Fixed Point Theory, Volume 8, No. 1, 2007, 69-85
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

ON SOME GENERALIZATIONS OF THE LANDESMAN-LAZER THEOREM

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Abstract. We apply the coincidence degree theory for compact multivalued perturbations of Fredholm operators to obtain necessary and sufficient conditions for the existence of solutions for an equation containing a linear Fredholm operator with an one-dimensional kernel and a discontinuous nonlinearity. Further we consider the extension to the case when the kernel is multi-dimensional and the Fredholm operator is not necessarily self-adjoint. Some examples are given.
Key Words and Phrases: Landesman–Laser equation, Fredholm operator, resonance, degeneracy, discontinuous nonlinearity, multivalued map, coincidence point, coincidence degree, topological degree.
2000 Mathematics Subject Classification: 47J05, 35J25, 35R05, 47H04, 47H11, 54H25.

1. Introduction

In the work of E.M. Landesman and A.C. Lazer [14] it was observed that the boundary value problem for a nonlinear elliptic equation at the presence

First and third author are supported by the Russian FBR Grants 04-01-00081 and 05-01-00100. The second author is supported by the Italian Cofin 04-05. All authors work is partially supported by the NATO Grant ICS.NR.CLG 981757.

of resonance is solvable not for each nonlinear part even in case when this term is bounded. The authors of this work presented necessary and sufficient conditions for the solvability of such problems for the case of one-dimensional degeneracy of the linear part of equation. In the sequel these ideas and methods were widely extended to the cases of equations of higher order and multi-dimensional degeneracy (see, e.g., [17]) as well as to equations with discontinuous nonlinearities (see, e.g. [1], [2], [15], [18], [20], [21] and others). Let us mention that equations with discontinuous nonlinearities are the subject of interest of many researchers since they find interesting applications to problems of mathematical physics (free boundary problems, in particular, obstacle problem, the seepage surface problem etc.)(see, e.g. [5]-[7], [11]-[13]).

In the present paper we consider a new class of equations containing an abstract linear Fredholm operator with one-dimensional kernel and a discontinuous nonlinearity. Such equation is reduced to an operator inclusion and the topological coincidence degree theory for compact multivalued perturbations of Fredholm operators is applied to justify the Landesman-Lazer type conditions for the existence of solutions to the initial problem. We present two examples of problems of mathematical physics in which such equations appear. The first example deals with the equilibrium position of membrane at the presence of resonance and nonlinear deformation, the second one concerns the Lavrentiev's problem on detachable currents. In conclusion we give, in terms of topological degree for multivalued maps, the extensions of conditions for solvability to the case when the degeneracy is multi-dimensional and the Fredholm operator is not necessarily self-adjoint.

2. Preliminaries

By Ω we will denote a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary. For $p \geq 1$, we let $L_p(\Omega)$ denote the Banach space of p-integrable functions on Ω with the norm

$$\|u\|_p = \|u\|_{L_p} = (\int_\Omega |u|^p \, dx)^{1/p}.$$

For an integer $k > 0$ *the Sobolev space* $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) = \{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega) \text{ for all } |\alpha| \leq k\},$$

where $D^{\alpha}u$ denotes the distributional derivative of u of order α. We will assume that $W_p^k(\Omega)$ is equipped with the norm

$$\|u\|_{p,k} = \sum_{|\alpha|\leq k} \|D^{\alpha}u\|_p.$$

By $\overset{\circ}{W}{}_p^k(\Omega)$ we will denote the subset of $W_p^k(\Omega)$ consisting of all functions vanishing on the boundary $\partial\Omega$.

Let us recall (see, e.g. [8]) that in accordance with the Sobolev embedding theorem in case $pk > n$ the space $W_p^k(\Omega)$ is compactly embedded into $C(\Omega)$.

