Fixed Point Theory, Volume 8, No. 1, 2007, 97-104
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINT THEORY FOR MULTIVALUED OPERATORS ON A SET WITH TWO METRICS

ADRIAN PETRUŞEL AND IOAN A. RUS

Department of Applied Mathematics
Babeş-Bolyai University Cluj-Napoca
Kogălniceanu 1, 400084, Cluj-Napoca, Romania
E-mails: petrusel@math.ubbcluj.ro, iarus@math.ubbcluj.ro

Abstract. The purpose of this work is to present some fixed point results for multivalued operators on a set with two metrics. A multivalued version of Maia's fixed point theorem is proved. The data dependence and the well-posedness of the fixed point problem are also discussed. Some extensions to generalized multivalued contractions are pointed out.
Key Words and Phrases: set with two metrics, Maia fixed point theorem, multivalued operator, fixed point, strict fixed point, well-posed fixed point problem, generalized contraction, data dependence.
2000 Mathematics Subject Classification: 47H10, 54H25, 47H04.

1. Introduction

Throughout this paper, the standard notations and terminologies in nonlinear analysis (see [14], [15], [7]) are used. For the convenience of the reader we recall some of them.

Let (X, d) be a metric space. In the sequel we will use the following symbols: $P(X) := \{Y \subset X |\ Y \text{ is nonempty}\}$, $P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\}$.

Let A and B be nonempty subsets of the metric space (X, d). The gap between these sets is

$$D(A, B) = \inf\{d(a, b) |\ a \in A,\ b \in B\}.$$

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

Also, if $A, B \in P_b(X)$, then one denote

$$\delta(A,B) := \sup\{d(a,b)|\ a \in A,\ b \in B\}.$$

The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets A and B of the metric space (X,d) is defined by the following formula:

$$H(A,B) := \max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}.$$

The symbol $T : X \multimap Y$ means $T : X \to P(Y)$, i. e. T is a multivalued operator from X to Y. We will denote by $G(T) := \{(x,y) \in X \times Y | y \in T(x)\}$ the graph of T. The multivalued operator T is said to be closed if $G(T)$ is closed in $X \times Y$.

For $T : X \to P(X)$ the symbol $F_T := \{x \in X|\ x \in T(x)\}$ denotes the fixed point set of the multivalued operator T, while $(SF)_T := \{x \in X|\ \{x\} = T(x)\}$ is the strict fixed point set of T. Also, for $x \in X$, we denote $F^n(x) := F(F^{n-1}(x))$, $n \in \mathbb{N}^*$, where $F^0(x) := \{x\}$.

If $F : X \to P_{cl}(X)$ is a multi-valued operator then F is said to be α-contraction if

$$\alpha \in [0,1[\text{ and for } x, y \in X \Rightarrow H(F(x), F(y)) \leq \alpha d(x,y).$$

The aim of this paper is to give some fixed point theorems for multivalued operators on a set endowed with two metrics. For the singlevalued case, see R. P. Agarwal, D. O'Regan [1], I. A. Rus, A. Petruşel, G. Petruşel [14] and the references therein.

2. Multivalued contraction on a set with two metrics

Our first main result is a multivalued version of Maia's fixed point theorem.

Theorem 2.1 *Let X be a nonempty set, d and ρ two metrics on X and $T : X \to P(X)$ be a multivalued operator. We suppose that:*

(i) (X,d) is a complete metric space;

(ii) there exists $c > 0$ such that $d(x,y) \leq c\rho(x,y)$, for each $x, y \in X$;

(iii) $T : (X,d) \to (P(X), H_d)$ is closed;

(iv) there exists $\alpha \in [0,1[$ such that $H_\rho(F(x), F(y)) \leq \alpha\rho(x,y)$, for each $x, y \in X$.

