_{n+1} \leq (1-\alpha_n)\rho_n + b_n,$$

then

$$\lim_{n\to\infty} \rho_n = 0,$$

where $\alpha_n \in [0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and* $b_n = o(\alpha_n)$.

2. Main Results

Following [2, 3, 13], we obtain such a result without employing any fixed point theorem.

Theorem 3. *Let* C *be a nonempty closed convex subset of a normed space* E. *Let* S, $T : C \to C$ *be two operators satisfying condition* Z. *Let* $\{x_n\}_{n=1}^{\infty}$ *be defined through the iterative process* (1.9). *If* $F = F(S) \cap F(T) \neq \varphi$, $\sum_{n=1}^{\infty} b_n = \infty$, $c_n = o(b_n)$ *and* $\lim_{n\to\infty} c'_n = 0$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a common fixed point of* S *and* T.

Proof. Since S, $T : C \to C$ be two operators satisfying condition Z, then at least one of the conditions $(z_1), (z_2)$ and (z_3) is satisfied. If (z_2) holds, then for x, $y \in C$

$$\begin{aligned} \|Sx - Ty\| &\leq b\left[\|x - Sx\| + \|y - Ty\|\right] \\ &\leq b\left[\|x - Sx\| + \|y - x\| + \|x - Sx\| + \|Sx - Ty\|\right], \end{aligned}$$

implies

$$(1-b)\|Sx - Ty\| \leq b\|x - y\| + 2b\|x - Sx\|,$$

which yields (using the fact that $0 \leq b < 1$)

$$\|Sx - Ty\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Sx\|. \tag{2.1}$$

If (z_3) holds, then similarly we obtain

$$\|Sx - Ty\| \leq \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Sx\|. \tag{2.2}$$

Denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{2.3}$$

Then we have $0 \leq \delta < 1$ and, in view of (z_1), (2.1-2.3) it results that the inequality

$$\|Sx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Sx\| \tag{2.4}$$

holds for all $x, y \in C$.

In a similar fashion, we can find

$$\|Sx - Ty\| \leq \delta \|x - y\| + 2\delta \|y - Ty\|. \tag{2.5}$$

Assume that $F \neq \varphi$ and $w \in F$, then

$$M = \max\left\{\sup_{n\geq 1} \|u_n - w\|, \sup_{n\geq 1} \|v_n - w\|\right\}.$$

Using (1.9), we have

$$\begin{aligned} \|x_n - w\| &= \|a_n x_{n-1} + b_n S y_n + c_n u_n - (a_n + b_n + c_n) w\| \\ &= \|a_n(x_{n-1} - w) + b_n(Sy_n - w) + c_n(u_n - w)\| \\ &\leq a_n \|x_{n-1} - w\| + b_n \|Sy_n - w\| + c_n \|u_n - w\| \\ &\leq (1 - b_n) \|x_{n-1} - w\| + b_n \|Sy_n - w\| + Mc_n. \end{aligned} \tag{2.6}$$

Now for $x = y_n$ and $y = w$, (2.5) gives

$$\|Sy_n - w\| \leq \delta \|y_n - w\|. \tag{2.7}$$

In a similar fashion, we can get

$$\begin{aligned} \|y_n - w\| &= \left\|a'_n x_{n-1} + b'_n T x_n + c'_n v_n - (a'_n + b'_n + c'_n) w\right\| \\ &= \left\|a'_n(x_{n-1} - w) + b'_n(Tx_n - w) + c'_n(v_n - w)\right\| \\ &\leq a'_n \|x_{n-1} - w\| + b'_n \|Tx_n - w\| + c'_n \|v_n - w\| \\ &\leq (1 - b'_n) \|x_{n-1} - w\| + b'_n \|Tx_n - w\| + Mc'_n. \end{aligned} \tag{2.8}$$

Again by (2.4), if $x = w$ and $y = x_n$, we get

$$\|Tx_n - w\| \leq \delta \|x_n - w\|. \tag{2.9}$$

From (2.6-2.9), we obtain

$$\begin{aligned} \|x_n - w\| \leq\ & (1 - b_n) \|x_{n-1} - w\| + b_n \delta [(1 - b'_n) \|x_{n-1} - w\| \\ & + b'_n \delta \|x_n - w\| + Mc'_n] + Mc_n, \end{aligned}$$

implies with the help of the condition $c_n = o(b_n)$ ($c_n = b_n t_n$; $t_n \to 0$ as $n \to \infty$)

$$\|x_n - w\| \leq \frac{1 - b_n + \delta b_n(1 - b'_n)}{1 - \delta^2 b_n b'_n} \|x_{n-1} - w\| + M \frac{b_n(\delta c'_n + t_n)}{1 - \delta^2 b_n b'_n}. \qquad (2.10)$$

Let

$$\begin{aligned} A_n &= 1 - b_n + \delta b_n(1 - b'_n), \\ B_n &= 1 - \delta^2 b_n b'_n, \end{aligned}$$

and consider

$$\begin{aligned} \beta_n &= 1 - \frac{A_n}{B_n} \\ &= 1 - \frac{1 - b_n + \delta b_n(1 - b'_n)}{1 - \delta^2 b_n b'_n} \\ &= \frac{b_n(1-\delta)(1 + \delta b'_n)}{1 - \delta^2 b_n b'_n} \\ &\geq b_n(1-\delta)(1 + \delta b'_n). \qquad (2.11) \end{aligned}$$

Because $0 \leq \delta < 1$ and $0 \leq b_n \leq 1$, implies $1 + \delta b'_n \geq 1$ and $\frac{1}{1-\delta^2 b_n b'_n} \leq \frac{1}{1-\delta^2}$. Now from (2.11) we get

$$\beta_n \geq (1-\delta) b_n,$$

implies

$$\frac{A_n}{B_n} \leq 1 - (1-\delta) b_n.$$

Thus from (2.10), we get

$$\|x_n - w\| \leq [1 - (1-\delta)b_n] \|x_{n-1} - w\| + M \frac{b_n(\delta c'_n + t_n)}{1 - \delta^2}.$$

With the help of lemma 1 and using the fact that $\sum_{n=1}^{\infty} b_n = \infty$ and $c_n = 0(b_n)$, it results that

$$\lim_{n\to\infty} \|x_n - w\| = 0.$$

Consequently $x_n \to w \in F$ and this completes the proof. □

Corollary 1. *Let C be a nonempty closed convex subset of a normed space E. Let $S, T : C \to C$ be two operators satisfying (2.4-2.5). Let $\{x_n\}_{n=1}^{\infty}$ be defined through the iterative process* (1.10). *If $F = F(S) \cap F(T) \neq \varphi$, $\sum_{n=1}^{\infty} b_n = \infty$ and $c_n = o(b_n)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of S and T.*

Remark. *1. Similar results can be found for the implicit iteration processes in the sense of (1.1-1.8).*

2. The Chatterjea's and the Kannan's contractive conditions (1.13) *and* (1.12) *are both included in the class of Zamfirescu operators and so their convergence theorems for the implicit iteration process in the sense of Ishikawa [6] can be obtained.*

3. Theorem 4 of Rhoades [15] *in the context of Mann iteration on a uniformly convex Banach space has been extended in Corollary* 1.

4. In Corollary 1, Theorem 8 of Rhoades [16] *is generalized to the setting of normed spaces.*

5. Our result also generalizes Theorem 5 of Osilike [12] *and Theorem 2 of Osilike* [13].

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Received: May 2, 2006; Accepted: January 26, 2007.