Let X and Z be Banach spaces; $Cv(Z)$ $[Kv(Z)]$ denote a collection of all nonempty closed convex [respectively, compact convex] subsets of Z. A multivalued map (multimap) $\Phi : X \to Cv\,(Z)$ is said to be: (i) *upper semi-continuous (u.s.c.)* if for every open set $V \subset Z$, the set

$$\Phi_+^{-1}\,(V) = \{x \in X : \Phi\,(x) \subset V\}$$

is open in X; (ii) *closed* if its graph

$$\Gamma_\Phi = \{(u,f) \in X \times Z : f \in \Phi(u)\}$$

is the closed subset of $X \times Z$.

For a linear operator $A : domA \subseteq X \to Z$, let $P : X \to X$ and $Q : Z \to Z$ be projectors such that $ImP = KerA$ and $KerQ = ImA$. If the operator

$$A_P : domA \cap KerP \to ImA$$

is defined as the restriction of A on $domA \cap KerP$ then it s clear that A_P is an algebraic isomorphism and we may define $K_P : ImA \to domA$ by $K_P = A_P^{-1}$. Now let $CokerA = Z/ImA$ and $\Pi : Z \to CokerA$ be canonical surjection:

$$\Pi(z) = z + ImA$$

and $K_{P,Q} : Z \to X$ be defined by

$$K_{P,Q} = K_P(I - Q).$$

Let us recall (see, for example, [4], [9], [16]) that a linear operator A : $domA \subseteq X \to Z$ is called *Fredholm of zero index* if

(i) ImA is closed in Z;

(ii) $KerA$ and $CokerA$ have finte dimension and

$$dimKerA = dimCokerA.$$

We will assume also that

(iii) the operator $K_{P,Q} : Z \to domA$ is continuous.

Now, let $U \subset X$ be an open bounded subset, $\Phi : \overline{U} \to Cv(Z)$ a multimap, and $A : domA \subseteq X \to Z$ a linear Fredholm operator of zero index. Let $\Lambda : CokerA \to KerA$ be any isomorphism. Suppose that the pair (A, Φ) is *compact* on $\overline{U}$, i.e. the composition $(\Lambda\Pi + K_{P,Q}) \circ \Phi : \overline{U} \to Kv(X)$ is the compact u.s.c. multimap.

The set of coincidence points

$$Coin(A, \Phi) = \{u \in domA : Au \in \Phi(u)\}$$

is equal to the set of fixed points of the multimap $F : \overline{U} \to Kv(X)$ defined by

$$F(u) = Pu + (\Lambda\Pi + K_{P,Q}) \circ \Phi(u).$$

From the definition and properties of multivalued maps (see, e.g. [3], [10]) it follows that the multimap F is compact and u.s.c.

Now under assumption that $Coin(A, \Phi) \cap \partial U = \emptyset$, *the coincidence index* $Ind(A, \Phi, \overline{U})$ is defined as

$$Ind(A, \Phi, \overline{U}) = deg(i - F, \overline{U}),$$

where the right-hand part of the above equality denotes the topological degree of the compact multivalued vector field $i - F$ corresponding to the multimap F (see, e.g. [3], [10]).

It is known (see, e.g. [9], [16], [19]) that the coincidence index has all usual properties of topological characteristic of that type. Let us select two of them.

(i) If $Ind(A, \Phi, \overline{U}) \neq 0$ then $\emptyset \neq Coin(A, \Phi) \subset U$.

(ii) If $\Psi : \overline{U} \times [0, 1] \to Cv(Z)$ is a multimap such that

$$Coin(A, \Psi(\cdot, \lambda)) \cap \partial U = \emptyset$$

for all $\lambda \in [0, 1]$ and the pair (A, Ψ) is compact on $\overline{U} \times [0, 1]$ then

$$Ind(A, \Psi(\cdot, 0), \overline{U}) = Ind(A, \Psi(\cdot, 1), \overline{U}).$$

3. Results

3.1. **The statement of the problem.** We will study the existence of generalized solutions to the following equation

$$(Au)(x) + g(u(x)) = \varphi(x, u(x)) \tag{1}$$

where x belongs to a bounded domain $\Omega \in \mathbb{R}^n$ with a smooth boundary.

We assume that the following main hypothesis are satisfied:

(A1) $$A : domA := W_p^2(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega) \to L_p^1(\Omega)$$
is a linear Fredholm operator of zero index, $p \geq 2$, $2p > n$.

