FIXED POINT THEORY FOR MULTIVALUED OPERATORS ON A SET WITH TWO METRICS

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Abstract. The purpose of this work is to present some fixed point results for multivalued operators on a set with two metrics. A multivalued version of Maia’s fixed point theorem is proved. The data dependence and the well-posedness of the fixed point problem are also discussed. Some extensions to generalized multivalued contractions are pointed out.

Key Words and Phrases: set with two metrics, Maia fixed point theorem, multivalued operator, fixed point, strict fixed point, well-posed fixed point problem, generalized contraction, data dependence.

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1. INTRODUCTION

Throughout this paper, the standard notations and terminologies in nonlinear analysis (see [14], [15], [7]) are used. For the convenience of the reader we recall some of them.

Let \((X, d)\) be a metric space. In the sequel we will use the following symbols:

\[ P(X) := \{ Y \subset X \mid Y \text{ is nonempty} \}, \quad P_d(X) := \{ Y \in P(X) \mid Y \text{ is closed} \}. \]

Let \(A\) and \(B\) be nonempty subsets of the metric space \((X, d)\). The gap between these sets is

\[ D(A, B) = \inf \{ d(a, b) \mid a \in A, \ b \in B \}. \]

In particular, \(D(x_0, B) = D(\{x_0\}, B)\) (where \(x_0 \in X\)) is called the distance from the point \(x_0\) to the set \(B\).
Also, if $A, B \in P_b(X)$, then one denote

$$\delta(A, B) := \sup\{d(a, b) | a \in A, b \in B\}.$$ 

The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets $A$ and $B$ of the metric space $(X, d)$ is defined by the following formula:

$$H(A, B) := \max\{\sup_{a \in A, b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.$$ 

The symbol $T : X \to Y$ means $T : X \to P(Y)$, i.e. $T$ is a multivalued operator from $X$ to $Y$. We will denote by $G(T) := \{(x, y) \in X \times Y | y \in T(x)\}$ the graph of $T$. The multivalued operator $T$ is said to be closed if $G(T)$ is closed in $X \times Y$.

For $T : X \to P(X)$ the symbol $F_T := \{x \in X | x \in T(x)\}$ denotes the fixed point set of the multivalued operator $T$, while $(SF)_T := \{x \in X | \{x\} = T(x)\}$ is the strict fixed point set of $T$. Also, for $x \in X$, we denote $F^n(x) := F(F^{n-1}(x)), n \in \mathbb{N}^*$, where $F^0(x) := \{x\}$.

If $F : X \to P_{cl}(X)$ is a multi-valued operator then $F$ is said to be $\alpha$-contraction if

$$\alpha \in [0, 1[ \text{ and for } x, y \in X \Rightarrow H(F(x), F(y)) \leq \alpha d(x, y).$$

The aim of this paper is to give some fixed point theorems for multivalued operators on a set endowed with two metrics. For the singlevalued case, see R. P. Agarwal, D. O’Regan [1], I. A. Rus, A. Petrușel, G. Petrușel [14] and the references therein.

2. Multivalued contraction on a set with two metrics

Our first main result is a multivalued version of Maia’s fixed point theorem.

**Theorem 2.1** Let $X$ be a nonempty set, $d$ and $\rho$ two metrics on $X$ and $T : X \to P(X)$ be a multivalued operator. We suppose that:

(i) $(X, d)$ is a complete metric space;

(ii) there exists $c > 0$ such that $d(x, y) \leq c \rho(x, y)$, for each $x, y \in X$;

(iii) $T : (X, d) \to (P(X), H_d)$ is closed;

(iv) there exists $\alpha \in [0, 1[ \text{ such that } H_\rho(F(x), F(y)) \leq \alpha \rho(x, y)$, for each $x, y \in X$.