Then we have:

(a) $F_T \neq \emptyset$;

(b) for each $x \in X$ *and each* $y \in T(x)$ *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *such that:*

(1) $x_0 = x,\ x_1 = y$*;*

(2) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$*;*

(3) $x_n \xrightarrow{d} x^* \in T(x^*)$*, as* $n \to \infty$*.*

Proof. As in the proof of Avramescu-Markin-Nadler's theorem (see [3], [6], [12]) hypothesis (iv) implies that there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, ρ), such that (1) and (2) hold. From (ii) it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Denote by $x^* \in X$ the limit of this sequence. From (i) and (iii) we get that $x_n \xrightarrow{d} x^* \in T(x^*)$, as $n \to \infty$. The proof is complete. □

Remark 2.1 *In terms of the multivalued weakly Picard operators theory (see* [15], [12]*), the conclusion of the above result takes the following form:*

(a')-(b') $T : (X, d) \multimap (X, d)$ *is a multivalued weakly Picard operator.*

The second main result of this section is the following theorem.

Theorem 2.2 *Let* X *be a nonempty set,* d *and* ρ *two metrics on* X *and* $T : X \to P(X)$ *be a multivalued operator. We suppose that:*

(i) (X, d) *is a complete metric space;*

(ii) there exists $c > 0$ *such that* $d(x, y) \le c\rho(x, y)$*, for each* $x, y \in X$*;*

(iii) $T : (X, d) \to (P(X), H_d)$ *is closed;*

(iv) there exists $\alpha \in [0, 1[$ *such that* $H_\rho(T(x), T(y)) \le \alpha\rho(x, y)$*, for each* $x, y \in X$*;*

(v) $(SF)_T \ne \emptyset$*.*

Then we have:

(a) $F_T = (SF)_T = \{x^*\}$*;*

(b) $H_\rho(T^n(x), x^*) \le \alpha^n \cdot \rho(x, x^*)$*, for each* $n \in \mathbb{N}$ *and each* $x \in X$*;*

(c) $\rho(x, x^*) \le \frac{1}{1-\alpha} \cdot H_\rho(x, T(x))$*, for each* $x \in X$*;*

(d) the fixed point problem is well-posed for T *with respect to* D_ρ*.*

Proof. (a)-(b) From (iv) we have that if $x^* \in (SF)_T$ then $(SF)_T = \{x^*\}$, see I. A. Rus [12], pp. 87. Also, by taking $y := x^*$ in (iv) we have that $H_\rho(T(x), x^*) \le \alpha\rho(x, x^*)$, for each $x \in X$. By induction we get that $H_\rho(T^n(x), x^*) \le \alpha^n\rho(x, x^*)$, for each $x \in X$. Consider now $y^* \in F_T$. Then:

$\rho(y^*, x^*) \le H_\rho(T^n(x), x^*) \le \alpha^n\rho(x, x^*) \to 0$, as $n \to \infty$. Hence $y^* = x^*$.

(c) We successively have: $\rho(x, x^*) \leq H_\rho(x, T(x)) + H_\rho(T(x), x^*) \leq H_\rho(x, T(x)) + \alpha\rho(x, x^*)$. Hence $\rho(x, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(x, T(x))$, for each $x \in X$.

(d) Let $(x_n)_{n\in\mathbb{N}}$ be such that $D_\rho(x_n, T(x_n)) \to 0$, as $n \to \infty$. We have to prove that $\rho(x_n, x^*) \to 0$, as $n \to \infty$ (see [10]).

Then we have:

$$\rho(x_n, x^*) \leq D_\rho(x_n, T(x_n)) + H_\rho(T(x_n), T(x^*)) \leq D_\rho(x_n, T(x_n)) + \alpha\rho(x_n, x^*).$$

Hence we get $\rho(x_n, x^*) \leq \frac{1}{1-\alpha} \cdot D_\rho(x_n, T(x_n)) \to 0$, as $n \to \infty$. □

Remark 2.2 *For the implication $(SF)_T \neq \emptyset \Rightarrow (SF)_T = \{x^*\}$ in the theory of multivalued generalized contractions see I. A. Rus* [13] *and A. Sîntămărian* [16].

Remark 2.3 *In the conditions of Theorem 2.2 we also have that:*

(1) $T : (X, d) \multimap (X, d)$ is a multivalued weakly Picard operator;

(2) $T : (X, \rho) \multimap (X, \rho)$ is a multivalued Picard operator (see A. Petruşel, I. A. Rus [9]*).*

A data dependence result is the following theorem.