Further, we suggest that

(A2) A is *selfajoint* in the sense that

$$(Au, v)_{L_2} = (u, Av)_{L_2}$$

for all $u, v \in domA$, where $(u, v)_{L_2} = \int_\Omega uv\, dx$;

(A3) $dimKerA = 1$ and $\omega \in domA$ is *the basic element* of $KerA$.

Concerning the function $g : \mathbb{R} \to \mathbb{R}$ we assume that

(g) g is continuous; there exist finite limits

$$g(-\infty) = \lim_{r \to -\infty} g(r); \qquad g(+\infty) = \lim_{r \to +\infty} g(r),$$

and

$$g(-\infty) \leq g(r) \leq g(+\infty)$$

for all $r \in \mathbb{R}$.

At last, the function $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(φ1) for a.e. $x \in \Omega$ there exist finite limits

$$\underline{\varphi}(x, \xi) = \liminf_{\xi' \to \xi} \varphi(x, \xi'); \qquad \overline{\varphi}(x, \xi) = \limsup_{\xi' \to \xi} \varphi(x, \xi')$$

and the functions $\underline{\varphi}$, $\overline{\varphi}$ are superpositionally measurable;

(φ2) there exist functions $f_*, f^* \in L_p(\Omega)$ such that

$$f_*(x) \leq \varphi(x, \xi) \leq f^*(x)$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}$.

Remark 1. *Let us recall (see, e.g.* [12]*) that Carathéodory functions, pointwise limits of continuous functions, and Borel measurable functions belong to the class of superpositionnaly measurable measurable functions.*

Denote by $[f_*, f^*] \subset L_p(\Omega)$ the interval

$$[f_*, f^*] = \{f \in L_p(\Omega) : f_*(x) \leq f(x) \leq f^*(x) \text{ for a.e. } x \in \Omega\}.$$

Now define the multimap $\Phi : C(\Omega) \to Cv(L_p(\Omega))$ by the rule

$$\Phi(u) = [\underline{\varphi}(x, u(x)), \overline{\varphi}(x, u(x))]. \tag{2}$$

According to [5], Theorem 1.1 let us mention the following property.

Proposition 2. *The multimap Φ is u.s.c.*

So, denoting by $\widetilde{g} : C(\Omega) \to C(\Omega)$ the function

$$\widetilde{g}(u)(x) = g(u(x)), \;\; x \in \Omega$$

we may substitute equation (1) by the following operator inclusion

$$Au + \widetilde{g}(u) \in \Phi(u) \tag{3}$$

Definition 3. *A function $u \in domA$ satisfying inclusion (3) is called the generalized solution to equation (1).*

Suppose that function $u \in domA$ is a generalized solution to the problem. Then for some $f \in \Phi(u)$ we have

$$Au + \widetilde{g}(u) = f. \tag{4}$$

Multiplying the both sides of equality (4) by ω in $L_2(\Omega)$ and using property $(A2)$ we obtain

$$(\widetilde{g}(u), \omega)_{L_2} = (f, \omega)_{L_2}.$$

Denoting

$$\Omega_+ = \{x \in \Omega : \omega(x) > 0\}; \;\; \Omega_- = \{x \in \Omega : \omega(x) < 0\}$$

we can rewrite the last equality in the integral form

$$\int_\Omega f\omega \, dx = \int_{\Omega_+} \widetilde{g}(u)\omega \, dx + \int_{\Omega_-} \widetilde{g}(u)\omega \, dx$$

from which it obviously follows that

$$\int_\Omega f\omega \, dx \le g(+\infty) \int_{\Omega_+} \omega \, dx + g(-\infty) \int_{\Omega_-} \omega \, dx \tag{5}$$

and

$$\int_\Omega f\omega \, dx \ge g(-\infty) \int_{\Omega_+} \omega \, dx + g(+\infty) \int_{\Omega_-} \omega \, dx. \tag{6}$$

Inequalities (5) and (6) form the necessary conditions for the solvability of problem (1).