Then we have:

(a) $F_T \neq \emptyset$;
(b) for each \( x \in X \) and each \( y \in T(x) \) there exists a sequence \((x_n)_{n \in \mathbb{N}}\) such that:

1. \( x_0 = x, \ x_1 = y; \)
2. \( x_{n+1} \in T(x_n), \ n \in \mathbb{N}; \)
3. \( x_n \xrightarrow{d} x^* \in T(x^*), \text{ as } n \to \infty. \)

**Proof.** As in the proof of Avramescu-Markin-Nadler’s theorem (see [3], [6], [12]) hypothesis (iv) implies that there exists a Cauchy sequence \((x_n)_{n \in \mathbb{N}}\) in \((X, \rho)\), such that (1) and (2) hold. From (ii) it follows that the sequence \((x_n)_{n \in \mathbb{N}}\) is Cauchy in \((X, d)\). Denote by \(x^* \in X\) the limit of this sequence. From (i) and (iii) we get that \( x_n \xrightarrow{d} x^* \in T(x^*), \text{ as } n \to \infty. \) The proof is complete. \( \square \)

**Remark 2.1** In terms of the multivalued weakly Picard operators theory (see [15], [12]), the conclusion of the above result takes the following form:

(a’)-(b’): \( T : (X, d) \to (X, d) \) is a multivalued weakly Picard operator.

The second main result of this section is the following theorem.

**Theorem 2.2** Let \( X \) be a nonempty set, \( d \) and \( \rho \) two metrics on \( X \) and \( T : X \to P(X) \) be a multivalued operator. We suppose that:

(i) \((X, d)\) is a complete metric space;
(ii) there exists \( c > 0 \) such that \( d(x, y) \leq c \rho(x, y) \), for each \( x, y \in X \);
(iii) \( T : (X, d) \to (P(X), H_d) \) is closed;
(iv) there exists \( \alpha \in [0, 1[ \) such that \( H_{\rho}(T(x), T(y)) \leq \alpha \rho(x, y) \), for each \( x, y \in X \);
(v) \((SF)_T \neq \emptyset. \)

Then we have:

(a) \( F_T = (SF)_T = \{x^*\}; \)
(b) \( H_{\rho}(T^n(x), x^*) \leq \alpha^n \cdot \rho(x, x^*) \), for each \( n \in \mathbb{N} \) and each \( x \in X \);
(c) \( \rho(x, x^*) \leq \frac{1}{1 - \alpha} \cdot H_{\rho}(x, T(x)) \), for each \( x \in X \);
(d) the fixed point problem is well-posed for \( T \) with respect to \( D_{\rho} \).

**Proof.** (a)-(b) From (iv) we have that if \( x^* \in (SF)_T \) then \((SF)_T = \{x^*\}\), see I. A. Rus [12], pp. 87. Also, by taking \( y := x^* \) in (iv) we have that \( H_{\rho}(T(x), x^*) \leq \alpha \rho(x, x^*) \), for each \( x \in X \). By induction we get that \( H_{\rho}(T^n(x), x^*) \leq \alpha^n \rho(x, x^*) \), for each \( x \in X \). Consider now \( y^* \in F_T \). Then:

\[ \rho(y^*, x^*) \leq H_{\rho}(T^n(x), x^*) \leq \alpha^n \rho(x, x^*) \to 0, \text{ as } n \to \infty. \]

Hence \( y^* = x^* \).
(c) We successively have: $\rho(x, x^*) \leq H_\rho(x, T(x)) + H_\rho(T(x), x^*) \leq H_\rho(x, T(x)) + \alpha \rho(x, x^*)$. Hence $\rho(x, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(x, T(x))$, for each $x \in X$.

(d) Let $(x_n)_{n \in \mathbb{N}}$ be such that $D_\rho(x_n, T(x_n)) \to 0$, as $n \to \infty$. We have to prove that $\rho(x_n, x^*) \to 0$, as $n \to \infty$ (see [10]).