Theorem 2.3 *Let X be a nonempty set, d and ρ two metrics on X and $T, S : X \to P(X)$ be two multivalued operators. We suppose that:*

(i) (X, d) is a complete metric space;

(ii) there exists $c > 0$ such that $d(x, y) \leq c\rho(x, y)$, for each $x, y \in X$;

(iii) $T : (X, d) \to (P(X), H_d)$ is closed;

(iv) there exists $\alpha \in [0, 1[$ such that $H_\rho(T(x), T(y)) \leq \alpha\rho(x, y)$, for each $x, y \in X$

(v) $(SF)_T \neq \emptyset$;

(vi) $F_S \neq \emptyset$

(vii) there exists $\eta > 0$ such that $H_\rho(T(x), S(x)) \leq \eta$, for each $x \in X$.

Then $H(F_T, F_S) \leq \frac{\eta}{1-\alpha}$.

Proof. Let $y^* \in F_S$. From the conclusion (c) of the above theorem we have that:

$\rho(y^*, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(y^*, T(y^*)) \leq \frac{1}{1-\alpha} \cdot H_\rho(S(y^*), T(y^*)) \leq \frac{\eta}{1-\alpha}$.

Hence $H(F_T, F_S) = \sup\limits_{y^* \in F_S} \rho(y^*, x^*) \leq \dfrac{\eta}{1-\alpha}$. The proof is complete. □

3. Multivalued generalized contraction on a set with two metrics

Let (X, d) be a metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. For $x, y \in X$, let us denote

$$M_d^T(x,y) := \max\{d(x,y), D_d(x,T(x)), D_d(y,T(y)), \frac{1}{2}[D_d(x,T(y)) + D_d(y,T(x))]\}.$$

Ćirić proved that if the space (X, d) is complete and if the multivalued operator $T : X \to P_{cl}(X)$ satisfies the following condition:

there exists $\alpha \in [0, 1[$ such that $H_d(T(x), T(y)) \leq \alpha \cdot M_d^T(x,y)$, for each $x, y \in X$,

then $F_T \neq \emptyset$ and for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(1) $x_0 = x,\ x_1 = y$;
(2) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;
(3) $x_n \xrightarrow{d} x^* \in T(x^*)$, as $n \to \infty$.

Next result is a multivalued version of Maia's theorem for Ćirić-type multivalued operators.

Theorem 3.1 *Let X be a nonempty set, d, ρ two metrics on X and $T : X \to P(X)$ be a multivalued operator. We suppose that:*

(i) (X, d) is a complete metric space;
(ii) there exists $c > 0$ such that $d(x, y) \leq c\rho(x, y)$, for each $x, y \in X$;
(iii) $T : (X, d) \to (P(X), H_d)$ is closed;
(iv) there exists $\alpha \in [0, 1[$ such that $H_\rho(T(x), T(y)) \leq \alpha M_\rho^T(x, y)$, for each $x, y \in X$.

Then we have:

(a) $F_T \neq \emptyset$;
(b) for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(1) $x_0 = x,\ x_1 = y$;
(2) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;
(3) $x_n \xrightarrow{d} x^* \in T(x^*)$, as $n \to \infty$.

Proof. As in the proof of Ćirić's theorem (see [2], Theorem 2), hypothesis (iv) implies that there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, ρ), such that (1)and (2) hold. From (ii) it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in

(X, d). Denote by $x^* \in X$ the limit of this sequence. From (i) and (iii) we get that $x_n \overset{d}{\to} x^* \in T(x^*)$, as $n \to \infty$. The proof is complete. □

Remark 3.1 *In terms of the multivalued weakly Picard operators theory, the conclusion of the above result takes the following form:*
(a')-(b') $T : (X, d) \multimap (X, d)$ *is a multivalued weakly Picard operator.*

For the next results, let us denote

$$N_d^T(x, y) := \max\{d(x, y), D_d(y, T(y)), \frac{1}{2}[D_d(x, T(y)) + D_d(y, T(x))]\}.$$

The second main result of this section is the following theorem.