3.2. **Sufficient conditions for the solvability of the problem.** We will show that relations similar to (5) and (6) but written in the form of strong inequalities are sufficient for the existence of a generalized solution to our problem. In fact, suppose that for functions f_* and f^* from condition ($\varphi 2$) the following relations hold true:

$$\int_{\Omega_+} f^*\omega\,dx + \int_{\Omega_-} f_*\,\omega\,dx < g(+\infty)\int_{\Omega_+} \omega\,dx + g(-\infty)\int_{\Omega_-} \omega\,dx \tag{7}$$

and

$$\int_{\Omega_+} f_*\,\omega\,dx + \int_{\Omega_-} f^*\omega\,dx > g(-\infty)\int_{\Omega_+} \omega\,dx + g(+\infty)\int_{\Omega_-} \omega\,dx. \tag{8}$$

It is easy to verify that these two relations are equivalent to the following condition:

$$g(-\infty)\int_{\Omega_+} \omega\,dx + g(+\infty)\int_{\Omega_-} \omega\,dx$$
$$< \int_{\Omega} f\omega\,dx < g(+\infty)\int_{\Omega_+} \omega\,dx + g(-\infty)\int_{\Omega_-} \omega\,dx \tag{9}$$

for each $f \in [f_*, f^*]$.

Now we may formulate the following existence result.

Theorem 4. *Under conditions* (A1)-(A3), (g), (φ1)-(φ2), *and (7)-(8) (or, equivalently (9)) there exists a generalized solution to problem (1).*

Before proving the theorem let us mention that from the properties of the operator A it follows that the spaces $E = domA$ and $Z = L_p(\Omega)$ may be decomposed as

$$E = E_0 \oplus E_1,$$

where $E_0 = KerA$ and

$$Z = Z_0 \oplus Z_1,$$

where $Z_0 = KerA$ and $Z_1 = ImA$. The corresponding decompositions of elements $u \in E$ and $f \in Z$ will be denoted by

$$u = u_0 + u_1$$

and

$$f = f_0 + f_1.$$

Consider the multimap $\Psi : C(\Omega) \times [0,1] \to Cv(Z)$ given by

$$\Psi(u,\lambda) = \alpha(\Phi(u),\lambda) - \widetilde{g}_0(u) - \lambda \widetilde{g}_1(u),$$

where $\alpha : Z \times [0,1] \to Z$ is defined as

$$\alpha(f_0 + f_1, \lambda) = f_0 + \lambda f_1.$$

Lemma 5. *For each bounded set $D \subset C(\Omega)$ the multimap $\Sigma : D \times [0,1] \to Kv(C(\Omega))$,*

$$\Sigma(u,\lambda) = Pu + (\Lambda\Pi + K_{P,Q}) \circ \Psi(u,\lambda)$$

is compact u.s.c.

Proof. Denote by Z_w the space Z endowed with weak topology. From the definition of multimap Φ and condition ($\varphi 2$) it follows that the multimap Φ and hence $\hat{\Psi}$, $\hat{\Psi}(u,\lambda) = \alpha(\Phi(u),\lambda)$ have w-compact values and are w-compact. From Proposition 2 we have that the multimap $\Phi : C(\Omega) \to Kv(Z_w)$ is u.s.c. Decomposing $\hat{\Psi}$ as

$$(u,\lambda) \longrightarrow \Phi(u) \times \{\lambda\} \xrightarrow{\alpha} \hat{\Psi}(u,\lambda)$$

and using the properties of operations over multivalued maps (see, e.g. [3], [10]) we come to the conclusion that the multimap $\hat{\Psi} : C(\Omega) \times [0,1] \to Kv(Z_w)$ and hence $\Psi : C(\Omega) \times [0,1] \to Kv(Z_w)$ are u.s.c.