Then we have:

$$
\rho(x_n, x^*) \leq D_\rho(x_n, T(x_n)) + H_\rho(T(x_n), T(x^*)) \leq D_\rho(x_n, T(x_n)) + \alpha \rho(x_n, x^*).
$$

Hence we get $\rho(x_n, x^*) \leq \frac{1}{1-\alpha} \cdot D_\rho(x_n, T(x_n)) \to 0$, as $n \to \infty$. □

Remark 2.2 For the implication $(SF)_T \neq \emptyset \Rightarrow (SF)_T = \{x^*\}$ in the theory of multivalued generalized contractions see I. A. Rus [13] and A. Sîntămărian [16].

Remark 2.3 In the conditions of Theorem 2.2 we also have that:

(1) $T : (X, d) \to (X, d)$ is a multivalued weakly Picard operator;

(2) $T : (X, \rho) \to (X, \rho)$ is a multivalued Picard operator (see A. Petruşel, I. A. Rus [9]).

A data dependence result is the following theorem.

**Theorem 2.3** Let $X$ be a nonempty set, $d$ and $\rho$ two metrics on $X$ and $T, S : X \to P(X)$ be two multivalued operators. We suppose that:

(i) $(X, d)$ is a complete metric space;

(ii) there exists $c > 0$ such that $d(x, y) \leq c \rho(x, y)$, for each $x, y \in X$;

(iii) $T : (X, d) \to (P(X), H_d)$ is closed;

(iv) there exists $\alpha \in [0, 1]$ such that $H_\rho(T(x), T(y)) \leq \alpha \rho(x, y)$, for each $x, y \in X$

(v) $(SF)_T \neq \emptyset$;

(vi) $F_S \neq \emptyset$

(vii) there exists $\eta > 0$ such that $H_\rho(T(x), S(x)) \leq \eta$, for each $x \in X$.

Then $H(F_T, F_S) \leq \frac{\eta}{1-\alpha}$.

**Proof.** Let $y^* \in F_S$. From the conclusion (c) of the above theorem we have that:

$$
\rho(y^*, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(y^*, T(y^*)) \leq \frac{1}{1-\alpha} \cdot H_\rho(S(y^*), T(y^*)) \leq \frac{\eta}{1-\alpha}.
$$

Hence $H(F_T, F_S) = \sup_{y^* \in F_S} \rho(y^*, x^*) \leq \frac{\eta}{1-\alpha}$. The proof is complete. □
3. MULTIVALUED GENERALIZED CONTRACTION ON A SET WITH TWO METRICS

Let \((X, d)\) be a metric space and \(T : X \rightarrow P_{cl}(X)\) be a multivalued operator. For \(x, y \in X\), let us denote

\[
M^T_d(x, y) := \max\{d(x, y), D_d(x, T(x)), D_d(y, T(y)), \frac{1}{2}[D_d(x, T(y)) + D_d(y, T(x))]/\}
\]

\(\dot{C}\)irić proved that if the space \((X, d)\) is complete and if the multivalued operator \(T : X \rightarrow P_{cl}(X)\) satisfies the following condition:

there exists \(\alpha \in [0, 1]\) such that \(H_d(T(x), T(y)) \leq \alpha \cdot M^T_d(x, y)\), for each \(x, y \in X\), then \(F_T \neq \emptyset\) and for each \(x \in X\) and each \(y \in T(x)\) there exists a sequence \((x_n)_{n \in \mathbb{N}}\) such that:

(1) \(x_0 = x, x_1 = y\);
(2) \(x_{n+1} \in T(x_n), n \in \mathbb{N}\);
(3) \(x_n \xrightarrow{d} x^* \in T(x^*)\), as \(n \to \infty\).

Next result is a multivalued version of Maia’s theorem for \(\dot{C}\)irić-type multivalued operators.

**Theorem 3.1** Let \(X\) be a nonempty set, \(d, \rho\) two metrics on \(X\) and \(T : X \rightarrow P(X)\) be a multivalued operator. We suppose that:

(i) \((X, d)\) is a complete metric space;
(ii) there exists \(c > 0\) such that \(d(x, y) \leq c\rho(x, y)\), for each \(x, y \in X\);
(iii) \(T : (X, d) \rightarrow (P(X), H_d)\) is closed;
(iv) there exists \(\alpha \in [0, 1]\) such that \(H_\rho(T(x), T(y)) \leq \alpha M^T_\rho(x, y)\), for each \(x, y \in X\).