Theorem 3.2 *Let* X *be a nonempty set,* $T : X \to P(X)$ *be a multivalued operator and* d, ρ *two metrics on* X. *We suppose that:*

(i) (X, d) *is a complete metric space;*

(ii) there exists $c > 0$ *such that* $d(x, y) \leq c\rho(x, y)$, *for each* $x, y \in X$;

(iii) $T : (X, d) \to (P(X), H_d)$ *is closed;*

(iv) there exists $\alpha \in [0, 1[$ *such that* $H_\rho(T(x), T(y)) \leq \alpha N_\rho^T(x, y)$, *for each* $x, y \in X$;

(v) $(SF)_T \neq \emptyset$.

Then we have:

(a) $F_T = (SF)_T = \{x^*\}$;

(b) $H_\rho(T^n(x), x^*) \leq \alpha^n \cdot \rho(x, x^*)$, *for each* $n \in \mathbb{N}$ *and each* $x \in X$;

(c) $\rho(x, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(x, T(x))$, *for each* $x \in X$;

(d) the fixed point problem is well-posed for T *with respect to* D_ρ.

Proof. (a)-(b) From (iv) we have that if $x^* \in (SF)_T$ then $(SF)_T = \{x^*\}$. Indeed, if $y \in (SF)_T$ then $\rho(x^*, y) = H_\rho(T(x^*), T(y)) \leq \alpha N_\rho^T(x^*, y) = \alpha \cdot \max\{\rho(x^*, y), \frac{1}{2}[\rho(x^*, y) + \rho(y, x^*)]\} = \alpha \cdot \rho(x^*, y)$. Hence $y = x^*$.

For the second conclusion let's take, in the condition (iv), $y := x^*$. Then, for each $x \in X$, we have: $H_\rho(T(x), x^*) = H_\rho(T(x), T(x^*)) \leq \alpha N\rho^T(x, x^*) = \alpha \cdot \max\{\rho(x, x^*), \frac{1}{2}[D_\rho(x^*, T(x)) + D_\rho(T(x^*), x)]\}$. We distinguish the following two cases:

1) If the above maximum is $\rho(x, x^*)$ then we have $H_\rho(T(x), x^*) \leq \alpha \cdot \rho(x, x^*)$.

2) If the maximum is $\frac{1}{2}[D_\rho(x^*, T(x)) + D_\rho(T(x^*), x)]$, then $H_\rho(T(x), x^*) \leq \alpha \cdot \frac{1}{2}[H_\rho(x^*, T(x)) + \rho(x^*, x)]$. Hence $H_\rho(T(x), x^*) \leq \frac{\alpha}{2-\alpha}\rho(x^*, x)$.

Since $\max\{\alpha, \frac{\alpha}{2-\alpha}\} = \alpha < 1$, from both cases, we have that: $H_\rho(T(x), x^*) \leq \alpha\rho(x, x^*)$, for each $x \in X$.

By induction we get that $H_\rho(T^n(x), x^*) \leq \alpha^n\rho(x, x^*)$, for each $x \in X$. Consider now $y^* \in F_T$. Then: $\rho(y^*, x^*) \leq H_\rho(T^n(x), x^*) \leq \alpha^n\rho(x, x^*) \to 0$, as $n \to \infty$. Hence $y^* = x^*$.

(c) We successively have: $\rho(x, x^*) \leq H_\rho(x, T(x)) + H_\rho(T(x), x^*) \leq H_\rho(x, T(x)) + \alpha\rho(x, x^*)$. Hence $\rho(x, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(x, T(x))$, for each $x \in X$.

(d) Let $(x_n)_{n\in\mathbb{N}}$ be such that $D_\rho(x_n, T(x_n)) \to 0$, as $n \to \infty$. We have to prove that $\rho(x_n, x^*) \to 0$, as $n \to \infty$.

Then we have:

$$\rho(x_n, x^*) \leq D_\rho(x_n, T(x_n)) + H_\rho(T(x_n), T(x^*)) \leq D_\rho(x_n, T(x_n)) + \alpha\rho(x_n, x^*).$$

Hence we get $\rho(x_n, x^*) \leq \frac{1}{1-\alpha} \cdot D_\rho(x_n, T(x_n)) \to 0$, as $n \to \infty$. □

Remark 3.2 *In the conditions of Theorem 3.2 we also have that:*

(1) $T : (X, d) \multimap (X, d)$ is a multivalued weakly Picard operator;

(2) $T : (X, \rho) \multimap (X, \rho)$ is a multivalued Picard operator.