Now, let us demonstrate that the multimap $\Theta \circ \Psi : C(\Omega) \times [0,1] \multimap C(\Omega)$, where $\Theta = \Lambda\Pi + K_{P,Q}$ is closed. In fact, let $\{(u_n,\lambda_n)\} \subset C(\Omega) \times [0,1]$, $(u_n,\lambda_n) \to (u_0,\lambda_0)$, $\{y_n\} \subset C(\Omega)$, $y_n \in \Theta \circ \Psi(u_n,\lambda_n)$, and $y_n \to y_0$. Take a sequence $z_n \in \Psi(u_n,\lambda_n)$ such that $y_n = \Theta(z_n)$. We may assume w.l.o.g. that $z_n \underset{w}{\to} z_0$. Since Θ is the continuous linear operator, we have that $y_0 = \Theta(z_0)$. From the other side, the multimap Ψ is closed with respect to the weak topology of Z (see, e.g. [3], [10]) and hence $z_0 \in \Psi(u_0,\lambda_0)$. So $y_0 \in \Theta \circ \Psi(u_0,\lambda_0)$.

Further, the range of Ψ is a bounded subset of Z. But then the range of $\Theta \circ \Psi$ is a bounded subset of E, and by the Sobolev embedding theorem it is relatively compact subset of $C(\Omega)$. Closed and compact multimap $\Theta \circ \Psi$ is u.s.c. (see, e.g. [3], [10]) and now the assertion follows from the fact that P is continuous and have a finite-dimensional range. □

Lemma 6. *The set of coincidence points $Coin(A, \Psi)$ is a priori bounded in the space $C(\Omega)$.*

Proof. In fact, suppose that for any $(u, \lambda) \in E \times [0, 1]$ we have that

$$Au \in \Psi(u, \lambda). \tag{10}$$

Decomposing $u = u_0 + u_1$ we can write this inclusion in the following component form:

$$\begin{cases} Au_1 + \lambda\widetilde{g_1}(u) = \lambda f_1 \\ \widetilde{g_0}(u_0 + u_1) = f_0, \end{cases} \tag{11}$$

where $f = f_0 + f_1 \in \Phi(u)$. In turn, the first equality may be expressed as

$$u_1 = \lambda K_{P,Q}(f_1 - \widetilde{g_1}(u))$$

and since $f_1 - \widetilde{g_1}(u)$ belongs to a bounded subset of Z, applying the embedding theorem we come to the conclusion that the component u_1 is a priori bounded in $C(\Omega)$.

Further, multiplying the both sides of the second equality from (11) by ω in L_2 and using the orthogonality of ω to components $\widetilde{g_1}$ and f_1 we come to the equality

$$\int_\Omega \widetilde{g}(u_0 + u_1)\omega \, dx = \int_\Omega f\omega \, dx$$

Expressing $u_0 = a\omega$ we can write the above relation as

$$\int_{\Omega_+} \widetilde{g}(a\omega + u_1)\omega \, dx + \int_{\Omega_-} \widetilde{g}(a\omega + u_1)\omega \, dx = \int_\Omega f\omega \, dx$$

Now suppose to the contrary that there exist sequences $a_n \to +\infty$, $\{u_1^{(n)}\} \subset E_1$, and $f^{(n)} \in \Phi(a_n\omega + u_1^{(n)})$ such that

$$\int_{\Omega_+} \widetilde{g}(a_n\omega + u_1^{(n)})\omega \, dx + \int_{\Omega_-} \widetilde{g}(a_n\omega + u_1^{(n)})\omega \, dx = \int_\Omega f^{(n)}\omega \, dx \tag{12}$$

Since the sequence $\{u_1^{(n)}\}$ is bounded it is easy to see that the first integral in (12) tends to

$$g(+\infty) \int_{\Omega_+} \omega \, dx$$

while $n \to +\infty$ whereas the second one has the limit

$$g(-\infty) \int_{\Omega_-} \omega \, dx.$$

From the other side, the sequence $\{f^{(n)}\}$ belongs to a bounded subset $[f_*, f^*]$ and so we may assume w.l.o.g. that it weakly converges in $L_2(\Omega)$ to a function $\widetilde{f} \in [f_*.f^*]$. Passing to the limit in (12) we obtain

$$g(+\infty)\int_{\Omega_+}\omega\,dx + g(-\infty)\int_{\Omega_-}\omega\,dx = \int_{\Omega}\widetilde{f}\omega\,dx$$

that contradicts to the right part of condition (9).