Then we have:

(a) \(F_T \neq \emptyset\);
(b) for each \(x \in X\) and each \(y \in T(x)\) there exists a sequence \((x_n)_{n \in \mathbb{N}}\) such that:

(1) \(x_0 = x, x_1 = y\);
(2) \(x_{n+1} \in T(x_n), n \in \mathbb{N}\);
(3) \(x_n \xrightarrow{d} x^* \in T(x^*)\), as \(n \to \infty\).

**Proof.** As in the proof of \(\dot{C}\)irić’s theorem (see [2], Theorem 2), hypothesis (iv) implies that there exists a Cauchy sequence \((x_n)_{n \in \mathbb{N}}\) in \((X, \rho)\), such that (1) and (2) hold. From (ii) it follows that the sequence \((x_n)_{n \in \mathbb{N}}\) is Cauchy in
(X, d). Denote by \( x^* \in X \) the limit of this sequence. From (i) and (iii) we get that \( x_n \xrightarrow{d} x^* \in T(x^*) \), as \( n \to \infty \). The proof is complete. □

**Remark 3.1** In terms of the multivalued weakly Picard operators theory, the conclusion of the above result takes the following form:

(a')-(b') \( T : (X, d) \to (X, d) \) is a multivalued weakly Picard operator.

For the next results, let us denote

\[
N^T_d(x, y) := \max\{d(x, y), D_d(y, T(y)), \frac{1}{2}[D_d(x, T(y)) + D_d(y, T(x))]\}.
\]

The second main result of this section is the following theorem.

**Theorem 3.2** Let \( X \) be a nonempty set, \( T : X \to P(X) \) be a multivalued operator and \( d, \rho \) two metrics on \( X \). We suppose that:

(i) \( (X, d) \) is a complete metric space;

(ii) there exists \( c > 0 \) such that \( d(x, y) \leq c\rho(x, y) \), for each \( x, y \in X \);

(iii) \( T : (X, d) \to (P(X), H_d) \) is closed;

(iv) there exists \( \alpha \in [0, 1] \) such that \( H_\rho(T(x), T(y)) \leq \alpha N^T_\rho(x, y) \), for each \( x, y \in X \);

(v) \( (SF)_T \neq \emptyset \).

Then we have:

(a) \( F_T = (SF)_T = \{x^*\} \);

(b) \( H_\rho(T^n(x), x^*) \leq \alpha^n \cdot \rho(x, x^*) \), for each \( n \in \mathbb{N} \) and each \( x \in X \);

(c) \( \rho(x, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(x, T(x)) \), for each \( x \in X \);

(d) the fixed point problem is well-posed for \( T \) with respect to \( D_\rho \).

**Proof.** (a)-(b) From (iv) we have that if \( x^* \in (SF)_T \) then \( (SF)_T = \{x^*\} \).

Indeed, if \( y \in (SF)_T \) then \( \rho(x^*, y) = H_\rho(T(x^*), T(y)) \leq \alpha N^T_\rho(x^*, y) = \alpha \cdot \max\{\rho(x^*, y), \frac{1}{2}[\rho(x^*, y) + \rho(y, x^*)]\} = \alpha \cdot \rho(x^*, y) \). Hence \( y = x^* \).

For the second conclusion let’s take, in the condition (iv), \( y := x^* \). Then, for each \( x \in X \), we have: \( H_\rho(T(x), x^*) = H_\rho(T(x), T(x^*)) \leq \alpha N^T_\rho(x, x^*) = \alpha \cdot \max\{\rho(x, x^*), \frac{1}{2}[\rho(x^*, T(x)) + \rho(T(x^*), x)]\} \). We distinguish the following two cases:

1) If the above maximum is \( \rho(x, x^*) \) then we have \( H_\rho(T(x), x^*) \leq \alpha \cdot \rho(x, x^*) \).

