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A COMPARISON OF PICARD AND MANN ITERATIONS FOR QUASI-CONTRACTION MAPS

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Abstract. For a class of quasi-contractive operators defined on an arbitrary Banach space, it has been shown that the Picard iteration technique converges faster than the Mann iteration technique. In this paper we make a comparison of the Picard and Mann iterations with respect to their convergence rate for a more general class of operators called quasi-contractions in metrizable topological vector spaces. It was observed that the Picard iteration converges faster than the Mann iteration for this class of maps. This answers the question posed by Berinde in his paper.
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1. Introduction

Several iteration techniques for approximating fixed points of various classes of Lipschitzian maps defined on a metric space have been introduced by several authors over a span of about four decades. Some of them are Picard iteration technique, Mann iteration technique, Krasnoselskij iteration technique, Bruck iteration technique and Newton iteration technique. The Picard iteration technique, the Mann iteration technique and the Ishikawa iteration technique are the most used of all those methods. For a discussion of those techniques and their limitations, see [6], [7] and [18-19].

Let X be a metrizable topological space space and C be a nonempty subset of X. Let $T : C \to C$ be a mapping. The iteration scheme called $Ishikawa-$

type scheme is defined as follows

$$x_o \in C, \tag{1}$$

$$y_n = \beta_n T x_n + (1 - \beta_n) x_n, \ n \geq 0, \tag{2}$$

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \ n \geq 0 \tag{3}$$

$\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 \leq \alpha_n, \ \beta_n \leq 1$ for all n, and $\sum_{n=0}^{\infty} \alpha_n = \infty$

For other variants of Ishikawa Iteration scheme see [6], [7] and [19]. If $\beta_n = 0$ for all n, then the Ishikawa Iteration scheme reduces to Mann Iteration scheme.

The most general *Mann type iterative scheme* now studied is the following:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \ x_o \in C, \ n \geq 0 \tag{4}$$

where $\{\alpha_n\}$ satisfy $0 \leq \alpha_n \leq 1$ for all n. We shall also assume that $\sum_{n=0}^{\infty} \alpha_n = \infty$. For other variants of Mann iteration scheme, see [6], [7] and [19].

Obviously, if $\alpha_n = 1$ for each n in the Mann iterative scheme, then we have the *Picard iteration technique*.

Definition 1. [8]. Let $T : M \to M$ be a mapping of a metric space (M, d) into itself. A mapping T will be called quasi-contraction if for some $0 \leq k < 1$ and all $x, y \in M$,

$$d(Tx, Ty) \leq k.max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\}$$

Obviously, it is a generalization of those maps studied by Kannan [11-12], Hardy and Rogers [10], Reich [17], Sehgal [21], Chatterjea [5], Olaleru [14-15] and Rhoades [18] among others. It also generalizes the class of quasi-contractive maps introduced by Zamfirescu in 1972 [22]. The convergence of *Picard iterates* to the unique fixed point of T when T is a Zamfirescu operator is proved in [22] while the convergence of the Ishikawa and the Mann iteration schemes to the unique fixed point of the Zamfirescu operator defined on a Banach space were proved by Berinde [3]. The result [3] was generalized to a metrisable locally convex space by author in [14]. Furthermore, Berinde [4] showed the Picard iteration of a Zamfirescu operator converges faster than the Mann iteration. He then asked whether the same result can be obtained for quasi-contraction maps which are more general than the Zamfirescu's operators. This paper answers the question in the affirmative, i.e. the Picard iteration of a quasi-contraction map converges faster than the Mann iteration scheme.

The existence of a unique fixed point for a quasi-contraction map T and the fact that the $Picard's\ iterates$ converge to the unique fixed point of T are proved by Ciric [8]. The convergence of the Ishikawa iteration scheme for a quasi-contraction map T when $X = L^p,\ 1 < p < 2$ was proved by Chidume [7]. This was later extended to when X is a complete convex metric space by Ciric [9]. As shown in recent papers, see, for example Chidume [6], at present it is no more of real interest to consider Ishikawa-type iterations in order to approximate fixed points of those classes of mappings where simpler iterative techniques (like Mann) could be sufficient. As a result of this observation, the author [16] proved the convergence of the Mann iteration scheme for a quasi-contraction map T when X is a complete metrizable (not necessarily convex) topological vector space.