Assuming that $a_n \to -\infty$ we analogously obtain he contradiction to the left part of (9). $\square$

Proof of Theorem 4. Let $U \subset C(\Omega)$ be a bounded open domain containing the set $Coin(A, \Psi)$. Let us mention that while $\lambda = 1$ inclusion (10) turns into initial inclusion (3).

Applying Lemma 5 and the property of homotopy invariance of the coincidence index we obtain

$$Ind(A, \Psi(\cdot, 1), \overline{U}) = Ind(A, \Psi(\cdot, 0), \overline{U}).$$

Let us evaluate $Ind(A, \Psi(\cdot, 0), \overline{U})$. To do this, notice that the map $\Psi(\cdot, 0)$ has the form

$$\Phi_0(u) - \widetilde{g_0}(u),$$

where $\Phi_0 = Q \circ \Phi$.

By definition, $Ind(A, \Psi(\cdot, 0), \overline{U}) = deg(i - F, \overline{U})$, where $deg(i - F, \overline{U})$ is the topological degree of a compact multivalued vector field $i - F$ corresponding to the multimap F

$$F(u) = Pu + (\Lambda\Pi + K_{P,Q})\Psi(\cdot, 0) = Pu + (\Lambda\Pi + K_{P,Q})(Q \circ \Phi(u) - \widetilde{g_0}(u))$$

$$= Pu + \Lambda\Pi(\Phi_0(u) - \widetilde{g_0}(u)).$$

W.l.o.g. we may assume that the maps $\Pi|_{Z_0}$ and Λ are identities. Then the multimap F has it range in E_0, and in accordance with the principle of map restriction (see, e.g. [3], [10])

$$deg(i - F, \overline{U}) = deg(i - F_0, \overline{U}_0),$$

where $\overline{U}_0 = \overline{U} \cap E_0$ and F_0 is the restriction of F to $\overline{U}_0$. The multifield $i - F_0$ has the form

$$\Phi_0(u_0) - \widetilde{g_0}(u_0).$$

A point u_0 may be expressed as $u_0 = a\omega$, $a \in \mathbb{R}$, where we may assume w.l.o.g. that $\|\omega\| = 1$. Take any $f_0 \in \Phi_0(u_0)$ and let $f = f_0 + f_1 \in \Phi(u_0)$. Let

$$f_0 - \widetilde{g}_0(u_0) = l\omega, \ l \in \mathbb{R}.$$

To estimate the coefficient l we can use the following:

$$l = (l\omega, \omega)_{L_2} = \int_\Omega (f_0 - \widetilde{g}_0(u_0))\omega \, dx = \int_\Omega f\omega \, dx - \int_\Omega \widetilde{g}(u_0)\omega \, dx$$

$$= \int_\Omega f\omega \, dx - \int_{\Omega_+} \widetilde{g}(a\omega)\omega \, dx - \int_{\Omega_-} \widetilde{g}(a\omega)\omega \, dx$$

$$\leq \int_{\Omega_+} f^*\omega \, dx + \int_{\Omega_-} f_*\omega \, dx - \int_{\Omega_+} \widetilde{g}(a\omega)\omega \, dx - \int_{\Omega_-} \widetilde{g}(a\omega)\omega \, dx.$$

And now from (7) it follows that $l < 0$ for a sufficiently large. At the same time applying (8) we have

$$l \geq \int_{\Omega_+} f_*\omega \, dx + \int_{\Omega_-} f^*\omega \, dx - \int_{\Omega_+} \widetilde{g}(a\omega)\omega \, dx - \int_{\Omega_-} \widetilde{g}(a\omega)\omega \, dx > 0$$

if $a < 0$ and $|a|$ is sufficiently large.