2) If the maximum is \( \frac{1}{2}[\rho(T(x^*), x) + \rho(T(x^*), x)] \), then \( H_\rho(T(x), x^*) \leq \alpha \cdot \frac{1}{2}[H_\rho(x^*, T(x)) + \rho(x^*, x)] \). Hence \( H_\rho(T(x), x^*) \leq \frac{\alpha}{2} \rho(x^*, x) \).
Since \( \max\{\alpha, \frac{\alpha}{2^{m-1}}\} = \alpha < 1 \), from both cases, we have that: \( H_\rho(T(x), x^*) \leq \alpha \rho(x, x^*) \), for each \( x \in X \).

By induction we get that \( H_\rho(T^n(x), x^*) \leq \alpha^n \rho(x, x^*) \), for each \( x \in X \). Consider now \( y^* \in F_T \). Then: \( \rho(y^*, x^*) \leq H_\rho(T^n(x), x^*) \leq \alpha^n \rho(x, x^*) \to 0 \), as \( n \to \infty \). Hence \( y^* = x^* \).

(c) We successively have: \( \rho(x, x^*) \leq H_\rho(x, T(x)) + H_\rho(T(x), x^*) \leq H_\rho(x, T(x)) + \alpha \rho(x, x^*) \). Hence \( \rho(x, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(x, T(x)) \), for each \( x \in X \).

(d) Let \( (x_n)_{n \in \mathbb{N}} \) be such that \( D_\rho(x_n, T(x_n)) \to 0 \), as \( n \to \infty \). We have to prove that \( \rho(x_n, x^*) \to 0 \), as \( n \to \infty \).

Then we have:

\[
\rho(x_n, x^*) \leq D_\rho(x_n, T(x_n)) + H_\rho(T(x_n), T(x^*)) \leq D_\rho(x_n, T(x_n)) + \alpha \rho(x_n, x^*).
\]

Hence we get \( \rho(x_n, x^*) \leq \frac{1}{1-\alpha} \cdot D_\rho(x_n, T(x_n)) \to 0 \), as \( n \to \infty \). □

**Remark 3.2** In the conditions of Theorem 3.2 we also have that:

1. \( T : (X, d) \to (X, d) \) is a multivalued weakly Picard operator;
2. \( T : (X, \rho) \to (X, \rho) \) is a multivalued Picard operator.

A data dependence result for Ćirić-type multivalued operators is the following theorem.

**Theorem 3.3** Let \( X \) be a nonempty set, \( d \) and \( \rho \) two metrics on \( X \) and \( T, S : X \to P(X) \) be two multivalued operators. We suppose that:

(i) \( (X, d) \) is a complete metric space;
(ii) there exists \( c > 0 \) such that \( d(x, y) \leq c \rho(x, y) \), for each \( x, y \in X \);
(iii) \( T : (X, d) \to (P(X), H_d) \) is closed;
(iv) there exists \( \alpha \in [0,1] \) such that \( H_\rho(T(x), T(y)) \leq \alpha N_\rho^T(x, y) \), for each \( x, y \in X \)

(v) \( (SF)_T \neq \emptyset \);
(vi) \( F_S \neq \emptyset \)
(vii) there exists \( \eta > 0 \) such that \( H_\rho(T(x), S(x)) \leq \eta \), for each \( x \in X \).

Then \( H(F_T, F_S) \leq \frac{\eta}{1-\alpha} \).

**Proof.** Let \( y^* \in F_S \). From the conclusion (c) of the previous theorem we have that:

\[
\rho(y^*, x^*) \leq \frac{1}{1-\alpha} \cdot H_\rho(y^*, T(y^*)) \leq \frac{1}{1-\alpha} \cdot H_\rho(S(y^*), T(y^*)) \leq \frac{\eta}{1-\alpha}.
\]

Hence \( H(F_T, F_S) = \sup_{y^* \in F_S} \rho(y^*, x^*) \leq \frac{\eta}{1-\alpha} \). The proof is complete. □
References


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