Definition 2. [1, p10]. An $F-norm$ on a vector space E is a real-valued function $\|\| : E \to \Re$, such that for all $x, y \in E$, we have
(1). $\|x\| \geq 0$,
(2). $\|x\| = 0 \Rightarrow x = 0$,
(3). $\|x + y\| \leq \|x\| + \|y\|$,
(4). $\|\lambda x\| \leq \|x\|$ for all $\lambda \in K$ with $|\lambda| \leq 1$,
(5). If $\lambda_n \to 0$, and $\lambda_n \in K$, then $\|\lambda_n x\| \to 0$.

The following theorem is fundamental to our results. The proof will be omitted.

Theorem A [1, &2]. *A topological vector space is metrizable if and only if it has a countable base of neighborhoods of zero. The topology of a metrizable topological vector space can always be defined by an* $F-norm$.

For the same result see Kothe [13, &15.11]

Henceforth, unless otherwise indicated, F shall denote $F - norm$ if it is characterizing a metrizable topological vector space. Observe that an $F{-}norm$ will be a norm if it is defining a normed space.

The author proved the following Theorem.

Theorem B. [16]. *Let K be a nonempty closed convex subset of a complete metrizable topological vector space X and let $T : K \to K$ be a quasi-contraction* *map. Suppose* $\{x_n\}$ *is a Mann iteration sequence defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \ n = 1, 2, \ldots$$

such that

$$\alpha_n < min \ \{1, \frac{1}{\delta}\}$$

for each n*, where*

$$\delta = \frac{k}{1 - k}$$

and

$$\sum \alpha_n = \infty.$$

Then $\{x_n\}$ *converges to unique fixed point of* T*.*

2. The main result

In comparing two fixed point iteration procedures $\{u_n\}_{n=0}^{n=\infty}$ and $\{v_n\}_{n=0}^{n=\infty}$ that converge to the same fixed point p of an operator T, Rhoades [19] considered that $\{u_n\}$ is *better than* $\{v_n\}$ if

$$\|u_n - p\| \leq \|v_n - p\|, \ for \ all \ n.$$

We shall adopt the following terminologies and definitions used in [2] and [4] since they are more suitable for our purpose.

Definition 3. Let $\{a_n\}_{n=0}^{n=\infty}$ and $\{b_n\}_{n=0}^{n=\infty}$ be two sequences of real numbers that converge to a and b respectively, and assume there exists

$$l = lim_{n \to \infty} \frac{|a_n - a|}{|b_n - b|}$$

a) If $l = 0$, then we say that $\{a_n\}_{n=0}^{n=\infty}$ converges faster to a than $\{b_n\}_{n=0}^{n=\infty}$ to b, in which case we use the notation $a_n - a = o(b_n - b)$;

b) If $0 < l < \infty$, then we say that $\{a_n\}_{n=0}^{n=\infty}$ and $\{b_n\}_{n=0}^{n=\infty}$ have the same rate of convergence;

c) If $l = \infty$, then we say that $\{b_n\}_{n=0}^{n=\infty}$ converges faster to b than $\{a_n\}_{n=0}^{n=\infty}$ to a, in which case we use the notation $b_n - b = o(a_n - a)$.

Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^{n=\infty}$ and $\{v_n\}_{n=0}^{n=\infty}$, both converging to the same fixed point p in a metrizable topological vector space X, the error estimates

$$F(u_n - p) \leq a_n, \ n = 0, 1, 2, \ldots \tag{5}$$

and

$$F(v_n - p) \leq b_n,\ n = 0, 1, 2, \ldots \tag{6}$$

are available, where $\{a_n\}_{n=0}^{n=\infty}$ and $\{b_n\}_{n=0}^{n=\infty}$ are two sequences of positive numbers (converging to zero).