From the properties of the Brouwer degree it follows now that

$$deg(i - F_0, \overline{U}_0) = -1$$

and hence $Ind(A, \Psi(\cdot, 1), \overline{U}) = -1$ and $Coin(A, \Phi) \neq \emptyset$ proving the theorem.■

Remark 7. *It is easy to see that under above conditions the set $Coin(A, \Phi)$ is compact. So we can guarantee the existence of a generalized solution of our problem optimizing a given continuous quality functional $j : C(\Omega) \to \mathbb{R}$.*

Example 8. *Consider a membrane with fixed boundary, acted on by an external force $f \in L_p(\Omega)$, and obstructed by a fixed obstacle $\psi \in W_p^2(\Omega)$ satisfying $\psi|_{\partial\Omega} \geq 0$. Nonlinear effects of deformation are simulated by a function $g : \mathbb{R} \to \mathbb{R}$. At the presence of resonance the equilibrium position of membrane $u \in W_p^2(\Omega) \bigcap \overset{\circ}{W}{}_p^1(\Omega)$ satisfies the following partial differential equation with discontinuous nonlinearity (cf.* [5], [7]*).*

$$-\triangle u(x) + \lambda u(x) + g(u(x)) = \varphi(x, u(x))$$

where λ is the first eigenvalue of Laplasian Δ and

$$\varphi(x,\xi) = \begin{cases} \min\{f(x), (-\triangle + \lambda)\psi(x)\}, & \xi \geq \psi(x); \\ f(x), & \xi < \psi(x). \end{cases}$$

It is clear that here $f_*(x) = \min\{f(x), (-\triangle+\lambda)\psi(x)\}$ whereas $f^*(x) = f(x)$, and sufficient conditions for the existence of a solution may be written in form of (7)-(9).

Example 9. *Lavrentiev's problem on detachable currents at the presence of resonance and nonlinear perturbations may be described by the following equation (cf.* [13]*):*

$$-\triangle u(x) + \lambda u(x) + g(u(x)) = \mu\, sign(u(x)),$$

$$u(x)|_{\partial\Omega} = 0,$$

where $\mu > 0$.

We have

$$\underline{\varphi}(x,\xi) = \begin{cases} \mu, & \xi > 0, \\ -\mu, & \xi \leq 0; \end{cases}$$

and

$$\overline{\varphi}(x,\xi) = \begin{cases} \mu, & \xi \geq 0, \\ -\mu, & \xi < 0. \end{cases}$$

So

$$f_*(x) = -\mu; \qquad f^*(x) = \mu$$

and conditions (7)-(8) may be written in the following form

$$\mu(\int_{\Omega_+} \omega\, dx - \int_{\Omega_-} \omega\, dx) < g(+\infty) \int_{\Omega_+} \omega\, dx + g(-\infty) \int_{\Omega_-} \omega\, dx$$

and

$$\mu(\int_{\Omega_-} \omega\, dx - \int_{\Omega_+} \omega\, dx) > g(-\infty) \int_{\Omega_+} \omega\, dx + g(+\infty) \int_{\Omega_-} \omega\, dx.$$

4. On some generalizations

In this section we will consider the situation when the Fredholm operator A has a multidimensional kernel (i.e. $dimKerA \geq 1$) and is not necessarily selfadjoint. First we apply the coincidence degree theory to prove the abstract existence result for an operator inclusion which is the extension of Theorem 4.1.4 from [17] to the case when the Fredholm operator is not necessarily continuous and the nonlinear part is multivalued and not necessarily compact.

Let X be a Banach space; Z a reflexive Banach space;

$$A : domA \subseteq X \to Z$$

a linear Fredholm operator of zero index with $dimKerA = d \geq 1$. We will suppose that the Banach space $E = domA$ is compactly embedded into X. As earlier, we may consider the decompositions

$$E = E_0 \oplus E_1,$$

where $E_0 = KerA$ and

$$Z = Z_0 \oplus Z_1,$$

where $Z_1 = ImA$ and $dimZ_0 = d$.

For a continuous linear operator $K_{P,Q} : Z \to X$, let $C = \|K_{P,Q}\|$.

Denote by Z_w the space Z endowed with the weak topology and let

$$\Phi : X \to Kv(Z_w)$$

be an u.s.c. multimap.