A data dependence result for Ćirić-type multivalued operators is the following theorem.

Theorem 3.3 *Let X be a nonempty set, d and ρ two metrics on X and $T, S : X \to P(X)$ be two multivalued operators. We suppose that:*

(i) (X, d) is a complete metric space;

(ii) there exists $c > 0$ such that $d(x, y) \leq c\rho(x, y)$, for each $x, y \in X$;

(iii) $T : (X, d) \to (P(X), H_d)$ is closed;

(iv) there exists $\alpha \in [0, 1[$ such that $H_\rho(T(x), T(y)) \leq \alpha N_\rho^T(x, y)$, for each $x, y \in X$

(v) $(SF)_T \neq \emptyset$;

(vi) $F_S \neq \emptyset$

(vii) there exists $\eta > 0$ such that $H_\rho(T(x), S(x)) \leq \eta$, for each $x \in X$.

Then $H(F_T, F_S) \leq \frac{\eta}{1-\alpha}$.

Proof. Let $y^* \in F_S$. From the conclusion (c) of the previous theorem we have that:

$\rho(y^*, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(y^*, T(y^*)) \leq \frac{1}{1-\alpha} \cdot H_\rho(S(y^*), T(y^*)) \leq \frac{\eta}{1-\alpha}$.

Hence $H(F_T, F_S) = \sup\limits_{y^* \in F_S} \rho(y^*, x^*) \leq \dfrac{\eta}{1-\alpha}$. The proof is complete. □

References

[1] R. P. Agarwal, D. O'Regan, *Fixed point theory for generalized contractions on spaces with two metrics*, J. Math. Anal. Appl., **248**(2000), 402-414.

[2] L. B. Ćirić, *Fixed points for generalized multi-valued contractions*, Math. Vesnik, **9**(24)(1972), 265-272.

[3] H. Covitz, S.B. Nadler Jr., *Multi-valued contraction mapping in generalized metric spaces*, Israel J. Math., **8**(1970), 5-11.

[4] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis*, Vol. I-II, Kluwer Academic Publishers, Dordrecht, 1997, 1999.

[5] W. A. Kirk, B. Sims (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Acad. Publ., Dordrecht, 2001.

[6] S. B. Nadler Jr., *Multivalued contraction mappings*, Pacific J. Math., **30**(1969), 475-488.

[7] A. Petruşel, *Generalized multivalued contractions*, Nonlinear Analysis, **47**(2001), 649-659.

[8] A. Petruşel, *Multivalued weakly Picard operators and applications*, Scienticae Mathematicae Japonicae, **59**(2004), 167-202.

[9] A. Petruşel, I. A. Rus, *Multivalued Picard and weakly Picard operators*, Fixed Point Theory and Applications (J. Garcia Falset, E. LLorens Fuster, B. Sims eds.), Yokohama Publishers 2004, pp. 207-226.

[10] A. Petruşel, I. A. Rus, *Well-posedness of the fixed point problem for multivalued operators*, Applied Analysis and Differential Equations (O. Cârjă, I. I. Vrabie eds.), World Scientific 2007, pp. 295-306.

[11] S. Reich, *Fixed points of contractive functions*, Boll. U.M.I., **5**(1972), 26-42.

[12] I. A. Rus, *Generalized Contractions and Applications*, Transilvania Press Cluj-Napoca, 2001.

[13] I. A. Rus, *Strict fixed point theory*, Fixed Point Theory, **4**(2003), 177-183.

[14] I. A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory 1950-2000 : Romanian Contributions*, House of the Book of Science, Cluj-Napoca, 2002.

[15] I. A. Rus, A. Petruşel, A. Sîntămărian, *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Analysis, **52**(2003), 1947-1959.

[16] A. Sîntămărian, *Metrical strict fixed point theorems for multivalued mappings*, Seminar on Fixed Point Theory, 1997, 27-31.

Received: December 4, 2006; Accepted: February 20, 2007.