Definition 4. Let $\{u_n\}_{n=0}^{n=\infty}$ and $\{v_n\}_{n=0}^{n=\infty}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (0.5) and (0.6) respectively. If $\{a_n\}_{n=0}^{n=\infty}$ converges faster than $\{b_n\}_{n=0}^{n=\infty}$, then we say that $\{u_n\}_{n=0}^{n=\infty}$ converges faster to p than $\{v_n\}_{n=0}^{n=\infty}$.

Example ([4]). If we take $p = 0$, $u_n = \frac{1}{n^2+n}$, $v_n = \frac{1}{n^2}$, $n \geq 1$, then $\{u_n\}$ is better than $\{v_n\}$, but $\{u_n\}$ does not converge faster than $\{v_n\}$. In fact

$$lim_{n\to\infty} \frac{u_n}{v_n} = 1$$

Thus definition 4 is a sharper concept of the rate of convergence than the one introduced by Rhoades [19].

We now prove our main result. We follow the style and the technique of Berinde [4].

Theorem 1. *Let K be a nonempty closed convex subset of a complete metrizable topological vector space X and let $T : K \to K$ be a quasi-contraction map. Then:*
1) T has a unique fixed point p in X;
2) The Picard iteration $\{x_n\}$ defined by $Tx_n = x_{n+1}$ converges to p for any $x_o \in K$;
3) Suppose the Mann iteration sequence $\{x_n\}$, defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$, $n = 1, 2, \ldots$ such that $\sum \alpha_n = \infty$, converges. Then the Picard iteration converges faster than Mann iteration.

Proof. The proofs of 1) and 2) follows immediately from [8].
Since T is a quasi-contraction, then,

$$\begin{aligned} F(Ty_n - Tx_n) \ \leq \ & k\ max\{F(y_n - x_n), F(x_n - Tx_n), F(y_n - Ty_n), \\ & F(x_n - Ty_n), F(y_n - Tx_n)\}. \end{aligned}$$

If $F(Ty_n - Tx_n) \leq kF(y_n - Ty_n)$, then

$$F(Ty_n - Tx_n) \leq k\{F(y_n - x_n) + F(x_n - Tx_n) + F(Tx_n - Ty_n)\}$$

and so,

$$F(Ty_n - Tx_n) \leq \frac{k}{1-k}\{F(y_n - x_n) + F(x_n - Tx_n)\} \tag{7}$$

If $F(Ty_n - Tx_n) \leq kF(x_n - Ty_n)$, then,

$$F(Ty_n - Tx_n) \leq k\{F(x_n - Tx_n) + F(Tx_n - Ty_n)\}$$

which, after computing, gives

$$F(Tx_n - Ty_n) \leq \frac{k}{1-k}F(x_n - Tx_n) \tag{8}$$

If $F(Ty_n - Tx_n) \leq kF(y_n - Tx_n)$, then,

$$F(Ty_n - Tx_n) \leq k\{F(y_n - x_n) + F(x_n - Tx_n)\} \tag{9}$$

Denote $\delta = \max\{k, \frac{k}{1-k}\} = \frac{k}{1-k}$. Then (0.7), (0.8) and (0.9) give

$$F(Ty_n - Tx_n) \leq \delta\{F(y_n - x_n) + F(x_n - Tx_n)\} \tag{10}$$

Suppose p is a fixed point of T, then, if $x_n = p$ and $y_n = x_n$, from (0.10) we obtain

$$F(Tx_n - p) \leq \delta F(x_n - p) \tag{11}$$

If we assume Picard approximation technique in (0.11) by assuming that $Tx_n = x_{n+1}$ for all n, we obtain

$$F(x_{n+1} - p) \leq \delta.F(x_n - p)$$

which inductively gives

$$F(x_n - p) \leq \delta^n.F(x_1 - p),\ n \geq 0 \tag{12}$$

Following the same procedure in proving (0.10), it can be shown that

$$F(Tx_n - Ty_n) \leq \delta.F(x_n - y_n) + \delta.F(y_n - Tx_n) \tag{13}$$

for all $x, y \in K$ where $\delta = \frac{k}{1-k}$. Let $\{y_n\}_{n=0}^{n=\infty}$ be the Mann iteration as defined in the Theorem and $y_o \in K$ arbitrary. Then

$$\begin{aligned} F(y_{n+1} - p) &= F((1-\alpha_n)y_n + \alpha_n Ty_n - (1 - \alpha_n + \alpha_n)p) \\ &= F((1-\alpha_n)(y_n - p) + \alpha_n(Ty_n - p)) \\ &\leq (1-\alpha_n)F(y_n - p) + \alpha_n F(Ty_n - p)\ (**) \end{aligned}$$