Theorem 10. *Assume that the following conditions hold true:*

(i) *the mutimap Φ is bounded, i.e. there exists a constant $M > 0$ such that*

$$\|\Phi(u)\| := \sup\{\|f\| : f \in \Phi(u)\} \leq M$$

for all $u \in X$;

(ii) *there exists a constant $N > 0$ such that for each $u = u_0 + u_1 \in E$ with $\|u_1\|_X \leq CM$ and $\|u_0\|_X \geq N$ we have*

$$0 \notin Q \circ \Phi(u);$$

(*iii*)

$$deg(Q \circ \Phi|_{S_N}, S_N) \neq 0,$$

where $S_N = \{u_0 \in E_0 : \|u_0\|_X = N\}$.

Then the inclusion

$$Au \in \Phi(u)$$

has a solution.

Remark 11. *It is easy to see that that the multimap*

$$Q \circ \Phi|_{S_N} : S_N \to Kv(Z_0 \setminus \{0\})$$

is u.s.c. and hence the topological degree in condition (*iii*) *is well-defined.*

Remark 12. *Conditions* (*ii*) *and* (*iii*) *are fulfilled if*

(*iv*) *there exists an u.s.c. multimap*

$$G : S_1 \subset E_0 \to Kv(Z_0 \setminus \{0\})$$

with

$$deg(G, S_1) \neq 0$$

with the property that for each $\varepsilon > 0$ *there exists* $R > 0$ *such that*

$$Q \circ \Phi(ru_0 + u_1) \subset V_\varepsilon(G(u_0))$$

for all $u = u_0 + u_1 \in E$ *with* $u_0 \in S_1$, $\|u_1\|_X \leq CM$, *and* $r \geq R$, *where* V_ε *denotes the* ε*-neighborhood of a set.*

Proof of Theorem 10. Since we will follow the main lines of the proof of Theorem 4 we will restrict ourselves to the sketch. The multimap $\Psi : X \times [0,1] \to Kv(Z_w)$,

$$\Psi(u, \lambda) = \alpha(\Phi(u), \lambda)$$

is u.s.c. and for each bounded $D \subset X$ the multimap $\Sigma : D \times [0,1] \to Kv(X)$,

$$\Sigma(u, \lambda) = Pu + (\Lambda\Pi + K_{P,Q}) \circ \Psi(u, \lambda)$$

is compact u.s.c. The inclusion

$$Au \in \Psi(u, \lambda) \tag{13}$$

is equivalent to the system

$$\begin{cases} Au_1 = \lambda f_1 \\ 0 = f_0, \end{cases} \tag{14}$$

where $u = u_0 + u_1$ and $f = f_0 + f_1 \in \Phi(u)$. The first equality of (14) results

$$u_1 = \lambda K_{P,Q}(f_1)$$

and hence $\|u_1\|_X \leq CM$. Applying condition (*ii*) we conclude from the second equality of (14) that $\|u_0\|_X \leq N$.

So the ball $B_L \subset X$ centered at the origin of radius $L = CM + N + 1$ a priori contains all solutions $u \in X$ of inclusion (13) and we have

$$ind(A, \Psi(\cdot, 1), \overline{B}_L) = ind(A, \Psi(\cdot, 0), \overline{B}_L).$$

Applying condition (*iii*), the map restriction principle and other basic properties of topological degree we obtain

$$ind(A, \Psi(\cdot, 0), \overline{B}_L) \neq 0$$

that concludes the proof. ■

Corollary 13. *Let*

$$A : domA := W_p^2(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega) \to L_p^1(\Omega)$$

be a linear Fredholm operator of zero index, $p \geq 2$*,* $2p > n$*. Let a function* $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies conditions* ($\varphi 1$) *and* ($\varphi 2$)*. If the multimap*

$$\Phi : C(\Omega) \to Cv(L_p(\Omega))$$

generated by φ *(see (2)) satisfies conditions* (*ii*) *and* (*iii*) *(or, respectively,* (*iv*)*) of Theorem 10 (for* $X = C(\Omega)$ *and* $Z = L_p(\Omega)$*) then the equation*

$$(Au)(x) = \varphi(x, u(x))$$

has a generalized solution.

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Received: February 13, 2007; Accepted: February 22, 2007.