If $x_n = p$ in (0.13) we obtain

$$F(Ty_n - p) \leq \delta F(y_n - p) + \delta.F(y_n - p) = 2\delta F(y_n - p)$$

and therefore by (**) we obtain

$$F(y_{n+1} - p) \leq [1 - \alpha_n + 2\delta\alpha_n]F(y_n - p)$$

$$\leq [1 + 2\delta\alpha_n + \delta]F(y_n - p),\ n = 0, 1, 2, \dots$$

which implies that

$$F(y_{n+1} - p) \leq \prod_{k=1}^{n} [[1 + 2\delta\alpha_k + \delta]F(y_1 - p),\ n = 0, 1, 2, \dots \tag{14}$$

In order to compare $\{x_n\}$ and $\{y_n\}$ we must compare δ^n and

$$\prod_{k=1}^{n} [1 + 2\delta\alpha_k + \delta].$$

We first note that $\delta < 1 + 2\delta\alpha_k + \delta$ for each k. Therefore $\frac{\delta}{1+2\delta\alpha_k+\delta} < 1$ for each k. Hence

$$lim_{n\to\infty} \frac{\delta^k}{\prod_{k=1}^{n}[1 + 2\delta\alpha_k + \delta]} \to 0$$

This shows that the Picard iteration converges faster than the Mann iteration.

Corollary 1. *Let K be a nonempty closed convex subset of a complete metrizable locally convex space X and let $T : K \to K$ be a quasi-contraction map. Then:*

1) T has a unique fixed point p in X;

2) The Picard iteration $\{x_n\}$ defined by $Tx_n = x_{n+1}$ converges to p for any $x_o \in K$;

3) Suppose is a Mann iteration sequence $\{x_n\}$, defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$, $n = 1, 2, \dots$ such that $\sum \alpha_n = \infty$, converges. Then the Picard iteration converges faster than Mann iteration.

Corollary 2. *Let K be a nonempty closed convex subset of a Banach space X and let $T : K \to K$ be a quasi-contraction map. Then:*

1) T has a unique fixed point p in X;

2) The Picard iteration $\{x_n\}$ defined by $Tx_n = x_{n+1}$ converges to p for any $x_o \in K$;

3) Suppose is a Mann iteration sequence $\{x_n\}$, defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$, $n = 1, 2, \dots$ such that $\sum \alpha_n = \infty$, converges. Then the Picard iteration converges faster than Mann iteration.

Corollary 3 [4, Theorem 4]. *Let X be a Banach space, K a closed convex subset of X, and $T : K \to K$ an operator for which there exist the real numbers a, b, c satisfying $0 < a < 1,\ 0 < b,\ c < 1/2$ such that for each pair $x, y \in K$, at least one of the following is true:*

(i) $\|Tx - Ty\| \leq a\|x - y\|$*;*

(ii) $\|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|]$*;*

(iii) $\|Tx - Ty\| \leq c[\|x - Ty\| + \|y - Tx\|]$.

Then:

1) T *has a unique fixed point* p *in* X*;*

2) The Picard iteration $\{x_n\}$ *defined by* $Tx_n = x_{n+1}$ *converges to* p *for any* $x_o \in K$*;*

3) The Mann iteration $\{y_n\}$ *converges to* p *for any* $y_o \in K$ *and* $\{\alpha_n\}$ *satisfying (0.4);*

4) Picard iteration converges faster than Mann iteration.

Proof. The fact that the Mann iteration converges for the operators defined above is already proved by the author [14].

Remark. It is not yet known whether it is true that the Picard iteration scheme will still converge faster than the Mann iteration when T is generalized to generalized contraction maps [20].

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